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On Semisecond Submodules

Zainab Saadi **Ghaleb Ahmed**

Department of Mathematics, College of Science, University of Mustansiriyah

zoozaih89@gmail.com ghaleb.a.h@ihcoedu.uobaghdad.edu.iq

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Abstract

Let M be a right module over a ring R with identity. The semisecond submodules are studied in this paper. A nonzero submodule N of M is called semisecond if $Na = Na^2$ for each $a \in R$. More information and characterizations about this concept is provided in our work.

Keywords: semisecond submodules, weak semisecond submodules, S-semisecond submodules, regular modules.

1. Introduction

R is indicated a ring with identity and M is viewed as a non-zero S- R-bimodule where $S = End_R(M)$ the endomorphism ring of M. We use the notation " $\subseteq$ " to denote inclusion. A non-zero submodule N of M is said to be a second submodule if for any $a \in R$, the endomorphism $f_a: N \to N$ defined by $f_a(n) = na$ for each $n \in N$, is either surjective or zero (that is $Imf_a = Na = N$ or $Imf_a = Na = 0$) [1]. Equivalently $0 \neq N$ is a second submodule of M if $NI = N$ or $NI = 0$ for every ideal I of R [1]. In that situation, $ann_R(N)$ is a prime ideal of R[1]. A non-zero module M is second (or coprime) if M is a second submodule of itself [1]. As a new type of second submodules, the concept of weakly second submodules is presented in [2]. A non-zero submodule N of M is weakly second submodule whenever $Nab \subseteq K$ where $a, b \in R$ and K a submodule of M implies either $Na \subseteq K$ or $Nb \subseteq K$ [2]. Equivalently, a non-zero submodule N of M is called weakly second if $Nab = Na$ or $Nab = Nb$ for every $a, b \in R$ [2]. More characterizations of the weakly second concept are provided in [3]. In fact this idea as a dual notion of the concept weakly prime (sometimes is called classical prime) submodules. A proper submodule N of M is wekly prime whenever $Kab \subseteq N$ where $a, b \in R$ and K a submodule of M implies either $Ka \subseteq N$ or $Kb \subseteq N$ [4]. In [5]. We define the idea of weakly secondary as a generalization of weakly second concept and the same time, it is a new class of secondary submodules and a dual notion of classical primary submodules respectively. A nonzero submodule N of M is weakly secondary submodule if

$Nab \subseteq K$ where a, $b \in R$ and K is a submodule of M implies either $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t. A nonzero submodule N is a secondary submodule of M if for any $a \in R$, the endomorphism $f_a: N \to N$ defined by $f_a(n) = na$ for each $n \in N$, is either surjective or nilpotent (that is $Imf_a = Na = N$ or $Imf_a = Na^t = 0$ for some positive integer t) [1]. Equivalently, $0 \neq N$ is a secondary submodule of M if for every ideal I of R, $NI = N$ or $NI^t = 0$ for some positive integer t [1]. In this case, $ann_R(N)$ is a primary ideal of R (that is $\sqrt{ann_R(N)}$ is a prime ideal of R) [1]. A proper submodule K of M is classical primary if $Nab \subseteq K$ where a, $b \in R$ and N is a submodule of M then $Na \subseteq K$ or $Nb^t \subseteq K$ for some positive integer t [6]. A proper submodule K of M is called completely irreducible when $K = \bigcap_{i \in \wedge} H_i$ where $\{H_i\}_{i \in \wedge}$ is a family of submodules of M implies that $K = H_i$ for some $i \in \wedge$ [2]. It is not hard to see that every submodule is an intersection of completely irreducible submodules of M consequently the intersection of all completely irreducible submodules of M is zero. N is called simple (sometimes minimal) submodule of a module M if $N \neq 0$ and for each submodule L of M and N contains L properly implies $L = 0$ [7]. M is coquasi-dedekind if all nonzero endomorphism of M is epimorphism (in other word, $f(M) = M$ for every $0 \neq f \in S$) [8]. Let R be a commutative integral domain, M is called divisible module over R if $Ma = M$ for each $0 \neq a \in R$ [7]. A proper submodule N is maximal if it is not properly contained in any proper submodule of M [7]. A proper submodule N is called prime if $mr \in N$ implies $m \in N$ or $Mr \subseteq N$ [9]. M is called a prime module if the zero submodule is prime. A proper ideal I is prime if $ab \in I$ where a, $b \in R$ implies $a \in I$ or $b \in I$ [10]. Equivalently, a proper ideal I is prime if $AB \subseteq I$ where A and B are ideals of R implies $A \subseteq I$ or $B \subseteq I$ [10]. A ring in which every ideal prime is called fully prime [11]. Equivalently, a ring R is fully prime if and only if it is fully idempotent (a ring in which every ideal is an idempotent that is $I^2 = I$ for each ideal I of) and the set of ideals of R is totally ordered under inclusion [11]. A proper submodule N is called primary if $mr \in N$ implies $m \in N$ or $Mr^t \subseteq N$ for some positive integer t [6]. M is called a primary module if the zero submodule is primary. A proper ideal I is primary if $ab \in I$ where a, $b \in R$ implies $a \in I$ or $b^t \in I$ for some positive integer t [6]. $0 \neq M$ is called an S-second module if for every $f \in S$ implies $f(M) = M$ or $f(M) = 0$ [12]. $0 \neq M$ is called an S-weakly second module whenever $fg(M) \subseteq K$, where f, $g \in S$ and K a submodule of M implies either $f(M) \subseteq K$ or $g(M) \subseteq K$ [3]. Equivalently, M is an S-weakly second module if and only if for each ζ, $\vartheta \in S$ implies $\zeta\vartheta(M) = \zeta(M)$ or $\zeta\vartheta(M) \supseteq \vartheta(M)$ [3]. M is called multiplication when each submodule N of M, we have $N = MI$ for some ideal I of R [13]. We able to take $I = [N :_R M] = \{r \in R \text{ and } Mr \subseteq N\}$ is an ideal of R [13]. M is called faithful if $[0:_R M] = ann_R(M) = \{r \in R \text{ and } Mr = 0\} = 0$. M is a scalar module when for each $f \in End(M)$ there is $a \in R$ with $f(m) = ma$ for all $m \in M$ [14].

The aim of this research is to continue studying the concept of semisecond submodules. A nonzero submodule N of M is called semisecond if for each $a \in R$, $Na = Na^2$ [2]. A nonzero module M is said to be semisecond if M is semisecond submodule of itself. In fact this idea is the dual notion of the concept semiprime submodules. A proper submodule of M is called semiprime if for each $a \in R$, $m \in M$ such that $ma^2 \in N$ implies $ma \in N$ [9]. A proper ideal I of R is semiprime if for each $a \in R$ such that $a^2 \in I$ implies $a \in I$ [7]. Equivalently, a proper ideal I of R is semiprime if for each ideal A of R such that $A^2 \subseteq I$ implies $A \subseteq I$ [7]. It is well-known that R is fully semiprime (that is R in which every ideal is semiprime) if and only if R is von Neumann regular (that is for every $a \in R$, there is $b \in R$ such that $a = aba$) [15]. It is well-known if R is commutative then R is von Neumann regular if and only if $aR = a^2R$ if and only if every ideal of R is pure (that is $I \cap J = IJ$ for each ideal I and J of R) if and only if R is fully idempotent. And M is called regular if for every $m \in M$ and for every $a \in R$ we have $ma = mara$ for some $r \in R$. If M is regular then every submodule of M is pure (that is every submodule N of M satisfying $NI = MI \cap N$ for each ideal I of R) [15]. If R is commutative then M is regular if and only if for every $m \in M$ and for every $a \in R$ we have $ma = ma^2r$ for some $r \in R$. Also R is Boolean ring if $a^2 = a$ for every $a \in R$ [7]. Thus a Boolean ring is von Neumann. We call a module M is Rickart when for every $f \in End_R(M)$, $kerf$ is a direct summand of M [16]. M is a dual Rickart module when for every $f \in End_R(M)$, $\operatorname{Im} f$ is a direct summand of M [16]. It is well-known that for each $a \in R$ we can define $f_a: R \to R$ by $f_a(r) = ar$ for each $r \in R$ then $Imf_a = aR$. This means R is von Neumann regular if and only if R is dual Rickart as R-module. A nonzero submodule N of M is weak semisecond whenever $Na^2 \subseteq K$ where $a \in R$ and K a submodule of M implies either $Na \subseteq K$ or $a^2 \in ann_R(N)$ [17]. A nonzero submodule N of M is called a strongly 2-absorbing second submodule if for each $a, b \in R$, we have $Nab = Na$ or $Nab = Nb$ or $Nab = 0$ [18]. A module M is called cacellation if $MI = MJ$ implies $I = J$ for each ideal I and J of R [19]. Other works within [20-23]. Is related topics.

The paper contains five branches and better say "sections"). In second part, we give other descriptions of the semisecond submodules idea (Theorem 2.2, Theorem 2.4, and Proposition 2.8). More examples and information about this idea are provided (Remarks and Examples 2.3). We study the homomorphic image and the direct sum of this class of modules (Proposition 2.5 and Propsition 2.6). Section three includes (Theorem 3.1) is the most important tool to describe semisecond submodules. More characterizations are supplied (Corollary 3.9 and Theorem 3.12). Section four is devoted to finding any relationships between semisecond submodules and related modules. Among other observations, we see that every nonzero regular module over a commutative ring is semisecond (Theorem 4.1). The semisecond and von Neumann regular concepts are coincident in the commutative rings (Theorem 4.7). In section five, we present the concept S-semisecond submodules and the basic properties of this modules is investigated.

In what follows, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$ and $Mat_n(R)$ we denote respectively, integers, rational numbers, the p-Prüfer group, the residue ring modulo n and an $n \times n$ matrix ring over R .

2. Semisecond Submodules

We give a characterization of semisecond submodules, first we recall the main definition.

Definition (2.1) [2]. A nonzero submodule N of R-module M is called semisecond if $Na = Na^2$ for each $a \in R$.

Theorem (2.2): The following assertions are equivalent

(1) N is a semisecond submodule of an R-module M

(2) $N \neq 0$ and whenever $Na^2 \subseteq K$, where $a \in R$ and K a submodule of M implies $Na \subseteq K$

Proof. (1) $\Rightarrow$ (2) Let $a \in R$ and K a submodule of M with $Na^2 \subseteq K$. Because N is semisecond then $N \neq 0$ and $Na = Na^2$ for each $a \in R$ implies $Na = Na^2 \subseteq K$ as desired.

(3) $\Rightarrow$ (1) Assume $N \neq 0$ and $a \in R$ then $Na^2 \subseteq Na^2$. By hypothesis $Na \subseteq Na^2$ and hence $Na = Na^2$ as required.

Remarks and Examples (2.3)

(1) Obviously semisecond submodules are weak semisecond but the converse fails for more information see [17].

(2) It is clear that weakly second submodules are semisecond. The converse is not hold in general, $\mathbb{Z}_6$ as $\mathbb{Z}$-module is semisecond since $\mathbb{Z}_6 . a = \mathbb{Z}_6 . a^2$ for each $a \in \mathbb{Z}$ but $\mathbb{Z}_6$ is not weakly second because $\mathbb{Z}_6 . 3 \neq \mathbb{Z}_6 . 2.3 = 0 \neq \mathbb{Z}_6 . 2$.

(3) As another example of (2), let $N = < \frac{1}{p} + \mathbb{Z} > \oplus < \frac{1}{q} + \mathbb{Z} >$ be a submodule of $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$ as $\mathbb{Z}$-module where p and q prime numbers. Then N is semisecond since $Na = Na^2$ for each $a \in \mathbb{Z}$ but N is not a weakly second submodule of M because $N.p.q = 0_M$ while $N.p = 0 \oplus \mathbb{Z}_{q^\infty}$ and $N.q = \mathbb{Z}_{p^\infty} \oplus 0$.

(4) Clearly every module over Boolean ring is semisecond.

(5) Secondary and weakly secondary submodules not necessarily semisecond. Consider $\mathbb{Z}_4$ as $\mathbb{Z}$-module is secondary (and hence weakly secondary) see [4]. But M is not semisecond because $\mathbb{Z}_4 . 2 \neq \mathbb{Z}_4 . 2^2$.

(6) Semisecond submodules also need not be secondary or weakly secondary submodules. For example: $\mathbb{Z}_6$ as $\mathbb{Z}$-module is semisecond by (2) but $\mathbb{Z}_6$ is not weakly secondary and hence it is not secondary see [4].

(7) It is obvious that coquasi-dedekind (or simple or divisible) submodule $\Rightarrow$ second submodule $\Rightarrow$ strongly 2-absorbing second submodules $\Rightarrow$ weakly second submodules $\Rightarrow$ semisecond submodules $\Rightarrow$ weak semisecond submodules. The converse is not true in general, $M = \mathbb{Z}_6 \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is semisecond but it is not strongly 2-absorbing second,(and hence not weakly second) since $M.3 \neq M2.3 = 0 \oplus \mathbb{Z}_{p^\infty} \neq M.2$ and $M.2.3 \neq 0_M$.

(8) If N is a maximal (and hence prime) submodule then N may not be semisecond. For example, $N = p\mathbb{Z}$ is a maximal submodule of $\mathbb{Z}$ as $\mathbb{Z}$-module but N is not semisecond since $Na^2 \neq Na$ for every $a \in \mathbb{Z}$ and any prime number p.

(9) Let $N$and H be submodules of an R-module M with $N \subseteq H \subseteq M$. If N is a smisecond submodule of M then H needs not be a semisecond submodule of M. Let $N = \mathbb{Z}_4 . 2$ and $H = \mathbb{Z}_4 = M$ submodules of $M = \mathbb{Z}_4$ as $\mathbb{Z}$-module where N is a simple submodule so it is semisecond while H is not semisecond by (5).

(10) Let N and H be submodules of an R-module M with $N \subseteq H \subseteq M$. If H is a sermisecond submodule of , then N needs not be a semisecond submodule of M. Let $N = < \frac{1}{p^2} + \mathbb{Z} >$ be a submodule of $M = \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. Since M is a divisible module then M is semisecond but N is not semisecond because $N.p^2 = 0_M \neq Np = < \frac{1}{p} + \mathbb{Z} >$.

(11) As another example of (10), $\mathbb{Q}$ as $\mathbb{Z}$-module is divisible so it is semisecond but the submodule $\mathbb{Z}$ is not semisecond.

Theorem (2.4): The following assertions are equiavalent

(1) N is a semisecond submodule of an R-module M.

(2) $N \neq 0$ and for each a, $b \in R$ and K a finite intersection of completely irreducible submodules of M with $Na^2 \subseteq K$ implies $Na \subseteq K$.

Proof. (1) ⇒ (2) it is clear.

(3) ⇒ (1) Let $0 \neq N$ and K are submodules of M with $Na^2 \subseteq K$ where $a \in R$. Suppose $Na \nsubseteq K$ implies $K = \cap_{i\in\wedge} H_i$ for some collection $\{H_i\}_{i\in\wedge}$ of completely irreducible submodules of M. We have $Na \nsubseteq \cap_{i\in\wedge} H_i$. So there exists $i \in \wedge$ such that $Na \nsubseteq H_i$. On the other hand, $Na^2 \subseteq K = \cap_{i\in\wedge} H_i$ and hence $Na^2 \subseteq K \subseteq \bigcap_{i=1}^{n} H_i$ for some positive integer n because $K \subseteq H_i$ for each $i \in \wedge$. By hypothesis, $Na \subseteq \bigcap_{i=1}^{n} H_i$ .Then $Na \subseteq H_i$ which is a contradiction as required.

Proposition (2.5): Every nonzero homomorphic image of semisecond submodule is smisecond.

Proof. Let A and B be R-modules and $0 \neq f: A \to B$ an R-homomorphism. Let N be a semisecond submodule of A. Firstly, since $f \neq 0$ implies $f(N) \neq 0$. For each $a \in R$ then $f(N)a = f(Na) = f(Na^2) = f(N)a^2$.

Proposition (2.6): Let N_1 and N_2 be non-zero submodules of M_1 and M_2 R-modules respectively. Then $N = N_1 \oplus N_2$ is a semisecond submodule of $M = M_1 \oplus M_2$ if and only if N_1 and N_2 are semisecond submodules of M_1 and M_2 respectively.

Proof. (⇒) Let $a \in R$ then $(N_1 \oplus N_2)a = (N_1 \oplus N_2)a^2$ and hence $N_1a \oplus N_2a = N_1a^2 \oplus N_2a^2$ implies $N_1a = N_1a^2$ and $N_2a = N_2a^2$ as required.

(⇐) it is clear.

Corollary (2.7): Every non-zero direct summand of a semisecond module is semisecond.

Proposition (2.8): The following statements are equivalent

(1) N is a semisecond submodule of R-module M.

(2) $\frac{N}{H}$ is a semisecond submodule of R-module $\frac{M}{H}$ for each submodule H of M contained in N.

Proof. (1) ⇒ (2) Let N be a semisecond submodule M and $\pi: M \to \frac{M}{H}$ be the natural homomorphism for each submodule H of M contained in N so by Proposition 2.5, $\pi(N) = \frac{N}{H}$ is a semisecond submodule $\frac{M}{H}$.

(2) ⇒ (1) It is clear by taking $H = 0$.

3. More Characterizations and Facts About Semisecond Submodules

Theorem (3.1): The following statements are equivalent

(1) N is a semisecond submodule of an R-module M.

(2) $N \neq 0$ and $[K:_R N]$ is a semiprime ideal of R for each submodule $K \nsupseteq N$ in M.

Proof. (1) ⇒ (2) Assume N is a semisecond submodule of an R-module M and K a submodule of M such that $N \nsubseteq K$ implies $[K:_R N] \neq R$. Let $a \in R$ with $a^2 \in [K:_R N]$ implies $Na^2 \subseteq K$ thus $Na \subseteq K$ and hence $a \in [K:_R N]$ as required.

(2) $\Rightarrow$ (1) Let N and K be submodules of an R-module M such that $Na^2 \subseteq K$ where $a \in R$. In case $N \subseteq K$ then already $Na \subseteq K$. If $N \nsubseteq K$ then $[K:_R N]$ is a semiprime ideal of R by hypothesis and $a^2 \in [K:_R N]$ implies $Na \subseteq K$ as desired.

Corollary (3.2): Every submodule of a module over a fully semiprime (that is von Neumann regular) ring is semisecond.

Proof. Directly via Theorem 3.1.

Corollary (3.3): If N is a semisecond submodule of an R-module M then $ann_R(N)$ is a semiprime ideal of R.

Proof. Directly via Theorem 3.1.

Examples (3.4): $ann_R(N) = 0$ is a semiprime ideal of $\mathbb{Z}$ for every nonzero submodule N of the $\mathbb{Z}$-module $\mathbb{Z}$ while N is not semisecond.

Corollary (3.5): If N is a semisecond submodule of an R-module M then for every submodule $K \nsupseteq N$ in M we have $[K: N] = [K: Nb]$ for each $b \in R$.

Proof. Let $a \in [K:_R N]$ then $Na \subseteq K$ implies for each $b \in R$ $Nab \subseteq K$ so $a \in [K:_R Nb]$. Conversly, let $a \in [K:_R Nb]$ then $Nab \subseteq K$ implies $ab \in [K:_R N]$ and we can take $b = a$ then $a^2 \in [K:_R N]$. Via Theorem 3.1, $[K:_R N]$is a semiprime ideal of R implies $a \in [K:_R N]$ as required.

Corollary (3.6): If N is a semisecond submodule of an R-module M then $ann_R(N) = ann_R(Nb)$ for each $b \in R$.

Proof. Directly by Corollary 3.5.

Theorem (3.7): The following statements are equivalent

(1) N is a semisecond submodule of an R-module M.
(2) $N \neq 0$ and for each ideals I of R such that $NI^2 \subseteq K$ implies $NI \subseteq K$.

Proof. (1) $\Rightarrow$ (2) First since N is a semisecond submodule of an R-module M then $N \neq 0$. Let I be an ideal of R and K a submodule of M. If $N \nsubseteq K$ we have either $NI^2 \nsubseteq K$ and so nothing to prove or $NI^2 \subseteq K$ it follows $I^2 \subseteq [K:_R N]$ and by Theorem 3.1, $[K:_R N]$ is a semiprime ideal of R so $I \subseteq [K:_R N]$ and hence $NI \subseteq K$. In case $N \subseteq K$ then the result already is obtained.

(2) $\Rightarrow$ (1) Let $Na^2 \subseteq K$, where $a \in R$ and K a submodule of M, then $N < a^2 > \subseteq K$. By hypothesis $N < a > \subseteq K$ where $< a >$ is the principal ideal generated by a and hence $Na \subseteq K$ as dsired.

Corollary (3.8): The following statements are equivalent

(1) N is a semisecond submodule of an R-module M.
(2) $N \neq 0$ and for each ideal I of R and K a submodule of M such that $N \nsubseteq K$ and $I^2 \subseteq [K: N]$ implies $I \subseteq [K: N]$.

Proof. Directly via corollary 3.7.

Corollary (3.9): The following statements are equivalent

(1) N is a semisecond submodule of an R-module M.
(2) $N \neq 0$ and for each ideal I of R implies $NI^2 = NI$.

Proof. (1) $\Rightarrow$ (2) First since N is a semisecond submodule of an R-module M then $N \neq 0$. Let I be an ideal of R then $NI^2 \subseteq NI^2$ so by Theorem 3.7, we have $NI \subseteq NI^2$ and thus $NI^2 = NI$.

(2) $\Rightarrow$ (1) it is clear.

Theorem (3.10): Let N be a submodule of an R-module M. If for each $a \in R$, $a^2R + ann_R(N) = aR + ann_R(N)$ then N is semisecond.

Proof. Assume for each $a \in R$, $a^2R + ann_R(N) = aR + ann_R(N)$ then $a^2 + b = a + c$ for some $b, c \in ann_R(N)$ implies $a^2 - a \in ann_R(N)$ and hence $Na^2 = Na$.

Theorem (3.11): If N is a semisecond finitely generated submodule of an R-module M then for each $a \in R$, $a^2R + ann_R(N) = aR + ann_R(N)$.

Proof. Let $a \in R$ then $Na^2 = Na$ that is $N(aR)(aR) = N(aR)$. By hypothesis N is finitely generated. It is not hard to see that $N(aR)$ is also finitely generated. Via [23, Corollary 2.5], it follows that $x - 1 \in Ra$ and $Nx(aR) = 0$. Let $x - 1 = at$ for some $t \in R$ then $x = at + 1$ implies $N(at + 1)a = 0$. This means $a^2t + a \in ann_R(N)$ so $a^2t + a = b$ for some $b \in ann_R(N)$ implies $a = -a^2t + b$ and hence $aR + ann_R(N) \subseteq a^2R + ann_R(N)$. Then $a^2R + ann_R(N) = aR + ann_R(N)$.

Theorem (3.12): Let N be a finitely generated submodule of a module M over a commutative ring R. The following statements are equivalent

(1) N is semisecond.
(2) For each $a \in R$, $Na = Nr = Nr^2$ for some $r \in R$.

Proof. (1) $\Rightarrow$ (2) By Theorem 3.11, for each $a \in R$, $a^2R + ann_R(N) = aR + ann_R(N)$. Then $a^2t + b = as + c$ for some $s, t \in R$ and $b, c \in ann_R(N)$. By choosing $s = 1$ we have $a = a^2t + d$ for some $d = b - c \in ann_R(N)$ thus $aR \subseteq atR + ann_R(N)$ implies $aR + ann_R(N) \subseteq atR + ann_R(N)$ hence $aR + ann_R(N) = atR + ann_R(N)$. Put $r = at$ it follows $a - r \in ann_R(N)$. Therefore $Na = Nr$ but $at = a^2t^2 + dt$ that is $r - r^2 \in ann_R(N)$ thus $Na = Nr = Nr^2$ as desired.

(2) $\Rightarrow$ (1) for each $a \in R$, $Na^2 = Naa = Nra = Nar = Nrr = Nr = Na$ implies N is semisecond.

4. Semisecond Submodules and Related Concepts

Let us start by the following observation (observation)

Theorem (4.1): Every non-zero regular module over a commutative ring is semisecond.

Proof. Let M be a nonzero regular R-module. We show $Ma = Ma^2$ for each $a \in R$. Let $x \in Ma$ implies $x = ma$ for some $m \in M$ it follows $ma = mara = ma^2r \in Ma^2$ for some $r \in R$.

Example (4.2):

(1) Every regular ideal I of commutative ring R is a semisecond as R-module.
(2) $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ as $\mathbb{Z}$-modules are semisecond but not regular.

Corollary (4.3): Every non-zero module over commutative von Neumann regular ring is semisecond.

Proof. Since every module over von Neumann regular ring is regular so the result follows by Theorem 4.1.

Corollary (4.4): Every nonzero submodule of a regular module over commutative ring is semisecond.

Proof. Since every submodule of a regular module is regular, so by theorem 4.1 we already have the result.

Corollary (4.5): Every nonzero semisimple module over commutative ring is semisecond.

Corollary (4.6): Every submodule of a semisimple module over commutative ring is semisecond.

Theorem (4.7): The von Neumann regular and semisecond notions in the commutative rings are the same.

Proof. It is clear by definitions both notions.

Examples (4.8):

(1) The commutativity condition in Theorem and Theorem cannot be dropped. Consider the ring $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ as a right R-module. By simple calculation, we see that R is von Neumann regular and R is not commutative. On the other hand, if we take $a = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \in R$ implies $aR \neq a^2R = \left\{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}\right\}$, what follows R is not a semisecond ring.

(2) Consider the ring $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \bar{0} & \bar{0} \end{pmatrix} = \{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$ as a right R-module where R is not commutative. By simple steps, we have $aR = a^2R$ for each $a \in R$ it follows that R is semisecond but R is not von Neumann regular since $\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} b \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$ for each $b \in R$.

(3) Semisecond modules may not be semisimple. Consider $R = \prod_{i\in\Lambda} \mathbb{F}_i$ is commutative von Neumann regular ring (R is a regular as R-module) and hence R is semisecond but R is not semisimple since the submodule $R = \bigoplus_{i\in\Lambda} \mathbb{F}_i$ is not a direct summand of R.

Proposition (4.9): Let R be a commutative ring then we have the equivalent

(1) R is von Neumann regular.
(2) R is fully semiprime.
(3) R is fully idempotent.
(4) R is a dual Rickart as R-module.
(5) R is semisecond
(6) R is cosemisimple.

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) as we mentioned before where the commutativity condition is not necessary, (1) ⇔ (5) by Theorem 4.4 and (1) ⇔ (6) via [7].

Proposition (4.10): Every nonzero module over semisecond ring is semisecond.

Proof. Let $0 \neq M$ be a module over a semisecond ring R implies $Ra^2 = Ra$ and thus $Ma^2 = Ma$.

Example (4.11): Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \bar{0} & \mathbb{Z}_2 \end{pmatrix}$ and $M = \begin{pmatrix} \bar{0} & \mathbb{Z}_2 \\ \bar{0} & \mathbb{Z}_2 \end{pmatrix}$ be considered as a right R-module. By simple steps, we see that $Ma = Ma^2$ for each $a \in R$ that M is semisecond but R is not semisecond since if we take $a = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$ we have $Ra \neq Ra^2$. In fact if R is semisecond, then M is semisecond which is a contradiction by Proposition 4.3. Moreover, M is not semisimple since $\begin{pmatrix} \bar{0} & \mathbb{Z}_2 \\ \bar{0} & \bar{0} \end{pmatrix}$ is a cyclic submodule of M which is not a direct summand of M. Also, M is not regular since $\begin{pmatrix}\bar{0} & \mathbb{Z}_2\\ \bar{0} & \mathbb{Z}_2\end{pmatrix}\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{1}\end{pmatrix} \cap \begin{pmatrix}\bar{0} & \mathbb{Z}_2\\ \bar{0} & \bar{0}\end{pmatrix} = \begin{pmatrix}\bar{0} & \mathbb{Z}_2\\ \bar{0} & \bar{0}\end{pmatrix} \neq \begin{pmatrix}\bar{0} & \mathbb{Z}_2\\ \bar{0} & \bar{0}\end{pmatrix}\begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{1}\end{pmatrix} = \begin{pmatrix}\bar{0} & \bar{0}\\ \bar{0} & \bar{0}\end{pmatrix}$ thus $\begin{pmatrix}\bar{0} & \mathbb{Z}_2\\ \bar{0} & \bar{0}\end{pmatrix}$ is not a pure submodule of M.

Corollary (4.12): Let M be an R-module and I be an ideal of R such $I \subseteq ann_R(M)$. If $\frac{R}{I}$ is a semisecond ring then N is semisecond.

Proof. Since N is considered as $\frac{R}{I}$ –module so by Proposition 4.10, the result is obtained.

Proposition (4.13): Let M be an R-module and I be an ideal of R such that $I \subseteq ann_R(M)$. Then M is a semisecond R-module if and only if M is a semisecond $\frac{R}{I}$-module.

Proof. It is clear.

Examples (4.14):

(1) $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ as $\mathbb{Z}$-modules are semisecond but $\frac{\mathbb{Z}}{ann_{\mathbb{Z}}(\mathbb{Q})} \cong \mathbb{Z} \cong \frac{\mathbb{Z}}{ann_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})}$ is not semisecond.

(2) Consider $\mathbb{Z}_2$ as $\mathbb{Z}$-module implies $\frac{R}{ann_R(M)} = \frac{\mathbb{Z}}{ann_{\mathbb{Z}}(\mathbb{Z}_2)} = \mathbb{Z}_2$ is semisecond but $R = \mathbb{Z}$ is not semisecond.

Proposition (4.15): If N is a cancellation semisecond submodule of an R-module M then R is semisecond.

Proof. For each $a \in R$, we have $a^2N = aN$, then $(a^2R)N = (aR)N$ and since N is cancellation implies $a^2R = aR$ as desired.

Corollary (4.16): If M is a finitely generated faithful multiplication semisecond R-module then R is fully idempotent (and hence semisecond).

Proof. Let M be a semisecond R-module then $I^2M = IM$ for each ideal I of R. Since M is a finitely generated faithful multiplication so by [13], M is cancellation then $I^2 = I$ thus R is fully idempotent.

Corollary (4.17): If M is a cancellation (or finitely generated faithful multiplication) semisecond R-module such that the set of ideals of R is totally ordered under inclusion then R is fully prime.

Proof. By Corollary 4.15, R is fully idempotent so by [11]. R is fully prime.

Theorem (4.18): Let M be a multiplication R-module**.** If $[N:_R M]$ is a semisecond ideal of R then N is a semisecond submodule of M.

Proof. By hypothesis, $[N:_R M]I = [N:_R M]I^2$ for each ideal I of R then $M[N:_R M]I = M[N:_R M]I^2$. By hypothesis M is multiplication thus $NI = NI^2$ so N is semisecond.

Theorem (4.19): Let M be a finitely generated faithful multiplication R-module**.** If N is a semisecond submodule of M then $[N:_R M]$ is a semisecond ideal of R.

Proof. Since $NI = NI^2$ for each ideal I of R then $M[N:_R M]I = M[N:_R M]I^2$ because M is multiplication. But M is finitely generated faithful implies M is cancellation and hence $[N:_R M]I = [N:_R M]I^2$ thus $[N:_R M]$ is semisecond.

Remark (4.20): If I is a semisecond ideal of R then $I^2 = I^3$.

Proof. Since $IJ = IJ^2$ for each ideal J of R so if we choose $J = I$ implies $I^2 = I^3$.

Proposition (4.21): Every nonzero pure submodule of a semisecond module is semisecond.

Proof. Let N be a nonzero pure submodule of a semisecond R-module M. Then for each ideal I of R implies $NI = N \cap MI = N \cap MI^2 = NI^2$ as desired.

The following result is appeared in [2]. Without proof

Proposition (4.22): Every sum of second submodules is semisecond.

Proof. Let $a \in R$, N and H be second submodules of an R-module M implies either $(N + H)a = Na + Ha = Na^2 + Ha^2 = (N + H)a^2$ or $(N + H)a = Na + Ha = 0 + Ha^2 = Ha^2 \subseteq (N + H)a^2$ or $(N + H)a = Na + Ha = 0 + 0 = 0 \subseteq (N + H)a^2$ and hence $(N + H)a = (N + H)a^2$.

Example (4.23): The sum of second submodules may not be second. The submodules $\mathbb{Z}_6.2$ and $\mathbb{Z}_6.3$ are simple and hence second of $\mathbb{Z}_6$ as $\mathbb{Z}$-module while $\mathbb{Z}_6.2 + \mathbb{Z}_6.3 = \mathbb{Z}_6$ is semisecond but not second.

Proposition (4.24): Every semisecond submodule of prime module is second.

Proof. Let $a \in R$ and N be a semisecond submodule of a prime R-module M implies $Na = Na^2$ then for each $n \in N$ we have $na = ma^2$ for some $m \in N$ implies $(n - ma)a = 0$. But $< 0 >$ is a prime submodule in, it follows either $n - ma \in < 0 >$ implies $n = ma$ and hence $N = Na$ or $a \in [< 0 >: M] \subseteq [< 0 >: N]$ implies $Na = 0$ as desired.

Proposition (4.25): Every semisecond submodule of primary module is secondary.

Proof. Similarly of Proposition 4.24.

5. *S*-Semisecond Modules

Definition (5.1): A nonzero R-module M is called S-semisecond whenever $f^2(M) \subseteq K$, where $f \in S = End_R(M)$ and K a submodule of M implies $f(M) \subseteq K$.

Theorem (5.2): The following are equivalent

(1) M is a S-semisecond R-module.

(2) $M \neq 0$ and $f^2(M) = f(M)$ for each $f \in S$.

Proof. (1) ⇒ (2) Assume M is an S-semisecond R-module implies $M \neq 0$. Since $f^2(M) \subseteq f^2(M)$ implies $f(M) \subseteq f^2(M)$ and hence $f^2(M) = f(M)$ for each $f \in S$ as desired.
as desired.

(2) ⇒ (1) Assume $f^2(M) \subseteq K$, where $f \in S$ and K a submodule of M implies $f(M) = f^2(M) \subseteq K$ as required.

Proposition (5.3): Every semisecond multiplication module is S-semisecond.

Proof. Let M be a semisecond multiplication R-module and $f \in S$ with $f^2(M) \subseteq K$ for some K a submodule of M. Since M is multiplication then $f^2(M) = f(IM) = If(M) = IIM$ for some ideal I of R and hence $I^2M \subseteq K$. By Theorem 3.6, we have $IM \subseteq K$ it follows $f(M) \subseteq K$ that is M is S-semisecond.

Corollary (5.4): Every semisecond cyclic module is S-semisecond.

Remarks and Examples (5.5):

(1) Every S-semisecond module is semisecond.

Proof. Let M be an S-semisecond R-module, then $M \neq 0$. Let $Ma^2 \subseteq K$ for some $a \in R$ and K a submodule of M. Define the endomorphisms $f_a: M \to M$ by $f_a(m) = ma$ for each $m \in M$. Then, $f^2(M) = f(f(M)) = f(Ma) = f(M)a = Ma^2 \subseteq K$. By hypothesis, we have $f(M) \subseteq K$ that is $Ma \subseteq K$ as desired.

(2) The converse of (1) is not true in general. For example, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is semisecond where

(3) $S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \begin{pmatrix} End_{\mathbb{Z}}(\mathbb{Z}_2) & Hom_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2) \\ Hom_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Z}_2) & End_{\mathbb{Z}}(\mathbb{Z}_2) \end{pmatrix} \cong Mat_2(\mathbb{Z}_2) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ is not semisecond ring by Example 4.8(1) and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \{\binom{\bar{0}}{\bar{0}}, \binom{\bar{0}}{\bar{1}}, \binom{\bar{1}}{\bar{0}}, \binom{\bar{1}}{\bar{1}}\}$ so if we take $f = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \in S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ implies $f(M) = \{\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}\binom{x}{y}, x, y \in \mathbb{Z}_2\} = \left\{\binom{\bar{0}}{\bar{0}}, \binom{\bar{1}}{\bar{0}}\right\} \neq f^2(M) = \{\begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}\binom{x}{y}, x, y \in \mathbb{Z}_2\} = \left\{\binom{\bar{0}}{\bar{0}}\right\}$ it follows that M is not semisecond as S-module that is, M is not S-semisecond as $\mathbb{Z}$-module.

(4) If $0 \neq M$ is not a divisible $\mathbb{Z}$-module, then $M \oplus M$ can not be an S-semisecond $\mathbb{Z}$-module.

Proof. Let M be a not divisible $\mathbb{Z}$-module. Suppose that $M \oplus M$ is an S-semisecond $\mathbb{Z}$-module. We can define the maps $f: M \oplus M \to M \oplus M$ $f(x,y) = (y2, x)$for each $(x,y) \in M$. It is clear that $f \in S$ implies $f^2(M \oplus M) = ff(M \oplus M) = f(M2 \oplus M) = M2 \oplus M2 \neq f(M \oplus M) = M2 \oplus M$ which is a contradiction.

(5) As another example of the converse of (1), we have $\mathbb{Z}_p \oplus \mathbb{Z}_p$, $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ as $\mathbb{Z}$-modules are semisecond but they are not S-semisecond by (2). In fact, S is not semisecond ring so any module over S cannot be semisecond by Proposition 4.10 as we mentioned in Proposition.

(6) The direct sum of S-semisecond modules needs not be S-semisecond. For example, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ are not S-semisecond as $\mathbb{Z}$-modules

(7) It is clear every that S-weakly second module is S-semisecond. The converse is not hold in general, $\mathbb{Z}_6$ as $\mathbb{Z}$-module is S-semisecond since $\mathbb{Z}_6$ is multiplication and semisecond and hence it is S-semisecond but not weakly second and hence not S-weakly second.

(8) As another example of (6), consider $M = \mathbb{Q} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module. Then $S = End_{\mathbb{Z}}(\mathbb{Q} \oplus \mathbb{Z}_2) \cong \begin{pmatrix} End_{\mathbb{Z}}(\mathbb{Q}) & Hom_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Q}) \\ Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}_2) & End_{\mathbb{Z}}(\mathbb{Z}_2) \end{pmatrix} = \begin{pmatrix} \mathbb{Q} & 0 \\ \bar{0} & \mathbb{Z}_2 \end{pmatrix}$ is a commutative von Neumann regular ring and hence S is semisecond so by Proposition 4.10, $\mathbb{Q} \oplus \mathbb{Z}_2$ is semisecond as S-module; that is, $\mathbb{Q} \oplus \mathbb{Z}_2$ is S-semisecond as $\mathbb{Z}$-module. But $\mathbb{Q} \oplus \mathbb{Z}_2$ is not S-weakly second as $\mathbb{Z}$-module since if we take $f = \begin{pmatrix} 1 & 0 \\ \bar{0} & \bar{0} \end{pmatrix}$, $g = \begin{pmatrix} 0 & 0 \\ \bar{0} & \bar{1} \end{pmatrix}$ then $0 \oplus \mathbb{Z}_2 = g(M) \neq fg(M) = \left\{\begin{pmatrix} 1 & 0 \\ \bar{0} & \bar{0} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \bar{0} & \bar{1} \end{pmatrix}\binom{x}{y} \mid x \in \mathbb{Q}, y \in \mathbb{Z}_2\right\} = \begin{pmatrix} 0 & 0 \\ \bar{0} & \bar{0} \end{pmatrix} \neq f(M) = \mathbb{Q} \oplus \bar{0}$

(9) We have the implication Coquasi-dedekind modules $\Rightarrow$ S-second modules $\Rightarrow$ S-weakly second modules $\Rightarrow$ S-semisecond modules.

Proposition (5.6): Every semisecond scalar module is S-semisecond.

Proof. Let M be a semisecond scalar R-module and $f \in S$ with $f^2(M) \subseteq K$ for some K a submodule of M. Since M is scalar, then there exist $a \in R$ such that $f(m) = ma$ for all $m \in M$. Then $f^2(M) = Ma^2$ implies $Ma \subseteq K$ and hence $f(M) \subseteq K$ as desired.

Theorem (5.7): Let $0 \neq M$ be an R-module such that S is commutative. If M is a regular S-module then M is S-semisecond.

Proof. Similarly proof of Theorem 4.1.

Corollary (5.8): Every Rickart and dual Rickart module has a commutative endomorphism ring is S-semisecond.

Proof. By [16]. The endomorphism ring of Rickart and dual Rickart modules is von Neumann regular so by Theorem 5.7**,** the result is obtained.

Remark (5.9): The commutativity condition in Theorem 5.7 or Corollary 5.8 can not (cannot) (be) dropped as follows, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is Rickart and dual Rickart and hence $S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \begin{pmatrix} End_{\mathbb{Z}}(\mathbb{Z}_2) & Hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ Hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) & End_{\mathbb{Z}}(\mathbb{Z}_2) \end{pmatrix} \cong Mat_2(\mathbb{Z}_2) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ is von Neumann regular, but $Mat_2(\mathbb{Z}_2)$ not commutative ring. On the other hand, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \{\binom{\bar{0}}{\bar{0}}, \binom{\bar{0}}{\bar{1}}, \binom{\bar{1}}{\bar{0}}, \binom{\bar{1}}{\bar{1}}\}$, so if we take $f = \begin{pmatrix} \bar{0} & \bar{1} \\ 0 & \bar{0} \end{pmatrix} \in S = End_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ implies $f(M) = \{\begin{pmatrix} \bar{0} & \bar{1} \\ 0 & \bar{0} \end{pmatrix}\binom{x}{y}, x, y \in \mathbb{Z}_2\} = \left\{\binom{\bar{0}}{\bar{0}}, \binom{\bar{1}}{\bar{0}}\right\} \neq f^2(M) = \{\begin{pmatrix} \bar{0} & \bar{0} \\ 0 & \bar{0} \end{pmatrix}\binom{x}{y}, x, y \in \mathbb{Z}_2\} = \left\{\binom{\bar{0}}{\bar{0}}\right\}$ it follows that M is not semisecond as S-module that is, M is not S-semisecond as $\mathbb{Z}$-module.

Proposition (5.10): Every non-zero direct summand of S-semisecond module is S-semisecond.

Proof. Let N be a direct summand of an S-semisecond R-module M then $M = N \oplus H$ for some submodule H of M. Let $f \in End(N)$ with $f^2(N) \subseteq K$ for some K a submodule of N. We can define $\alpha(n + h) = f(n)$ where $n \in N$ and $h \in H$. It is easy to see that $\alpha \in S$, $\alpha(M) = f(N)$ implies $\alpha^2(M) = f^2(N) \subseteq K$. It follows $\alpha^2(M) \subseteq K$ implies $\alpha(M) \subseteq K$ and hence $f(N) \subseteq K$ as desired.

Theorem (5.11): The following statements are equivalent

(1) M is a S-semisecond R-module.

(2) $M \neq 0$ and $[K:_S M]$ is a semiprime ideal of S for each proper submodule K of M.

Proof. Similarly, proof of Theorem 3.1.

Corollary (5.12): If M is an S-semisecond R-module M then $ann_S(M) = \{f \in S: f(M) = 0\}$ is a semiprime ideal of S.

Proof. Directly By Theorem 5.11.

Examples (5.13): The opposite result is not held in general for example $\mathbb{Z}$ is not semisecond and hence not S-semisecond while $ann_S(\mathbb{Z}) = 0$ is a semiprime ideal of S.

Corollary (5.14): If M is an S-semisecond R-module then for every proper submodule K of M we have $[K:_S M] = [K:_S g(M)]$ for each $g \in S$.

Proof. Similarly, proof of Corollary 3.5.

Corollary (5.15): If M is an S-semisecond R-module then $ann_S(M) = ann_S(gM)$ for each $g \in S$.

Proof. Directly by Corollary 5.14.

Theorem (5.16): The following statements are equivalent

(1) M is an S-semisecond R-module.

(2) $M \neq 0$ and for each ideals I of S and K a submodule of M such that $I^2M \subseteq K$ implies $IM \subseteq K$.

Proof. Similarly, proof of Theorem 3.7.

Corollary (5.17): The following statements are equivalent

(1) M is an S-semisecond R-module.

(2) $M \neq 0$ and for each ideals I of S and K a proper submodule of M and $I^2 \subseteq [K:_S M]$ implies $I \subseteq [K:_S M]$.

Proof. Directly via Theorem 5.16.

Proposition (5.18): The following statements are equivalent

(1) M is an S-semisecond R-module.
(2) $M \neq 0$ and for each ideal I of S implies $I^2M = IM$.

Proof. By using Theorem 5.10 and Theorem 5.2.

6. Conclusion

In this research we present comprehensive study of semisecond submodules. We show that every regular module is semisecond, and the semisecond and regular concepts in the commutative rings are the same. Comprehensive study in this type of modules is introduced and numerous examples and basic properties are provided.

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