ANALYSIS OF A SQUARE COAXIAL LINE WITH ANISOTROPIC SUBSTRATES BY STRONG FEM FORMULATION

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Abstract. *In this paper the concept of the strong Finite Element Method (FEM) formulation is explained first. Next, a brief review of strong basis functions that are used for quasi-static analysis of transmission lines with piecewise homogeneous anisotropic medium is presented. As numerical examples, effective relative permittivities of square coaxial lines with two anisotropic layers or one isotropic and one anisotropic layer are calculated by using the Galerkin version of the strong FEM formulation. High accuracy of the method is demonstrated for the layer thicknesses ranging from 0 to 100% of the transmission line height. It is also shown that in the case of the half-filled line, effective relative permittivity computed by the FEM is practically equal to the one obtained by a simple formula.*

Key words: *strong FEM formulation, quasi-static analysis, anisotropic dielectric, square coaxial line.*

1. INTRODUCTION

One of the frequent and multidisciplinary methods used for calculation of electromagnetic (EM) fields is the Finite Element Method. FEM belongs to the group of numerical methods that are used for approximate solving of the boundary value problems in mathematical physics (partial differential equations whose order is two or higher, with the given boundary conditions). In almost all the available literature the *weak* FEM formulation [1]–[3] is used. Weak formulation is based on basis functions that are not in the domain of the original differential operator (usually of the second order). Furthermore, most often low order approximations (i.e. of the first order) are used. On the other hand, only several published papers deal with the *strong* FEM formulation. First studies in this research area [4]–[6] have shown that strong formulation may have certain advantages in comparison to weak formulation. These advantages are reflected in the conceptual simplicity, simpler and more natural inclusion of boundary conditions and inherent higher order of approximation (the lowest order being three). A convenient choice of basis functions satisfies both

boundary conditions and results in much smaller number of unknowns for the same approximation order.

For numerical examples in this paper, square coaxial lines with anisotropic layers are chosen, as a special case of the shielded planar lines. Shielded planar transmission lines with anisotropic dielectrics were subject to previous research, where their propagation characteristics were calculated by the use of various numerical and analytical methods. In [7] rectangular coaxial lines with homogeneous anisotropic dielectric were analyzed and their capacitance calculated by the use of an expanded charge simulation method and affine transformations. In [8] an analytical technique (the spectral domain technique in discrete Fourier variable) is presented for quasi-static analysis of rectangular lines with homogeneous and inhomogeneous anisotropic dielectric. In [9] several method classes (quasi-static, dynamic, empirical; analytical, numerical) and a number of methods (Method of Moments, Finite differences method, Transmission-line matrix technique, Modified Wiener-Hopf method, Fourier series techniques, Method of lines) were discussed and applied for calculation of the propagation characteristics of several typical planar structures. The paper, however, does not mention FEM. In [10] weak FEM formulation is applied to analysis of square coaxial lines with inhomogeneous anisotropic dielectric. In [4], [6] and [11] strong FEM formulation for anisotropic medium is presented and applied to analysis of shielded transmission lines with homogeneous anisotropic dielectric. Its high accuracy is demonstrated by comparison with results obtained by other numerical methods and by the commercial software.

This paper is aimed to generalize the strong FEM formulation for anisotropic homogeneous medium to anisotropic *inhomogeneous* (piecewise homogeneous) medium and apply the method to calculation of effective permittivity of such square coaxial lines. To the best of authors' knowledge, there are no published papers on the strong FEM formulation applied to those structures.

Considering square coaxial lines as an example of a simple geometry, they are not only an excellent benchmark for numerical methods, but are also advantageous for practical measurements of constitutive parameters of anisotropic materials [12].

2. STRONG AND WEAK FORM OF BOUNDARY VALUE PROBLEM, STRONG AND WEAK FORMULATION

Let the computational domain Ω, be filled with anisotropic (possibly inhomogeneous) medium of parameters $\bar{\bar{\varepsilon}}$ and β and bounded by $\Gamma = \Gamma_1 \bigcup \Gamma_2$. In Ω and on Γ, let the following differential equation and boundary conditions be given:

$$-\nabla \cdot (\bar{\bar{\varepsilon}} \nabla f) + \beta f = g \,, \tag{1}$$

$$f = V_0, \text{ on } \Gamma_1, \tag{2}$$

$$\mathbf{n} \cdot (\bar{\bar{\varepsilon}} \nabla f) = -A_{n0}, \text{ on } \Gamma_2 \,. \tag{3}$$

In (1) f denotes the unknown function, function g represents sources, i.e. excitations, V_0 and A_{n0} are known values on the boundary. If a domain is spatial (3-D), it is bounded by surfaces. If it is a surface (2-D), it is bounded by lines (contours). By introducing vector

$\mathbf{A} = -\overline{\overline{\varepsilon}}\,\mathrm{grad}\,f$, differential equation (1) is written in the form $\mathrm{div}\,\mathbf{A} + \beta f = g$. In a 3-D case and for a diagonal tensor $\overline{\overline{\varepsilon}} = \mathrm{diag}[\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}]$, equation (1) can be written in Cartesian coordinate system as a compact operator form,

$$Lf = g, \tag{4}$$

where operator L is defined by

$$L = -\frac{\partial}{\partial x}\left(\varepsilon_{xx}\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(\varepsilon_{yy}\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial z}\left(\varepsilon_{zz}\frac{\partial}{\partial z}\right) + \beta. \tag{5}$$

Expression (2) represents *Dirichlet* boundary condition, whereas expression (3) represents *Neumann* boundary condition. If the parameters of the medium have abrupt changes on the surface (in 3-D problems) or on the contour (in 2-D problems) that separates mediums 1 and 2, it is necessary to satisfy *Dirichlet* and *Neumann* boundary conditions on the boundary between mediums, i.e. continuity of both function f, $f_1 = f_2$, and its generalized first derivative, $\mathbf{A}_1 \cdot \mathbf{n} = \mathbf{A}_2 \cdot \mathbf{n}$ ($\mathbf{n} \cdot (\overline{\overline{\varepsilon}}_1 \nabla f_1) = \mathbf{n} \cdot (\overline{\overline{\varepsilon}}_2 \nabla f_2)$), where $\mathbf{n}$ is the unit vector normal to the boundary and directed into medium 1. In the literature [1]–[3], the problem defined by (1)–(3), which contains a differential equation with boundary conditions, is referred to as the *strong form* for a given contour problem. In the given case, the strong form contains second derivative.

The solution of equations (1)–(3) is adopted in the form of approximation function, usually a polynomial with initially unknown coefficients. In general, mathematical functions can exhibit different orders of continuity: C^0 represents the continuity of the function itself, C^1 – continuity of both function and its first derivative(s). In general, C^m continuity represents continuity of function and its derivatives up to the *m*-th order. When the problem is described by a system of partial differential equations, *strong formulation* requires that the approximation function be in the domain of the original differential operator, i.e. of the operator in the *strong form*, throughout the entire domain Ω [13]. This means the continuity of both function and its derivatives up to the order one less than that of the original differential operator [3]. The strong form requires the strong formulation. In case of the approximation function in equation (1), or alternatively eq. (4), we introduce the concept of a generalized C^1 continuity, as the continuity of the expression $\varepsilon_{xx}(\partial f / \partial x)$ with respect to x, $\varepsilon_{yy}(\partial f / \partial y)$ with respect to y and $\varepsilon_{zz}(\partial f / \partial z)$ with respect to z, so that $\nabla \cdot (\overline{\overline{\varepsilon}}\nabla f)$ is regular. This also implies that both boundary conditions (continuity of f and $\mathbf{n} \cdot \mathbf{A}$) must be satisfied on every boundary inside Ω, including boundaries between finite elements, if the problem is solved by FEM. Approximation function is represented as a sum of basis functions multiplied by initially unknown coefficients. Sufficient condition that the approximation function is a generalized C^1 function, i.e. that the formulation is strong, is that basis functions are generalized C^1 functions. Such functions are called strong basis functions for a given problem, both in homogeneous (constant $\overline{\overline{\varepsilon}}$ and β) and inhomogeneous ($\overline{\overline{\varepsilon}}(x, y, z)$, $\beta(x, y, z)$) medium [4],[6].

In order to solve problems easier, conditions of the strong form are commonly weakened, through integration, resulting in a wider class of possible approximate solutions. An arbitrary test (weighted) function w and the integral of the weighted residual for equation (1),

$$\int_\Omega w(-\nabla\cdot(\overline{\overline{\varepsilon}}\nabla f)+\beta f-g)\mathrm{d}\Omega=0\,, \tag{6}$$

are introduced. The first term inside the integral (the term that contains the second derivative) can be transformed by the use of the Gauss-Ostrogradski theorem [1],[14],

$$\int_\Omega w(-\nabla\cdot(\overline{\overline{\varepsilon}}\nabla f))\mathrm{d}\Omega=\int_\Omega w\,\mathrm{div}\,\mathbf{A}\mathrm{d}\Omega=\int_\Omega (\mathrm{div}(w\mathbf{A})-\mathbf{A}\cdot\nabla w)\mathrm{d}\Omega$$

$$=\oint_\Gamma w\mathbf{A}\cdot\mathbf{d\Gamma}-\int_\Omega \mathbf{A}\cdot\nabla w\mathrm{d}\Omega=-\oint_\Gamma w(\overline{\overline{\varepsilon}}\nabla f)\cdot\mathbf{d\Gamma}+\int_\Omega (\overline{\overline{\varepsilon}}\nabla f)\cdot\nabla w\mathrm{d}\Omega, \tag{7}$$

so that equation (6) can now be written in the form

$$-\oint_\Gamma w(\overline{\overline{\varepsilon}}\nabla f)\cdot\mathbf{d\Gamma}+\int_\Omega ((\overline{\overline{\varepsilon}}\nabla f)\cdot\nabla w+w\beta f-wg)\mathrm{d}\Omega=0\,, \tag{8}$$

where only first derivatives exist. Equation (8) represents the weak form.

Weak form of equations, instead of using the original operator L, in which C^1 is required, uses the extended operator (not explicitly expressed) that requires only C^0 continuity. This means that on interelement boundaries only C^0 continuity is required. Such approximation is the weak formulation. In weak formulations approximation functions have one order lower continuity in comparison with the strong formulation. In the weak formulation, boundary condition for normal components of vector $\mathbf{A}$ are not exactly satisfied. They are satisfied in approximate sense, through the weighted residuals. This introduces artificial charges at interelement boundaries. Details on this can be seen in [13]. Both strong and weak formulations can be applied in the weak form.

3. BASIC FEM METHODOLOGY FOR A 2-D CASE AND AN ANISOTROPIC DIELECTRIC

Let now the domain Ω be two-dimensional, e.g., uniform with respect to the z-axis, filled with linear, anisotropic dielectric without free charges, in which the distribution of electrostatic potential, $V(x,y)$, is the unknown function. Let dielectric be discontinuously inhomogeneous. In this case $\beta=0$ and differential equation for potential V is

$$\mathrm{div}_\mathrm{s}(\overline{\overline{\varepsilon}}\,\mathrm{grad}_\mathrm{s}V)=0\,, \tag{9}$$

where div_s and grad_s are surface divergence and surface gradient, respectively. In case of 2-D problems and for the particular choice of coordinate axis along the crystallographic axis, $\overline{\overline{\varepsilon}}=\mathrm{diag}[\varepsilon_{xx}\ \varepsilon_{yy}]$ [9]. In this paper we will consider only such (diagonal) permittivity tensors. Computational domain is divided into M finite elements. Let the elements be rectangular and homogeneous, so that the surfaces of discontinuity of $\overline{\overline{\varepsilon}}$ coincide with interelement boundaries. Exact solution $V(x,y)$ is substituted by approximate solution expressed as a linear combination of basis function with unknown coefficients, $V\approx f=\sum_{j=1}^{N}a_jf_j$. Following the Galerkin procedure [1], the system of linear algebraic equations for unknown coefficients is obtained,

$$[K_{ij}][a_j]=[G_i],\quad i,j=1,...,N, \tag{10}$$

$$K_{ij} = \int\limits_{S} (\text{grad } f_i)(\bar{\bar{\varepsilon}} \text{ grad } f_j) \text{d}S \ , \quad G_i = \int\limits_{\Gamma_2} f_i D_{n0} \text{d}l \ , \tag{11}$$

where $D_{n0} = -\varepsilon_{xx,yy}(\partial V/\partial n)$ is a known normal component of the electric induction vector **D** on the contour Γ_2 with indices xx and yy corresponding to the orientation of the unit normal **n**, i and j are global indices of basis functions and S is the union of all the finite elements surfaces, $S = \bigcup_{e=1}^{M} S^e$. The approximate solution f is obtained by solving the system of equations (10) [5]. The matrix of this system, $[K_{ij}]$, is a sparse matrix. Such systems are usually solved by using specialized computer routines for sparse systems (in this paper we used [15]).

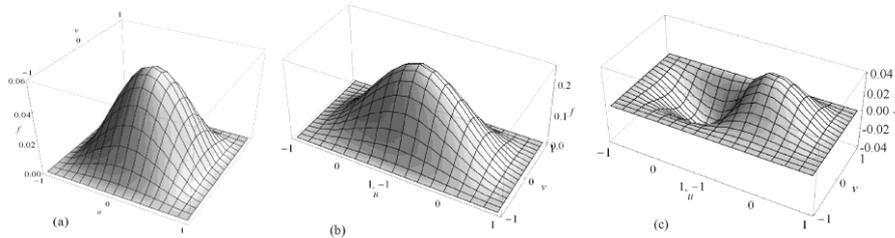## 4. STRONG BASIS FUNCTIONS FOR ANISOTROPIC MEDIUM

Unlike conventional node-based functions [1]–[3], we will use basis functions that are not node-based and are of the strong type. 2-D basic functions for strong formulation are obtained by mutual multiplying pairs of 1-D basis functions for the strong formulation [16],

$$f_k(u) = \frac{1}{4} \begin{cases} (u-1)^2(u+2), & k=1 \\ (u-1)^2(u+1)\dfrac{L^e}{2}, & k=2 \\ (u-1)^2(u+1)^{k-1}, & k=3,...,n-1 \\ (u+1)^2(2-u), & k=n \\ (u+1)^2(u-1)\dfrac{L^e}{2}, & k=n+1. \end{cases} \tag{12}$$

In 2-D strong formulation all basis functions are products of the two 1-D basis functions of the two orthogonal coordinates u and v. Thus, continuity of the function's first derivative (C^1 continuity) on all the boundaries between elements is automatically satisfied. Basis functions in this case (except the first two, for $j=1$, $j=2$ and last two, for $j=n$, $j=n+1$) are polynomials of different order and as such they are linearly independent. The order of all the other four polynomials is 3, but it is easy to show that they are linearly independent, i.e. none of them is a linear combination of the others. Strong formulation of 2-D problems has the minimal order of basis function equal to 3. A complete set of 2-D strong basis functions consists of singlets (basis functions defined over a single finite element), doublets (basis functions defined over two adjacent elements) and quadruplets (basis functions defined over a maximum of four adjacent elements that have a common node) [5], [17].

In Fig. 1 are shown one singlet and two kinds of doublets in the x-direction, respectively. Analogously, there are two kinds of doublets in the y-direction. D1-x doublet provides C^0 continuity, whereas D2-x doublet provides C^1 continuity. In Fig. 2 are shown four kinds

of quadruplets, where Q1 provides C^0 continuity whereas Q2-x and Q2-y provide C^1 continuity in the x and y direction, respectively.

Fig. 1 (a) Singlet, (b) D1-x doublet, (c) D2-x doublet
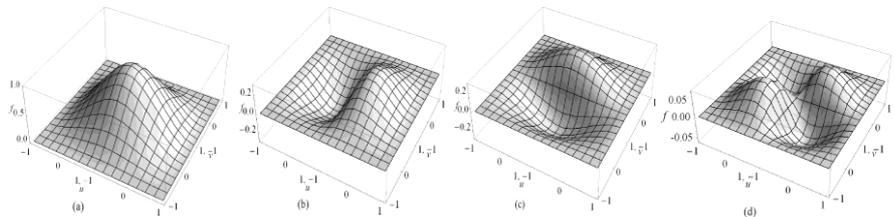
**Fig. 2** Quadruplets: (a) Q1, (b) Q2-x, (c) Q2-y, (d) Q3

Two-dimensional strong basis functions for anisotropic mediums can be formed as [6]

$$f_j(u,v) = f_{k,l}(u,v) = f_k(u)f_l(v)F_{k,l}^e, \tag{13}$$

where $F_{k,l}^e$ is a constant factor defined within e-th element, providing D_n continuity across the interelement boundaries (retaining continuity of potential V over the boundaries at the same time). This factor is required only if k and/or l are equal to 2 or $n + 1$. From this condition is derived general continuity of strong basis functions.

For piecewise homogeneous anisotropic dielectric, a complete set of strong basis functions can be applied under certain conditions. We will consider here a case where dielectric is homogeneous in the x-direction and piecewise homogeneous in the y-direction (Fig. 3). Let us find which basis functions (singlets, doublets and quadruplets) can be accepted in this case. All the functions that have zero derivative at their boundaries can be automatically accepted, as they do not participate in the *Neumann* boundary condition. These are all the singlets, doublets D1-x and D1-y and the quadruplet Q1. Doublet D2-x is defined in the homogeneous region, so no modification is needed; it is directly accepted into the set of basis functions. Doublet D2-y spans over two homogeneous elements of different $\bar{\bar{\varepsilon}}$. In order that it can be accepted in the collection of strong basis functions, it must provide continuity of D_n between the two elements. One way to establish this is to multiply it by the factor $F_{k,l}^e = 1/\varepsilon_{ryy}^e$, which is different for the two elements [6]. (Relative permittivity is used here for simplicity.) Considering the inclusion

of the quadruplet Q2-x the sufficient condition $\varepsilon_{rxx}^1/\varepsilon_{rxx}^2 = \varepsilon_{rxx}^3/\varepsilon_{rxx}^4$ is automatically satisfied, as both fractions are equal to 1 and no additional factor is needed. Considering the inclusion of the quadruplet Q2-y the sufficient condition $\varepsilon_{ryy}^1/\varepsilon_{ryy}^3 = \varepsilon_{ryy}^2/\varepsilon_{ryy}^4$ is also automatically satisfied. However, now the factors are needed, as direction of derivative is across the boundary between two different mediums. The simplest choice is $F_{k,l}^e = 1/\varepsilon_{rxx}^e$ for all the four elements in Fig. 3. Finally, for the quadruplet Q3 sufficient condition is $(\varepsilon_{rxx}^1 \varepsilon_{ryy}^2)/(\varepsilon_{rxx}^2 \varepsilon_{ryy}^1) = (\varepsilon_{rxx}^3 \varepsilon_{ryy}^4)/(\varepsilon_{rxx}^4 \varepsilon_{ryy}^3)$. As both fractions are equal to one, this condition is satisfied. Then, the simplest choice is $F_{k,l}^e = 1/\varepsilon_{ryy}^e$. With these additional factors all the singlets, doublets and quadruplets are approved to participate in the set of strong basis functions. Some of those doublets and quadruplets are in the next step (enforcing *Dirichlet* boundary conditions on the conductor surfaces) simply omitted from the set. The remaining basis functions enter the Galerkin procedure to determine the unknown coefficients.
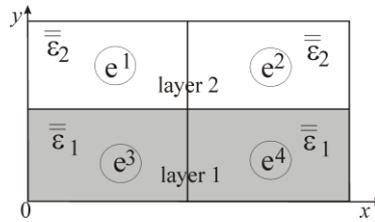

Fig. 3 Two-layer anisotropic medium divided into four finite elements: e^1, e^2, e^3 and e^4

In this way it is shown that in the case when the boundary lines between domains of different parameters are straight, mutually parallel lines, the complete set of strong basis functions can be applied.

5. NUMERICAL RESULTS

Applying the Galerkin version of FEM [4]–[6],[10],[18], using the complete set of strong basis functions, we have calculated effective relative permittivity, ε_{re}, of square coaxial lines shown in inset of Fig. 4. We performed verification comparing results obtained by the strong FEM formulation with those obtained by the weak FEM formulation from [10], by the other available software FEMM [19] and, in special cases, with simple analytical formulas.

In Fig. 4 dependence of ε_{re} on the relative height of a dielectric layer, h/b, above which is air, is shown for the three anisotropic dielectrics. Fig. 5 shows the same dependence but in the case where both layers are dielectrics; in three cases both dielectrics are anisotropic and in one case combination of isotropic-anisotropic dielectrics is applied.
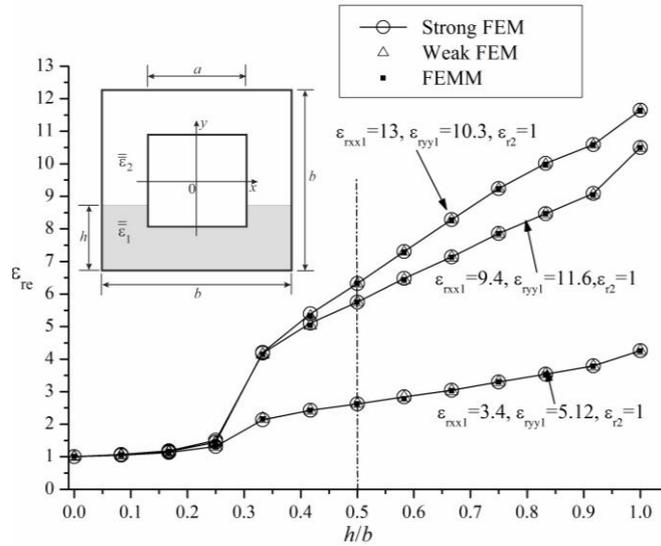
Fig. 4 Effective relative permittivity as a function of the dielectric relative height, h/b, for three different anisotropic dielectrics over which is air, obtained by using strong FEM formulation, weak FEM formulation [10] and the commercial software FEMM [19], for $b/a = 3$
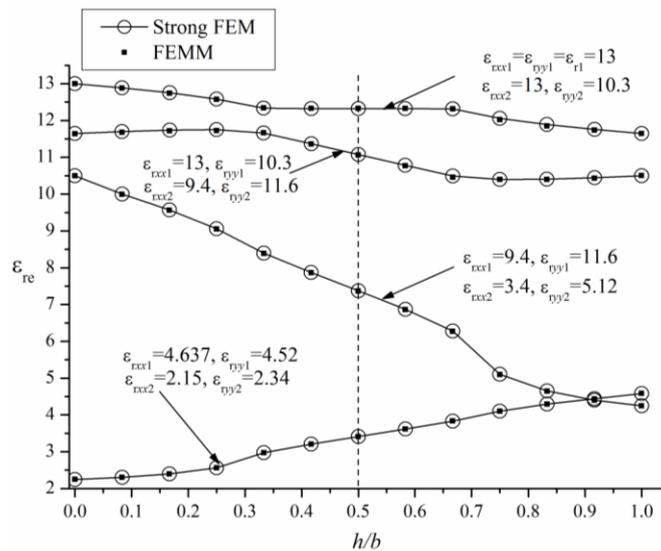
Fig. 5 Effective relative permittivity ε_{re} as a function of the dielectric relative height, h/b, for three different cases of double-layered anisotropic dielectrics and one example of isotropic-anisotropic dielectrics, for $b/a = 3$

These dependencies are shown for the following anisotropic dielectrics[*]: Boron Nitride, BN ($\varepsilon_{rxx} = 3.4$, $\varepsilon_{ryy} = 5.12$), Sapphire ($\varepsilon_{rxx} = 9.4$, $\varepsilon_{ryy} = 11.6$), Epsilam 10 ($\varepsilon_{rxx} = 13$, $\varepsilon_{ryy} = 10.3$), α-quartz ($\varepsilon_{rxx} = 4.52$, $\varepsilon_{ryy} = 4.637$) and PTFE Glass ($\varepsilon_{rxx} = 2.15$, $\varepsilon_{ryy} = 2.34$). These dielectrics are uniaxial crystals, for which cutting the layers perpendicularly to the axis of symmetry (optical axis, here a y-axis) results in a diagonal permittivity tensor [9]. Relative height of the dielectric substrate, h/b, is varied from 0 to 1. Excellent mutual agreement of all the groups of results on both diagrams can be observed.

Another comparison can also be made. If the line is completely filled with an anisotropic dielectric, effective relative permittivity is $\overline{\varepsilon}_{re} \approx (\varepsilon_{rxx} + \varepsilon_{ryy})/2$. Here, a dash over ε_{re} denotes both the average value and that it is obtained by the formula and not by the FEM. Applying this expression, the effective relative permittivity is $\varepsilon_{re} = 11.65$ for Epsilam 10, $\varepsilon_{re} = 4.26$ for BN, $\varepsilon_{re} = 10.5$ for Sapphire, $\varepsilon_{re} = 4.5787$ for α-quartz and for PTFE Glass is $\varepsilon_{re} = 2.245$. Differences between values obtained by strong FEM formulation and the formula, although both groups of values are approximate, were found to be less than 0.5% (see results from Fig. 5, $h/b = 0.0, 1.0$).

For the half-filled line ($h/b = 0.5$) with two anisotropic dielectrics (or with one isotropic and one anisotropic dielectric), approximate formula

$$\overline{\varepsilon}_{re} \approx \frac{\varepsilon_{rxx1} + \varepsilon_{ryy1} + \varepsilon_{rxx2} + \varepsilon_{ryy2}}{4} \qquad (14)$$

can be used. Table 1 shows relative permittivity of the square coaxial transmission line half-filled with two anisotropic dielectrics, $\varepsilon_{re}^{strongFEM}$ ($h/b = 0.5$), compared with $\overline{\varepsilon}_{re}$, calculated from the above formulas. Excellent agreement can be observed. (Derivations of the above formulas are given in Appendix III.)

Table 1 Relative permittivity of the square coaxial transmission line half-filled with two anisotropic dielectrics or one anisotropic and one isotropic dielectric

Dielectrics	Sapphire / Epsilam 10	BN / Sapphire	Epsilam 10 / isotropic dielectric, $\varepsilon_r = 13$
$\varepsilon_{re}^{strong\,FEM}$ ($h/b = 0.5$)	11.0786	7.4004	12.3249919
$\overline{\varepsilon}_{re}$	11.075	7.38	12.325
$\delta(\%)$	0.033	0.276	0.0008

Comparison between the number of unknowns required for the accuracy better than 0.5% can be made. For the weak FEM formulation the number of (rectangular) mesh elements was 512 and the number of unknowns was 4416 [10]. For the strong FEM formulation in this paper and the same mesh of 512 elements the number of unknowns is 1472. The order of basis functions was $n_x = n_y = 3$ for both formulations. For low-order weak formulation [18] number of (triangular) mesh elements was between 5895 and 6019 and the number of unknowns between 3097 and 3177.

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[*] Permittivity values taken from [9] and [20].

6. CONCLUSION

In this paper the strong FEM formulation for piecewise homogeneous anisotropic medium with a diagonal permittivity tensor was defined for the closed quasistatic 2-D problems. It is shown that for such a medium, the complete set of strong basis functions (singlets, doublets and quadruplets) can be used. The square coaxial transmission line has been analyzed by the presented method. Obtained results for effective relative permittivity for the line partly or completely filled with anisotropic dielectric have shown that the strong FEM formulation of the third order is exceptionally accurate. Calculated values are found to be in excellent agreement (better than 0.5%) with those obtained by the other available software and by simple approximate formulas. For the same accuracy, number of unknowns for the strong formulation was less than one half of the number of unknowns required for the weak formulation.

Practical scope of the method is the analysis of all closed (shielded) planar transmission lines with anisotropic dielectric whose permittivity tensor is diagonal. Perspectives of the method are its generalizations to 3-D and open problems.

APPENDIX I.
POSSIBILITY OF EXTENSION OF THE METHOD TO A NON-DIAGONAL PERMITTIVITY TENSOR

Presented strong FEM formulation is based on basis functions that automatically satisfy both boundary conditions (for E_t and D_n) at interelement boundaries, provided that permittivity discontinuities coincide with those boundaries. Let us examine, for example, doublet D1-x (Fig. 1b) in this regards. This doublet should provide both non-zero function value and zero normal component of the field at the interelement boundary. For non-diagonal permittivity tensors of the two elements forming a doublet, $\bar{\bar{\varepsilon}}_e = [\varepsilon_{xx}^e \ \varepsilon_{xy}^e ; \varepsilon_{yx}^e \ \varepsilon_{yy}^e]$, $e = 1,2$, from continuity of the function along the boundary follows $E_{1t} = E_{2t}$ and from the doublet property follows $E_{1n} = E_{2n} = 0$. Next, $D_{1n} = \varepsilon_{xx}^1 E_{1n} + \varepsilon_{xy}^1 E_{1t}$, $D_{1n} = \varepsilon_{xx}^2 E_{2n} + \varepsilon_{xy}^2 E_{2t}$. As, in general, $\varepsilon_{xy}^1 \neq \varepsilon_{xy}^2$, it follows that $D_{1n} \neq D_{2n}$, thus this doublet in this case does not satisfy boundary condition for D_n. Multiplying parts of the doublet inside each of the two elements by different factors in order to satisfy boundary condition for D_n would ruin the boundary condition for E_t. The same reasoning is valid for the quadruplet Q1 (Fig. 2a). Without doublets D1-x, D1-y and quadruplet Q1, approximation is not possible, as, e.g. the function value will be forced to zero in nodes. Thus, the presented method is not applicable in this case. This, however, is not a significant shortcoming, as anisotropic substrates are in practice often cut perpendicularly to their optical axis, which results in a diagonal permittivity tensor.

APPENDIX II.

POSSIBILITY OF EXTENSION OF THE METHOD TO ARBITRARY SHAPED 2-D GEOMETRIES

Presented strong FEM formulation is based on basis functions that automatically satisfy both boundary conditions at interelement boundaries which, by themselves, should coincide with u or v coordinate lines. Also, permittivity tensor must be diagonal with respect to the applied coordinate system. Thus, any deformation (transformation $(x, y) = f(u,v)$) of originally straight u-v coordinate lines that 1. preserves mutual orthogonality of lines, 2. provides that discontinuities of dielectric coincide with either u- or v-lines and 3. preserves the diagonal property of $\bar{\bar{\varepsilon}}$ ($\bar{\bar{\varepsilon}} = \mathrm{diag}[\varepsilon_{uu}\ \varepsilon_{vv}]$) is possible. E.g., for a coaxial line with *isotropic* dielectric and dielectric discontinuities either along the radial or along the angular coordinate of a polar coordinate system, the u-v mesh (and the corresponding finite elements) that coincides with polar coordinate lines enables the strong FEM formulation. For anisotropic dielectric, however, the diagonal property of $\bar{\bar{\varepsilon}}$ for the curved u-v-coordinates is very unlikely to be provided in practice, so the presented FEM formulation for the curved geometries and anisotropic dielectric is practically not possible.

APPENDIX III.

ON THE EXPLICIT FORMULAS FOR EFFECTIVE PERMITTIVITY USED IN THIS PAPER

Distribution of the electrostatic potential, $V(x,y)$, inside the TEM line with homogeneous *isotropic* dielectric is independent of its permittivity, as it is the solution of the Laplace equation, $\Delta V = 0$. For a square coaxial line, due to symmetry, along x- and y-axis (according to the coordinate system shown in inset of Fig. 4) is also $E_n = 0$ (component normal to the axis). Therefore, if this line is half-filled with isotropic dielectrics, potential distribution inside each of the two dielectrics is the same as the whole line is filled with that dielectric and independent of permittivities, so its capacitance per unit length is $C' = (C'_1 + C'_2)/2$, where $C'_{1,2}$ are values for the cases when the line is completely filled with dielectric 1 or 2. From this follows that effective relative permittivity is $\varepsilon_{re} = (\varepsilon_{r1} + \varepsilon_{r2})/2$.

For the TEM line with homogeneous *anisotropic* dielectric with a diagonal permittivity tensor, distribution $V(x,y)$ is the solution of equation $\varepsilon_{rxx}\partial^2 V/\partial x^2 + \varepsilon_{ryy}\partial^2 V/\partial y^2 = 0$, so it depends on ε_{rxx} and ε_{ryy} (more precisely on their ratio). In the case of the square coaxial line, along the x- and y-axis is again, due to symmetry, $E_n = 0$, but we note that now $V(x,0)$ and $V(0,y)$ depends on ε_{rxx} and ε_{ryy}. For a given ratio b/a, the capacitance per unit length of the line is $C' = f(\varepsilon_{rxx}, \varepsilon_{ryy}) = f(\varepsilon_{ryy}, \varepsilon_{rxx})$. After the change of variables, $p = \varepsilon_{rxx} + \varepsilon_{ryy}$, $q = \varepsilon_{rxx} - \varepsilon_{ryy}$, this transforms to $C' = g(p,q) = g(p,-q)$. In the special case of the isotropic dielectric ($q = 0$), $g(p,0) = \varepsilon_{rxx}C'_0 = pC'_0/2$. From those properties follows that the Taylor series of $g(p,q)$ around $q = 0$ and arbitrary p, up to linear terms is $g(p,q) \approx g(p,0)$. Thus, for small $\varepsilon_{rxx} - \varepsilon_{ryy}$, $\varepsilon_{re} = g(p,q)/C'_0 \approx p/2 = (\varepsilon_{rxx} + \varepsilon_{ryy})/2$.

For the same line, but half-filled with two anisotropic dielectrics with a diagonal permittivity tensor, potential distribution is not simply composed of the two distributions from the two cases when the line is completely filled with one or the other dielectric, because $V(x,0)$ and $V(0,y)$ (and thus, E_t on dielectric boundary) depend on elements of the two permittivity tensors. For the capacitance per unit length of such a line, formula

$C' \approx (C'_1 + C'_2)/2$ is now an approximation for small $\varepsilon_{rxx1} - \varepsilon_{ryy1}$ and $\varepsilon_{rxx2} - \varepsilon_{ryy2}$. After substitution $C'_{12} = \varepsilon_{re1,2} C'_0$ and $\varepsilon_{re1,2} \approx (\varepsilon_{rxx1,2} + \varepsilon_{ryy1,2})/2$, an approximate formula $\varepsilon_{re} \approx (\varepsilon_{rxx1} + \varepsilon_{ryy1} + \varepsilon_{rxx2} + \varepsilon_{ryy2})/4$ is obtained. This formula is exact in the limiting case $\varepsilon_{rxx1} \to \varepsilon_{ryy1}$ and $\varepsilon_{rxx2} \to \varepsilon_{ryy2}$ (isotropic dielectrics).

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