A Space of Fuzzy Orderings

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Abstract

In this paper the chain length of a space of fuzzy orderings is defined, and various properties of this invariant are proved. The structure theorem for spaces of finite chain length is proved.

Spaces of Fuzzy Orderings

Throughout X = (X,A) denoted a space of fuzzy orderings. That is, A is a fuzzy subgroup of abelian group G of exponent 2. (see [1] (i.e. $x^2 = 1$, $\forall\ x \in$ G), and X is a (non empty) fuzzy subset of the character group χ (A) = Hom(A,{1,−1}) satisfying:

1. X is a fuzzy closed subset of χ (A).
2. $\exists$ an element $e \in$ A such that $\sigma(e) = -1\ \forall\ \sigma \in$ X.
3. $X^{\perp} := \{a \in A \backslash\ \sigma(a) = 1\ \forall\ \sigma \in X\} = 1$.
4. If f and g are forms over A and if $x \in D(f \oplus g)$ then $\exists\ y \in D(f)$ and $z \in D(g)$ such that $x \in D<y, z>$.

Observe, by 3, that the element $e \in$ A whose existence is asserted by 2 is unique. Also, $e \neq 1$ (since $\sigma(1) = 1\ \forall\ \sigma \in$ X).

Notice that for $a \in$ A, the set $X(a) := \{\sigma \in X \mid \sigma(a) = 1\}$ is clopen (i.e. both closed and open) in X. Moreover, $\sigma(a) = -1 \Leftrightarrow \sigma(-a) = 1$ holds for any $\sigma \in$ X (by 2).

Definition 1

A forms f and g are said to be isometric (over X) if they have the same dimension and $\sigma(f) = \sigma(g)\ \forall\ \sigma \in$ X. This is denoted by writing $f \cong g$ or $g \cong f$ (over X).

Note A form f is said to represent the element X $\in$ A (over X) if $\exists$ elements $x_1,\ldots,x_n \in$ A such that $f \cong < x, x_2,\ldots, x_n > \cdot$ D(f) or D(f,X) will be used to denote the set of elements of A which are represented by f in this sense.

Definition 2

A form f is said to be isotropic if $\exists\ x_3, \ldots, x_n \in$ A, such that $f \cong <1,-1, x_3, \ldots, x_n>$. Notice, in particular, this implies dim(f) ≥ 2. A form which is not isotropic is said to be anisotropic, for any $x \in$ A, $< x,-x > \cong <1,-1>$.
any such form will be called a hyperbolic plane.

Theorem 1

The following are equivalent

(i) $\forall\ x \in G, x \neq -1 \Rightarrow D<1, x> = \{1, x\}$.
(ii) $X = \{\alpha \in \chi(A) \mid \alpha(-1) = -1\}$.

Proof: see [3].

A space of fuzzy ordering satisfying either of the equivalent conditions in theorem 1 will be referred to as a fan.

Corollary 1

Suppose X is a fan. Then every subspace of X is also a fan.

Proof: compare [3].

Recall, a space of fuzzy orderings (X,A) is said to be finite if X (or equivalently A) is finite fuzzy set; and two spaces of fuzzy orderings (X,A) and (X′,A′) are said to be isomorphic if there exists a group isomorphism $\alpha:A \longrightarrow A'$ such that the dual isomorphism $\alpha^*:\chi(A') \longrightarrow \chi(A)$ maps X′ on to X.

Definition 3

The chain length of X (denoted C1(X)) is the maximum integer $k \geq 1$ such that $\exists\ a_0, \ldots, a_k \in A$ satisfying: $X(a_{i-1}) \subset X(a_i)$, $i = 1, \ldots, k$ (or $C1(X) = \infty$ if no such maximum exists).

Remark 1

It is easily verified that C1(X) = 1 if and only if $|x| = 1$, and $C1(x) \leq 2$ if and only if X is a fan.

Recall that X is said to be decomposable if there exist non-empty subspaces X_i of X, i = 1,2 such that $X = X_1 \oplus X_2$.

Let us denote by gr(X) the translation fuzzy group of X, i.e.,

$gr(X) = \{T \in \chi(x) \mid TX = X\}$.

Thus gr(X) is a closed fuzzy subgroup of $\chi(A)$.

Let the residue space of X be defined to be X′ = (X′,A′) where $A'=gr(X)^{\perp} \subseteq A$, and where X′ denotes the image of X in $\chi(A')$ via restriction, X′ is a space of fuzzy orderings. Moreover gr(X′) = 1, and X is a fuzzy group extension of X′.

We can state the main theorem concerning spaces of finite chain length.

Theorem 2

Suppose $C1(X) < \infty$. Then either $|X| = 1$, or $gr(X) \neq 1$, or X is decomposable.

The proof of this key result is found in [4]. For now we concentrate on giving two important applications.

Theorem 3

Suppose a form f is anisotropic over a space of fuzzy ordering X_0. Then there exists a finite subspace $X \subset X_0$ such that f is an isotropic over X.

Proof: Let X=(X,A) be a subspace of X_0 chosen minimal subject to f is anisotropic over X. Let $a_0, \ldots, a_k \in A$ satisfy: $D<1,a_{i-1}> \subset D<1,a_i>$, $i = 1,\ldots,k$. Thus $<1,a_i> \cong < a_{i-1}, a_{i-1}\, a_i>$ and $a_{i-1} \neq a_i$ for $i = 1,\ldots,k$. We may assume $a_0 = 1$, $a_k = 1$. Let $b_i = a_{i-1}\, a_i$. Thus $b_i \neq 1$, so $X(b_i)$ is a proper subspace of X. Thus f is isotropic over $X(b_i)$, i.e. there exists a form g_i of dimension $n-2$ (where n denotes the dimension of f) such that $f \sim g_i$ over $X(b_i)$. Thus:

$f \otimes <1,b_i> \sim g_i \otimes <1,b_i>$ over X, so by addition

$$f \otimes \left(\sum_{i=1}^{k} <1,b_i> \right) : \sum_{i=1}^{k} g_i \otimes <1,b_i> \text{ (over X)} \qquad \ldots(1)$$

But using the assumptions on $a_0, \ldots, a_k$ we see that (over X) $< b_0, \ldots, b_k > \cong < a_0\, a_1, a_1\, a_2,\ldots, a_{k-1}\, a_k > \cong < a_1, a_1\, a_2,\ldots, a_{k-1}\, a_k > \cong < 1, a_2, a_2\, a_3,\ldots, a_{k-1}\, a_k > \cong\ldots\cong < 1,\ldots, 1, a_k > \cong < 1,\ldots,1, 1 >$.

Substituting this in (1) yields

$$(2k-2)f : \sum_{i=1}^{k} g_i <1,b_i>$$

Now f (and hence $(2k-2)f$, by (3, corollary 3.5(ii)) is anisotropic over X, so comparing dimensions, and using (3, lemma 2.4), $(2k-2)\,n \le k\,(n-2)(2)$, i.e., $k \le \frac{1}{2}n$. This proves $C1(X) < \infty$.

Now, we apply theorem 2. If $|X| = 1$ we are done.

Suppose $X = X_1 \oplus X_2$ where $X_i = (X_i, A/\Delta_i)$ is a non-empty subspace of X, $i = 1,2$. Thus there exist elements $a_{i3}, \ldots, a_{in} \in A$ such that

$f \cong \langle -1,1, a_{i3}, \ldots, a_{in} \rangle$ over X_i, $i = 1,2$.

Since $X = X_1 \oplus X_2$, the natural injection $A \longrightarrow A/\Delta_1 \times A/\Delta_2$ is surjective, so there exist $a_3, \ldots, a_n \in A$ such that $a_j \equiv a_{ij} \pmod{\Delta_i}$, $3 \le j \le n$, $i = 1,2$.

Then clearly $f \equiv \langle 1,-1, a_3, \ldots, a_n \rangle$ over X, a contradiction. Thus X is indecomposable, so $gr(X) \ne 1$. Let $X' = (X', A')$ denote the residue space of X and decompose f as $f \cong \pi_1 f_1 \oplus \ldots \oplus \pi_s f_s$ where $f_1, \ldots, f_s$ are forms over A', and $\pi_1, \ldots, \pi_s \in A$ are distinct modulo A'.

The assertion that f is anisotropic over X is equivalent to the assertion that each $f_1, \ldots, f_s$ is anisotropic over X'.

There are two cases to be considered.

Suppose $S = 1$. Let Δ be any fuzzy subgroup of A such that A is the direct product $A = \Delta \times A'$, and let $Y = \Delta^{\perp} \cap X$. Then one verifies easily that $Y = (Y, A/\Delta)$ is a subspace of X and that $(Y, A/\Delta) \sim (X', A')$, this equivalence being induced by the natural isomorphism $A/\Delta \cong A'$ Thus, since f_1 is anisotropic over X', it (and then $f \cong \pi_1 f_1$) is anisotropic over Y. But, on the other hand $gr(X) \ne 1$, i.e. $A' \ne A$, i.e. $\Delta \ne 1$, i.e., $Y \subset X$.

This contradicts the minimal choice of X.

Thus $S \ge 2$. It follows that each f_i has strictly lower dimension than f so by induction on the dimension, there exist finite subspaces $Z_{1'}, \ldots, Z_{s'} \subseteq X'$ such that f_i is anisotropic over $Z_{i'}$. Thus $f_1, \ldots, f_s$ are all anisotropic over the subspace of X' generated by $Z_{1'}, \ldots, Z_{s'}$. Denote this space by $Z' = (Z', A'/\Delta')$. Note Z' is still finite $Z = \Delta'^{\perp} \cap X$. Then $Z = (Z, A/\Delta')$ is a subspace of X, and a fuzzy group extension of $Z' = (Z', A/\Delta')$. Moreover, since $\pi_1, \ldots, \pi_s$ are distinct modulo A', f is anisotropic over Z. Thus, by minimal choice of X, $Z = X$, i.e. $\Delta' = 1$, i.e., $Z' = X'$ is finite. However, X itself could be infinite (since, a priori, $gr(x)$ could be infinite). Define A'' to be the fuzzy subgroup of A generated by A' and $\pi_1, \ldots, \pi_s$, and let X'' denote the restriction of X to A''.

Thus (X, A) is a fuzzy group extension see[2] of (X'', A'') which, inturn, is a fuzzy group extension of (X', A'). Moreover (X'', A'') is finite, and f is anisotropic over X''. Finally, let Δ be fuzzy subgroup of A so that $A = \Delta \times A''$, and let $Y = \Delta^{\perp} \cap X$. Then $Y = (Y, A/\Delta)$ is a subspace of X naturally equivalent to (X'', A''). Thus Y is finite, and f is anisotropic over Y. Thus $Y = X$ is finite.

Notice, the condition $X(a_{i-1}) \subset X(a_i)$ is equivalent to $D\langle 1, a_i \rangle \subset D\langle 1, a_{i-1} \rangle$.

Theorem 4

(i) Suppose $X_i = (X_i, A/\Delta_i)$, $i = 1, \ldots, n$ are subspaces of X generating X. Then: $CL(X) = \sum_{i=1}^{n} CL(X_i)$.

(ii) If, in addition, $X = X_1 \oplus \ldots \oplus X_n$, then: $C1(X) = \sum_{i=1}^{n} C1(X_i)$.

(iii) If X is a fuzzy group extension of X', then $CL(X) = CL(X')$, except in the case $|X'| = 1$ (in which case X is a fan).

Proof:

(i) Suppose $X(a_{j-1}) \subset X(a_j)$, $j = 1,\ldots,k$. Then for each i, $1 \le i \le n$, $X_i(a_{j-1}) \subset X_i(a_j)$. Moreover, since $X(a_{j-1}) \ne X(a_j)$, there exists i, $1 \le i \le n$ such that $X_i(a_{j-1}) \ne X_i(a_j)$. (for if $X_i(a_{j-1}) = X_i(a_j)$ for all $i \le n$, then $a_j\, a_{j-1} \in \bigcap_{i=1}^{n} \Delta_i = 1$, i.e., $a_j = a_{j-1}$ a contradiction). This holds for $j = 1,\ldots,k$. Simple counting yields $k \le \sum_{i=1}^{n} CL(X_i)$, i.e., $CL(X) \le \sum_{i=1}^{n} CL(X_i)$.

(ii) We are assuming $X = U_i\, X_i$ and the natural homomorphism from A into $\pi_i A/\Delta_i$ is an isomorphism. Suppose $X_i(a_{i,j-1}) \subset X_i(a_{i,j})$, $j = 1,\ldots,ki$, $i = 1,\ldots,n$.

We may as well assume $a_{i,0} = -1$, and $a_{i,k_i} = 1$. Choose elements $b_{ij} \in A$ such that: $b_{ij} = 1 \pmod{\Delta_k}$ for $k < i$.

$b_{ij} \equiv a_{ij} \pmod{\Delta_i}$, and $b_{ij} \equiv -1 \pmod{\Delta_k}$, for $k > i$.

Notice that $X(b_{ij}) = (U_{s<i}\, X_s)\ U\ X_i(a_{ij})$. It follows that $X(b_{10}) \subset\ldots\subset X(b_{1k_1}) = X(b_{20}) \subset\ldots\subset X(b_{nk_n})$. There are $\sum k_i$ inequalities in this chain, so $CL(X) \ge \sum k_i$, and hence $CL(X) \ge \sum CL(X_i)$.

The other inequality follows from (i).

(iii) Suppose $|X'| \ne 1$. Suppose $X'(a_{i-1}) \subset X'(a_i)$, $i = 1,\ldots,k$, with $a_i \in A'$. Then clearly $X(a_{i-1}) \subset X(a_i)$, $i = 1,\ldots,k$. Thus $CL(X) \ge CL(X')$. Now suppose $D<1,a_i> \subset D<1, a_{i-1}>$, $i = 1,\ldots,k$, with $a_1, \ldots, a_k \in A$. We may assume $a_0 = -1$, $a_k = 1$. Then $a_1 \ne -1$. There are two cases to be considered

1st Case: Suppose $a_1 \notin A'$. It follows (from the definition of fuzzy group extension) that $D<1,a_1> = \{1,a_1\}$. Thus $K \le 2$ in this case. Thus, since $|X'| \ne 1$, $CL(X') \ge 2 \ge k$.

2nd Case: Suppose $a_1 \in A'$. Then $D<1,a_1> \subset A'$ (e.g. by (5, lemma 4.9); notice $a_1 \ne -1$. Thus $a_1, \ldots, a_k$ are all in A', and $X'(a_{i-1}) \subset X'(a_i)$, $i = 1,\ldots,k$. Thus $CL(X') \ge K$. Thus, in any case $CL(X') \ge K$, so $CL(X') \ge CL(X)$.

Lemma 1

Suppose $b, a_0, \ldots, a_k \in A$ satisfy $D<1,b> = \{1,b\}$, and $D<1,a_{i-1}> <1,b> \subseteq D<1,a_i> <1,b>$, $i = 1,\ldots,k$. Then there exists $a'_i \in D< a_i,a_ib > = \{a_i,a_ib\}$ such that $D<1, a'_{i-1} > \subseteq D<1, a'_i >$, $i = 1,\ldots,k$.

Proof: compare [6].

We now proceed to prove a deeper property of chain length.

Theorem 5

Suppose Y is a subspace of X. Then $C1(Y) \le C1(X)$.

Proof: Suppose, to the contrary, $cl(Y) > cl(X)$. Then, in particular, $cl(X) < \infty$. Choose a subspace $Z \subseteq X$ minimal subject to (1)$Z \supseteq Y$ and (2) $C1(Z) \le C1(X)$. To show such Z exists. Suppose $\{Z_i\}$ is a collection of subspaces of X satisfying (1) and (2) and linearly ordered by inclusion. Let $z' = \bigcap_i Z_i$. Then z' is a subspace of X satisfying (1). To show z' satisfies (2) suppose $a_0, \ldots, a_k \in A$ satisfy $z'(a_j) \subset z'(a_{j-1})$, $j = 1,\ldots,k$. Thus the set $M = \{\sigma \in X \mid \sigma < 1, a_j \ge \sigma < a_{j-1}, a_{j-1}\, a_j >, j = 1,\ldots,k\}$ is open in X and contains Z'. By compactness, $Z_i \subseteq M$ for some i, so $Z_i(a_j) \subset Z_i(a_{j-1})$, $j = 1,\ldots,k$. These inclusions must be strict, since $Z' \subseteq Z_i$. Thus $k \le CL(Z_i) \le CL(X)$, so $CL(Z') \le CL(X)$. So Z exists as asserted. To simplify notation, we may assume $X = Z$. Let $Y = (Y, A/\Delta)$, since $Y \ne X(CL(Y) > CL(X))$. It follows that $\Delta \ne 1$, so there exists $a \in \Delta$, $a \ne 1$. Thus $Y \subseteq X(a) \subset X$. Since $CL(X) < \infty$, there exists $b \in A$, $b \ne 1$, such that

X(a) ⊆ X(b) ⊆ X, X(b) maximal. Thus D<1,b> is minimal, i.e., D<1,b> = {1,b}. By the minimal choice of X (=Z), it follows that CL(X(b)) > CL(X). On the other hand it follows from lemma (1) that CL(X(b)) ≤ CL(X). This is a contradiction.

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الفضاء الضبابي الترتيب

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الخلاصة

يعرض البحث تعريف طول سلسلة في فضاء ضبابي الترتيب ومن ثم عرض خواص وبرهنتها، ولقد تم برهان المبرهنة الأساسية لطول السلسلة المنتهية وعرض بعض النتائج المتعلقة بالموضوع.