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On the Space of Primary La-submodules

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Abstract

Suppose that F is a reciprocal ring which has a unity and suppose that H is an F-module. We topologize La-Prim(H), the set of all primary La-submodules of H , similar to that for FPrim(F), the spectrum of fuzzy primary ideals of F, and examine the characteristics of this topological space. Particularly, we will research the relation between La−Prim(H) and La-Prim(F/ Ann(H)) and get some results.

Keywords primary La-submodules, Fuzzy primary spectrum, La-top modules.

1. Introduction

Suppose that F is a reciprocal ring with a unity and H is a unitary F-module. The primary spectrum Prim (F) and the topological space acquired by inserting Zariski topology on the collection of primary ideals of a reciprocal ring with unity play an significant role in the fields of reciprocal algebra, algebraic geometry and lattice theory. As well, lately the concept of primary submodules and Zariski topology on Prim (H), the collection of all primary submodules of a module H on a reciprocal ring together identity F, were studied in a previous article [1]. As it is famous [2]. Inserted the concept of a fuzzy subset ϑ of a nonempty collection L as a mapping from L to $[0,1]$. Goguen JA [3]. Changed [0,1] by an entire lattice La in the definition of fuzzy collections while inserted the concept of La-fuzzy sets. Rosenfeld inserted the concept of fuzzy groups [4]. While fuzzy submodules of H over F were first inserted by [5]. Pan F-Z [6]. Elaborate fuzzy finitely created modules while fuzzy quotient modules (look at [7]). In previous years a large saucepan of labor has been completed on fuzzy ideals in common and primary fuzzy ideals in special, while several motivating topological features of the spectrum of fuzzy primary ideals of a ring were acquired (look at [8-15]).

Suppose that H is an F-module. By $G \leq H$, we mean that G is a submodule of H. For any $G \leq H$, we indicate the residual of G by H by [G:H], and define$[G:H] =\{ r' \in F \setminus r'H \subseteq G\}$. In special, $[(0) :H]$ is called the annihilator of H and is indicated by Ann(H), that is

$\mathrm{Ann}(H)=\{r' \in F \setminus r'H=0\}$. A primary submodule (or a q-primary submodule) of H is a proper submodule Q with Q:H=q, such that $r'h \in Q$ for $r' \in F$ and $h \in H$, either $h \in Q$ or $r' \in \sqrt{q}$.

The collection of all primary submodules of H is called the primary spectrum of H or, artlessly the p spectra of H and is indicated by Prim(H). Note that the Prim(H) may be empty for some module H . Such a module is said to be primary less (cf. [1]). Clearly, zero module is primary less, but in [1]. Some nontrivial examples are shown.

For example, the Prüfer group $\mathbb{Z}(p^{\infty})$ as a $\mathbb{Z}$-module has no primary submodule for any prime integer p. When $\mathrm{Prim}(H) \neq \emptyset$, the map φ:La-Prim(H)→ La-Prim(F/ Ann(H)) defined by $\varphi(\vartheta)=\overline{(\vartheta{:}\,1H)}$ for $\vartheta \in$ La-Prim(H), φ will be called the standard map.

In [1]. It is shown that for each multiplication module H,(An F-module H is called a multiplication module if every submodule B of H is of the form IH for some ideal I of F) the Prim(H) is non-empty. For any submodule G of H, V(G) indicates the collection of all primary submodules of H including G. Of course V(H) is just the empty set and V(0) is Prim(H). For any family of submodules $G_j (j \in J)$ of H , $\bigcap_{j\in J} V(G_j) = V(\sum_{j\in J} G_j)$

Thus if $\omega(H)$ indicates the set of all subsets V(G) of Prim(H), then $\omega(H)$ includes the empty set and Prim(H) and is closed beneath arbitrary intersection. If also $\omega(H)$ is closed beneath finite union, i.e. for any submodules G and K of H, there occurs a submodule J of H such that $V(G)\cup V(K)=V(J)$, for in this state $\omega(H)$ satisfies the axioms of closed subsets of a topological spaces, which is called Zariski topology. In [1]. A module with Zariski topology is called top module and it is shown that each multiplication module is a top module [1].

In [16,17]. Inserted the concept of primary La-submodules of a module H on a commutative ring together unity F, where La is a whole lattice. The collection of all primary La-submodules of H is called the primary La-spectrum of H or, artlessly the P-La-spectrum of H while is indicated via La-Prim(H). In this work, we follow [18]. And topologize La-Prim(H), which its surname is Zariski topology while examine the characteristics of this topological space. Thereafter, we discussed the relation between the topological spaces La-Prim(H) and La-Prim(F/Ann(H)). Finally, we located a basis for the Zariski topology on La-Prim(H).

2. Basic concepts

During this article via F, we mean a reciprocal ring together unity, and H is a unital F-module and La indicates a whole lattice. Via an La-subset ϑ of $Y\neq \emptyset$, we mean a mapping ϑ from Y to La while if La=[0,1] , then ϑ is a surname of a fuzzy subset of Y. La^{Y} indicates the collection of each La-subsets of Y . Suppose that C is a subset of Y and $b \in$ La. Define $b_C \in La^{Y}$ as follows:

$$b_C(y) = \begin{cases} b & \text{if } y \in C \\ 0 & \text{otherwise} \end{cases}$$

In particular case if C={c} we indicate $b_{\{c\}}$ by b_c , while its surname is an La-point of Y.

For $\vartheta \in La^{Y}$ while $c \in$ La, locate ϑ_c as follows:

$\vartheta_c = \{y \in Y \mid \vartheta(y) \geq c\}$,

ϑ_c is called the c-level subset of ϑ. The image of ϑ is indicated via Ima(ϑ) or $\vartheta(Y)$. In [18]. It was proved that $\vartheta = \bigcup_{c\in\vartheta(Y)} c_{\vartheta_c}$. For $\vartheta, \varpi \in La^Y$ we say that ϑ is included in ϖ while we write $\vartheta \subseteq \varpi$ if for every $y \in Y$, $\vartheta(y) \leq \varpi(y)$.

For $\vartheta, \varpi \in La$, $\vartheta \cup \varpi, \vartheta \cap \varpi \in La^Y$, are defined via

$(\vartheta \cup \varpi)(y) = \vartheta(y) \vee \varpi(y)$ and $(\vartheta \cap \varpi)(y) = \vartheta(y) \wedge \varpi(y)$, for each $y \in Y$.

If g is a function from H into G, $\vartheta \in La^H$ and $\varpi \in La^G$, then the La-subsets $g(\vartheta) \in La^G$ and $g^{-1}(\varpi) \in La^H$ are defined as follows:

$\forall\ d \in G$,

$$g(\vartheta)(d) = \begin{cases} \vee\{\vartheta(y) \mid y \in g^{-1}(d)\} & g^{-1}(d) \neq \emptyset; \\ 0 & \text{otherwise} \end{cases}$$

and $g^{-1}(\varpi)(h) = \varpi(g(h))\ \forall h \in H$.

Suppose that H, G are two F-modules while g: H→ G is an F-homomorphism. Then an La-subset ϑ of H is surname g-invariant if $g(a) = g(b)$ then $\vartheta(a) = \vartheta(b)$ for all $a, b \in H$.

Definition 2.1 Suppose that $\vartheta \in La^F$. Then ϑ is surname an La-ideal of F if for all $a, b \in F$ the following situations are satisfied:

(1) $\vartheta(a - b) \geq \vartheta(a) \wedge \vartheta(b)$;

(2) $\vartheta(ab) \geq \vartheta(a) \vee \vartheta(b)$.

The collection of every La-ideals of F is indicated via LaI (F).
For $\vartheta, \varpi \in LaI(F)$, $\vartheta\varpi(a) = \vee\{\vartheta(b) \wedge \varpi(c) \mid b, c \in F, a = bc\}\ \forall a \in F$, and in [18]. It was confirmed that $\vartheta\varpi \in LaI(F)$.

If La=[0, 1] , then an La-ideal is surname a fuzzy ideal while the collection of every fuzzy ideals of F is indicated via FI (F).

Definition 2.2 [18]. Suppose that ϑ is a La-subset of F. The radical of ϑ is indicated by $(\sqrt{\vartheta})$ and is defined by
$\sqrt{\vartheta}(y) = \bigvee_{n\in N} \vartheta(y^n)$ for all $y \in F$.

Definition 2.3 $\eta \in LaI(F)$ is surname a primary La-ideal of F if η is non-fixed and for all $\vartheta, \varpi \in LaI(F)$, if $\vartheta\varpi \subseteq \eta$ then $\vartheta \subseteq \eta$ or $\varpi \subseteq \sqrt{\eta}$.
Via La-Prim(F), we mean the collection of each primary La-ideals of F.

Proposition 2.4 [18]. Suppose that F and S′ are two rings while g: F ⟶ S′ is an epimorphism.

1) Suppose that $\vartheta \in$ La−Prim(F) and g-invariant, then $g(\vartheta) \in$ La−Prim(S').

2) If $\varpi \in$ La−Prim(S'), then $g^{-1}(\varpi) \in$ La−Prim(F).

For $\gamma \in$ LaI (F), V (γ) be the collection of all primary La-ideals of F such that includes γ, i.e

$V(\gamma) = \{ q \in \text{La-Prim}(F) \mid \gamma \subseteq q\}$.

collection $X(\gamma) = \text{La-Prim}(F) \setminus V(\gamma)$, the La-Prim(F) together the collection $\mathcal{T}' = \{X(\gamma) \mid \gamma \in \text{LaI}(F)\}$ is a topological space while the collection $\mathcal{B}' = \{X(y\alpha) \mid y \in F, \alpha \in (0, 1]$ formation a basis for $\mathcal{T}'$. As well, it can be shown that for two elements $X(y_\alpha)$, $X(x_{\alpha'})$; $X(y_\alpha) \cap X(x_{\alpha'}) = X((yx)_{\alpha\wedge\alpha'})$, $y, x \in F$, $\alpha, \alpha' \in La \setminus \{0\}$.

Definition 2.5 An element $z \in La \setminus \{1\}$ is surname a prime element of La if for $c, d \in La$, $c \wedge d \leq z$, then $c \leq z$ or $d \leq z$.

Definition 2.6 Suppose that $\varepsilon \in La^F$ and $\vartheta \in La^H$. Define $\varepsilon \cdot \vartheta \in La^H$ as follows:

$(\varepsilon.\vartheta)(y) = \vee\{ \varepsilon(r') \wedge \vartheta(b) \mid r' \in F, b \in H, r'b = y\}$ for all $y \in H$.

Definition 2.7 An La-subset $\vartheta \in La^H$ is a La-submodule of H if:

1) $\vartheta(0) = 1$;
2) $\vartheta(r'a) \geq \vartheta(a)$ for all $r' \in F$ and $a \in H$;
3) $\vartheta(a + b) \geq \vartheta(a) \wedge \vartheta(b)$ for all $a, b \in H$.

The collection of all La-submodules of H is indicated by La(H).

Definition 2.8 [18]. Suppose that $\{ \vartheta_j \mid j \in J\} \subseteq La(H)$. Define the La-submodule $\sum_{j\in J} \vartheta_j$ of H by

$(\sum_{j\in J} \vartheta_j)(y) = \vee\{\wedge_{j\in J} \vartheta_j(y_j) \mid y = \sum_{j\in J} y_j, y_j \in H, \forall j \in J\}$ $\forall y \in H$.

It is easy to look that $\sum_{j\in J} \vartheta_j \in La(H)$.

For $\vartheta, \varpi \in La^H$ and $\varepsilon \in La^F$, $\vartheta : \varpi \in La^F$ and $\vartheta : \varepsilon \in La^H$ are defined as follows:

$\vartheta : \varpi = \cup\{\gamma \mid \gamma \in La^F, \gamma.\varpi \subseteq \vartheta\}$.

$\vartheta : \varepsilon = \cup\{\varpi \mid \varpi \in La^H, \varepsilon.\varpi \subseteq \vartheta\}$.

In [18]. It was proved that if $\varpi \in La^H$, $\vartheta \in La(H)$, and $\varepsilon \in LaI(F)$, then

$\vartheta : \varpi = \cup\{\gamma \mid \gamma \in LaI(F), \gamma.\varpi \subseteq \vartheta\}$ and $\vartheta : \varepsilon = \cup\{\varpi \mid \varpi \in La(H), \varepsilon.\varpi \subseteq \vartheta\}$. Also it was shown that if $\vartheta \in La(H)$, $\varpi \in La^H$, $\varepsilon \in LaI(F)$, then $\vartheta : \varpi \in LaI(F)$ and $\vartheta : \varepsilon \in La(H)$.

Theorem 2.9 [18]. If $b \in La$ and G are a submodule of H, then $(1_G \cup b_H) : 1_H = 1_{[G:H]} \cup b_F$.

Definition 2.10[16]. A non- constant La-submodule ϑ of H is called primary if for $\varepsilon \in LaI(F)$ and $\varpi \in La(H)$ such that $\varepsilon.\varpi \subseteq \vartheta$ then either $\varpi \subseteq \vartheta$ or $\varepsilon \subseteq \sqrt{\vartheta : 1_H}$.

In the complement La-Prim(H) indicates the collection of all primary La-submodules of H.

Theorem 2.11 [16]. $\vartheta \in$ La-Prim(H) if and only if $\vartheta = 1_{\vartheta_*} \cup c_H$ such that $\vartheta_* = \{h \in H \mid \vartheta(h) = 1\}$ be a primary submodule of H while z is a prime element of La.

Theorem 2.12 [16]. If $\vartheta \in$ La-Prim(H), then $\vartheta : 1_H$ is a primary La-ideal of F.

3. Topologies on La-Prim(H)

In the complement via H we indicate a unitary module on a reciprocal ring together unity F. For $\vartheta \in La^H$ put $V^*(\vartheta) = \{Q \in \text{La-Prim}(H) \mid \vartheta \subseteq Q\}$.

Proposition 3.1 For family $\{\vartheta_j\}_{j\in J}$ in La(H), the following situations are satisfied:

1- $V^*(1_{\{0\}})$= La-Prim(H), $V^*(1_H) = \emptyset$;

2- $\bigcap_{j\in J} V^*(\vartheta_j)=V^*(\sum_{j\in J} \vartheta_j)$, for index collection J and $\vartheta_j \in$ La(H);

3- $V^*(\vartheta) \cup V^*(\varpi) \subseteq V^*(\vartheta \cap \varpi)$, for $\vartheta, \varpi \in$ La(H).

Proof (1) clearly.

(2) Let $Q \in \bigcap_{j\in J} V^*(\vartheta_j)$, then $Q \in V^*(\vartheta_j)$, $\forall j \in J$, and hence $Q \subseteq \vartheta_j$, $\forall j \in J$. Moreover, we have

$$(\textstyle\sum_{j\in J} \vartheta_j)(y) = \vee\{\wedge_{j\in J} \vartheta_j(y_j) \mid y=\sum_{j\in J} y_j, y_j \in H, \forall j \in J\}$$

$$= \leq \vee\{\wedge_{j\in J} Q(y_j) \mid y=\textstyle\sum_{j\in J} y_j, y_j \in H, \forall j \in J\} \leq Q(y).$$

Then $\sum_{j\in J} \vartheta_j \subseteq Q$ implies that $Q \in V^*(\sum_{j\in J} \vartheta_j)$, and hence $\bigcap_{j\in J} V^*(\vartheta_j) \subseteq V^*(\sum_{j\in J} \vartheta_j)$ (i).

For the converse, $Q \in V^*(\sum_{j\in J} \vartheta_j)$ then $\sum_{j\in J} \vartheta_j \subseteq Q$, and so $\vartheta_j \subseteq \sum_{j\in J} \vartheta_j$, $\forall j \in J$. So $\vartheta_j \subseteq Q$, $\forall j \in J$. Therefore, $Q \in V^*(\vartheta_j)$ $\forall j \in J$, and hence $Q \in \bigcap_{j\in J} V^*(\vartheta_j)$ then $V^*(\sum_{j\in J} \vartheta_j) \subseteq \bigcap_{j\in J} V^*(\vartheta_j)$ (ii). Now (2) instantly follows from (i) and (ii). For (3) let $\vartheta, \varpi \in$ La(H) and $Q \in V^*(\vartheta) \cup V^*(\varpi)$. Then $\vartheta \subseteq Q$, or $\varpi \subseteq Q$, and hence $\vartheta \cap \varpi \subseteq Q$. Thus $Q \in V^*(\vartheta \cap \varpi)$, while so $V^*(\vartheta) \cup V^*(\varpi) \subseteq V^*(\vartheta \cap \varpi)$. Suppose that $\vartheta \in La^H$. The La-submodule generated by ϑ, indicated via $< \vartheta >$, is the smallest La-submodule of H including ϑ. In fact, $< \vartheta > = \cap\{\varpi \in La(H) \mid \vartheta \subseteq \varpi\}$. For $\vartheta \in$ La(H), put $V(\vartheta)=\{Q \in \text{La-Prim}(H) \mid \vartheta{:}1_H \subseteq Q{:}\ 1_H\}$, while if $\varpi \in La^H$, by $V(\varpi)$ we mean $V(< \varpi >)$. Then we have the next outcomes.

Proposition 3.2. Suppose that $\{\vartheta_j\}_{j\in J}$, $\vartheta_j \in$ La(H). Then the following hold:

$V(1_H)=\emptyset, V(1_{\{0\}})$= La-Prim(H); (1)

$V(\vartheta)=V(< \vartheta >)$, for every $\vartheta \in La^H$; (2)

$V(\vartheta_j)= V(\sum_{j\in J} (\vartheta_j{:}\ 1_H).1_H)$; $\bigcap_{j\in J}$ (3)

$V(\vartheta) \cup V(\varpi) = V(\vartheta \cap \varpi)$, for $\vartheta, \varpi \in$ La(H). (4)

Proof (1) instant.

(2) It is an instant result of definition of $\langle\vartheta\rangle$. For (3) let $Q \in \bigcap_{j\in J} V(\vartheta_j)$, then $\vartheta_j : 1_H \subseteq Q : 1_H$, $\forall j \in J$. Thus for all $j \in J$ we have $(\vartheta_j : 1_H) . 1_H \subseteq (Q : 1_H) . 1_H \subseteq Q \Longrightarrow \sum_{j\in J} (\vartheta_j : 1_H) . 1_H \subseteq Q \Longrightarrow \left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right) : 1_H \subseteq Q : 1_H$

So $Q \in V\left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right)$, and hence $\bigcap_{j\in J} V(\vartheta_j) \subseteq V\left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right)$ (a)

Reciprocally, let $Q \in V\left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right)$, then $\left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right) : 1_H \subseteq Q : 1_H$.

Clearly, we have $((\vartheta_j : 1_H). 1_H) : 1_H = \vartheta_j : 1_H, \forall j \in J$. Also for each $j \in J$, we get that $((\vartheta_j : 1_H). 1_H) : 1_H \subseteq \left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right) : 1_H \subseteq Q : 1_H$. Thus for any $j \in J$ it deduces that

$\vartheta_j : 1_H \subseteq Q : 1_H \Longrightarrow \forall j \in J, Q \in V(\vartheta_j) \Longrightarrow Q \in \bigcap_{j\in J} V(\vartheta_j)$.

Thus $V\left(\sum_{j\in J} (\vartheta_j : 1_H) . 1_H\right) \subseteq \bigcap_{j\in J} V(\vartheta_j)$ (2). Now (3) follows by (a) and (b).

For(4) Let $\vartheta, \varpi \in La(H)$ and $Q \in V(\vartheta) \cup V(\varpi)$. Then $Q \in V(\vartheta)$ or $Q \in V(\varpi)$. Without loose of commonness, let $Q \in V(\vartheta)$ we have

$\vartheta : 1_H \subseteq Q : 1_H \Longrightarrow (\vartheta \cap \varpi) : 1_H \subseteq \vartheta : 1_H \subseteq Q : 1_H \Longrightarrow Q \in V(\vartheta \cap \varpi)$. Thus $V(\vartheta) \cup V(\varpi) \subseteq V(\vartheta \cap \varpi)$ (c)

For the opposite, let $Q \in V(\vartheta \cap \varpi)$ then $(\vartheta \cap \varpi) : 1_H \subseteq Q : 1_H$. But we have $(\vartheta \cap \varpi) : 1_H = (\vartheta : 1_H) \cap (\varpi : 1_H)$, and hence $(\vartheta : 1_H)(\varpi : 1_H) \subseteq (\vartheta : 1_H) \cap (\varpi : 1_H)$.

Thus $(\vartheta : 1_H)(\varpi : 1_H) \subseteq Q : 1_H$. Since $Q : 1_H$ is a primary La-ideal then $\vartheta : 1_H \subseteq Q : 1_H$ or $\varpi : 1_H \subseteq \sqrt{Q : 1_H}$,since Q is La-primary submodule then $Q : 1_H$ is prime thus $\sqrt{Q : 1_H} = Q : 1_H$. Thus $Q \in V(\vartheta)$ or $Q \in V(\varpi)$, so $Q \in V(\vartheta) \cup V(\varpi)$ and hence $V(\vartheta \cap \varpi) \subseteq V(\vartheta) \cup V(\varpi)$. (d)

Subsequently (2) follows via (c) and (d).

Now, we set

La- $\varepsilon^*(H) = \{V^*(\vartheta) \mid \vartheta \in La(H)\}$;

La- $\varepsilon'(H) = \{V^*(\gamma . 1_H) \mid \gamma \in LaI(F)\}$;

La- $\varepsilon(H) = \{V^*(\vartheta) \mid \vartheta \in La(H)\}$.

We consider the topologies of La-Prim(H) produced, respectively, via these three collections. From Proposition 3.1, we can facilely look that there occurs a topology τ^* say, over La-Prim(H) having La- $\varepsilon^*(H)$ as the set of every closed collections if and only if La- $\varepsilon^*(H)$ is closed beneath finite union.

In this state, we call the topology τ^* the near-Zariski topology on La-Prim(H). Following [17]. A module H is surname an La-P top module, if La- $\varepsilon^*(H)$ result the topology τ^* over La-Prim(H). In contrast with La- $\varepsilon^*(H)$, La- $\varepsilon'(H)$, permanently occurs a topology τ' on La-Prim(H), since

$V^*(\gamma_1 . 1_H) \cup V^*(\gamma_2 . 1_H) = V^*((\gamma_1 . \gamma_2) . 1_H)$.

Also, La- ε' (H) is closed beneath finite union. Obviously, τ' is coarser than the near-Zariski topology τ^*, when H be an La-P top module. For each F-module H while $\vartheta_1,\vartheta_2 \in$ La(H) we have the next outcome.

Proposition 3.3 If $\vartheta_1: 1_H = \vartheta_2: 1_H$, then $V(\vartheta_1) = V(\vartheta_2)$. The converse is true if both ϑ_1 and ϑ_2 are primary.

Proof First let $\vartheta_1: 1_H = \vartheta_2 : 1_H$, and $\varpi \in V(\vartheta_1)$. Then $\vartheta_1 : 1_H \subseteq \varpi : 1_H$ and hence $\vartheta_2 : 1_H \subseteq \varpi : 1_H$, that is $\varpi \in V(\vartheta_2)$. Therefore $V(\vartheta_1) \subseteq V(\vartheta_2)$. Similarly we get that $V(\vartheta_2) \subseteq V(\vartheta_1)$. Therefore $V(\vartheta_1) = V(\vartheta_2)$. For the opposite, let $\vartheta_1,\vartheta_2 \in$ La(H) are primary while $V(\vartheta_1) = V(\vartheta_2)$. Then

$\vartheta_1 \subseteq V(\vartheta_1) = V(\vartheta_2) \Longrightarrow \vartheta_2 : 1_H \subseteq \vartheta_1 : 1_H$ (a) and $\vartheta_2 \subseteq V(\vartheta_2) = V(\vartheta_1) \Longrightarrow \vartheta_1 : 1_H \subseteq \vartheta_2 : 1_H$ (b)
Then by (a) and (b) we get that $\vartheta_1: 1_H = \vartheta_2: 1_H$.

For q ∈ La-Prim(F), by La-Prim$_q$(H) we mean the collection of all $\vartheta \in$ La(H) such that $\vartheta : 1_H = q$. In other words La-Prim$_q$(H)= $\{ \vartheta \in$ La-Prim(H)| $\vartheta : 1_H = q\}$.

Proposition 3.4 (a) $V(\vartheta) = \bigcup_{q\in V(\vartheta:1_H)}$ La-Prim$_q$(H) for $\vartheta \in$ La(H)

(b) $V(\gamma.1_H) = V^*(\gamma.1_H)$ for every La-ideal γ of F.

Further, if $\vartheta \in$ La(H), then $V(\vartheta)= V((\vartheta: 1_H).1_H) = V^*((\vartheta: 1_H).1_H)$.

Proof (1) : Suppose that $\varpi \in V(\vartheta)$. Then $\vartheta : 1_H \subseteq \varpi : 1_H = q$, and hence $q \in V(\vartheta: 1_H)$. Also,

$\varpi \in$ La-Prim$_q$(H) $\Longrightarrow \varpi \in \bigcup_{q\in V(\vartheta:1_H)}$ La-Prim$_q$(H) $\Longrightarrow V(\vartheta) \subseteq \bigcup_{q\in V(\vartheta:1_H)}$ La-Prim$_q$(H)
(a)

Now suppose that $\varpi \in \bigcup_{q\in V(\vartheta:1_H)}$ La-Prim$_q$(H). Then there occurs q ∈ La-Prim$_q$(H) such that $\vartheta: 1_H \subseteq q$ and $\varpi \in$ La-Prim$_q$(H). Thus

$\varpi : 1_H = q \Longrightarrow \vartheta : 1_H \subseteq \varpi : 1_H \Longrightarrow \varpi \in V(\vartheta) \Longrightarrow \bigcup_{q\in V(\vartheta:1_H)}$ La-Prim$_q$(H) $\subseteq V(\vartheta)$ (b)

Now it follows from (a) and (b).

(2) Suppose that $Q \in V^*(\gamma.1_H)$. Then we have

$\gamma.1_H \subseteq Q \Longrightarrow \gamma.1_H : 1_H \subseteq Q : 1_H \Longrightarrow Q \in V(\gamma.1_H) \Longrightarrow V^*(\gamma.1_H) \subseteq V(\gamma.1_H)$ (c)

Let $Q \in V(\gamma.1_H)$, then $\gamma.1_H : 1_H \subseteq Q : 1_H$.

Clearly, $\gamma \subseteq \gamma.1_H$. Thus

$\gamma \subseteq Q : 1_H \Longrightarrow \gamma.1_H \subseteq Q \Longrightarrow Q \in V^*(\gamma.1_H) \Longrightarrow V(\gamma.1_H) \subseteq V^*(\gamma.1_H)$ (d)

Then from (c) , (d) the outcome satisfying.

As well, via the preceding debate instantly we get that
$V((\vartheta: 1_H).1_H)= V^*((\vartheta: 1_H).1_H)$.

Now for $Q \in V(\vartheta)$, it deduce that $\vartheta: 1_H \subseteq Q : 1_H$. Then we get that $(\vartheta: 1_H).1_H \subseteq \vartheta$, and so

$((\vartheta: 1_H).1_H) : 1_H \subseteq \vartheta : 1_H \subseteq Q : 1_H \Longrightarrow Q \in V((\vartheta: 1_H).1_H) \Longrightarrow V(\vartheta) \subseteq V((\vartheta: 1_H).1_H)$ (e)

Let $Q \in V^*((\vartheta: 1_H).1_H)$. Then $(\vartheta: 1_H).1_H) \subseteq Q$, so $\vartheta : 1_H \subseteq Q : 1_H$. Thus $Q \in V(\vartheta)$ and hence

$V^{*}((\vartheta: 1_H). 1_H) \subseteq V(\vartheta)$ (f)

Consequently from (e) and (f), we get that

$V^{*}((\vartheta: 1_H). 1_H) \subseteq V(\vartheta) \subseteq V((\vartheta: 1_H). 1_H)$.

Thus

$V(\vartheta)= V^{*}((\vartheta: 1_H). 1_H)= V((\vartheta: 1_H). 1_H)$.

Note that from Proposition 3.4 we get that La-ε (H)=La-ε'(H) $\subseteq$ La-ε^{*}(H).

Example 3.5 (1) Let H = $\mathbb{Z}$ as $\mathbb{Z}$-module and suppose that La is an arbitrary lattice. Let q $\in \mathbb{Z}$ is prime. For each prime element s $\in$ La, define T(s) $\in$ La($\mathbb{Z}$) by

$$T(s)(y)=\begin{cases} 1 \; if \; y \in \langle q\rangle; \\ s \; if \; y \in \mathbb{Z}\backslash\langle q\rangle \end{cases}$$

Then by Theorem 2.10, T(s) is a primary La-submodule of H. Therefore La-Prim(H)={T(s)|s is a prime element of La while q is prime element of $\mathbb{Z}$ }, and for La = [0, 1], then La-Prim(H) = {T(s)|s $\in$ [0, 1] while q is prime element of $\mathbb{Z}$ }.

(2) Let H = $\mathbb{R}$[x] as $\mathbb{R}$ [x]$-$module, where $\mathbb{R}$ is the field of real numbers. For each T $\in \mathbb{R}$ [x] while each s $\in$ La, defined the fuzzy subset T(s) of $\mathbb{R}$ [x] via

$$T(s)(y)=\begin{cases} 1 \; y \in \langle q\rangle; \\ s \; otherwise \end{cases}$$

Then by Theorem 2.10, T(s) primary La-submodule of H if and only if T is irreducible and s is a Prime element of La. Further, for La=[0,1], we have La-Prim(H)= {T(s)|q is irreducible in $\mathbb{R}$ [y] , s $\in$ [0, 1]}.

(3) Let H be an arbitrary F-module and T is a prime submodule of H . For each s $\in$ La, define

$$T(s)(y)=\begin{cases} 1 \; y \in T; \\ s \; otherwise \end{cases}$$

Then via Theorem 2.10, T(s) is a primary La-submodule of H if and only if s is a prime element of La. If Spec(La) indicate the collection of all prime elements of La, then La-Prim(H)={T(s)|s $\in$ Spec(La) and T be a primary submodule of H}.

(4) If we let H= $\mathbb{R}$ [y] as $\mathbb{R}$-module. Then all proper submodulesT of H , are indicated via T $<$ H , is primary. Then by part (3) La-Prim(H)= {T(s)| s $\in$ Spec(La) and T $<$ H}.

(5) Let La={0, x, y, 1} is a lattice which is not a chain, that is x and y are not similar. Then La-Prim(H)=$\emptyset$, for each F-module H , since La has not any prime element. This example display that La-Prim(H)=$\emptyset$, but Prim(H) may be non-empty.

4 .The relation b etween La-Prim(H) and La-Prim(F / Ann(H))

Suppose that ϑ is a primary La-submodule of H. Then by Corollary 2.11 we have (ϑ : 1_H) be a primary La-ideal of F. Let the quotient ring F/ Ann(H). We indicate a typical element of F/ Ann(H) by [y], where y $\in$ F. Consider the quotient map ρ:F $\rightarrow$ F / Ann(H), is defined via ρ(y)= [y], we indicate the image of ϑ :1_H beneath ρ by $(\overline{\vartheta: 1_H})$. In fact, $(\overline{\vartheta: 1_H})$([y])

$=\vee \{(\vartheta: 1_H)(a) \mid a \in [y]\}$.

Proposition 4.1 Suppose that $\vartheta \in La^H$. Then $\overline{(\vartheta: 1_H})$ is a primary La-ideal of $F/\mathrm{Ann}(H)$.

Proof The quotient function ρ is epimorphism, it is facile to prove that the $(\vartheta : 1_H)$ is ρ-invariant. Then via Proposition 2.3, $\overline{(\vartheta: 1_H})$ is primary La-ideal of $F/\mathrm{Ann}(H)$.

Define the function σ: La-Prim(H)→ La-Prim(F/ Ann(H)) by $\sigma(\vartheta) = \overline{(\vartheta: 1_H})$ for $\vartheta \in$ La-Prim(H), σ is called the standard function.

Lemma 4.2 Suppose that B is an ideal of F while $\varrho \in$ LaI (F/B). There occurs $\gamma \in$ LaI (F) such that $\varrho = \overline{\gamma}$.

Proof Let the quotient function ρ: F → F /B. Then it is to prove that $\varrho = o\ \rho$.

Proposition 4.3 The standard function σ be persistent for the topologies on La-Prim(H) while La-Prim (F/Ann(H)).

Proof Suppose that $\overline{\gamma} \in$ LaI (F/ Ann(H)). We claim that $\sigma^{-1}(V(\overline{\gamma})) = V(\gamma.1_H)$. For this let $Q \in V(\gamma.1_H)$. Then $\gamma.1_H \subseteq Q$ and $1_H \nsubseteq Q$. Thus $\gamma \subseteq Q{:}1_H$ and hence $\overline{\gamma} \subseteq \overline{Q : 1_H}$. Hence $\overline{Q : 1_H} \subseteq V(\overline{\gamma})$ and $\overline{Q : 1_H} = \sigma(Q)$, so $Q \in \sigma^{-1}(V(\overline{\gamma})) \Rightarrow V(\gamma.1_H) \subseteq \sigma^{-1}(V(\overline{\gamma}))$.

Identically we can prove that $\sigma^{-1}(V(\overline{\gamma})) \subseteq V(\gamma.1_M)$ and hence $\sigma^{-1}(V(\overline{\gamma})) = V(\gamma.1_H)$. Thus

σ is persistent.

Proposition 4.4 For each F-module H the following assertions are equivalent:

(1) σ be injective;

(2) for $\vartheta, \varpi \in$ La-Prim(H), if $V(\vartheta)=V(\varpi)$, then $\vartheta = \varpi$;

(3) for every $q \in$ La-Prim(F), $|\text{La-Prim}_p(H)| \leq 1$.

Proof (1) ⟹ (2): Suppose that $\vartheta, \varpi \in$ La-Prim(H). If $V(\vartheta)=V(\varpi)$ then $\vartheta{:}1_H = \varpi{:}\ 1_H$, by Proposition 3.3 and hence $\overline{\vartheta : 1_H} = \overline{\varpi : 1_H}$ which lead to that $\sigma(\vartheta)=\sigma(\varpi)$. Thus $\vartheta = \varpi$, since σ is injective by(1).

(2) ⟹ (3): Suppose that $\vartheta, \varpi \in \text{La-Prim}_p(H)$, then $\vartheta{:}1_H = \varpi : 1_H = q$. Therefore $V(\vartheta)=V(\varpi)$ via Proposition 3.3. Then by (2) we have $\vartheta = \varpi$, and hence $|\text{La-Prim}_p(H)| \leq 1$.

(3) ⟹ (1): Let $\vartheta, \vee \in$ La-Prim(H) and $\sigma(\vartheta)=\sigma(\varpi)$. Then

$\overline{\vartheta : 1_H} = \overline{\varpi : 1_H} \Rightarrow \vartheta : 1_H = \varpi : 1_H = q \Rightarrow \vartheta, \varpi \in \text{La-Prim}_p(H) \Rightarrow \vartheta = \varpi$.

That is σ injective.

In the complements we put Y= La-Prim(H) and $\overline{Y}$ = La-Prim(F /Ann(H)).

Theorem 4.5 Suppose that σ is the natural map. If σ is inclusive then σ is both closed while open.

Proof Let $\sigma : Y \to \overline{Y}$ is the standard function and $\vartheta \in Y$. Then via the proof of Proposition 4.3,

$\sigma^{-1}(\mathrm{V}\ (\overline{\vartheta :\ 1_{\mathrm{H}}}))=$ V((ϑ : 1_{H}). 1_{H})= V(ϑ) $\Longrightarrow$ σ (V(ϑ))=$\sigma\ o\ \sigma^{-1}(\mathrm{V}\ (\overline{\vartheta :\ 1_{\mathrm{H}}}))=$ V $(\overline{\vartheta :\ 1_{\mathrm{H}}})$,

That is σ is closed. Also we have
σ (Y-V(ϑ))=$\sigma(\sigma^{-1}(\overline{\mathrm{Y}})$- σ^{-1} (V$(\overline{\vartheta :\ 1_{\mathrm{H}}})$)) = $\sigma(\sigma^{-1}(\overline{\mathrm{Y}}$ - V$(\overline{\vartheta :\ 1_{\mathrm{H}}})$)) = $\overline{\mathrm{Y}}$ - V$(\overline{\vartheta :\ 1_{\mathrm{H}}})$,

That is σ be open.

Proposition 4.6 Suppose that σ is the standard function from Y into $\overline{\mathrm{Y}}$ and it is inclusive. Then Y is linked if and only if $\overline{\mathrm{Y}}$ is linked.

Proof Suppose that Y is linked. Then $\overline{\mathrm{Y}}$=σ(Y) is linked, since σ be persistent while inclusive. Conversely, let $\overline{\mathrm{Y}}$ is linked but Y is non-linked. Then Y includes a non-empty proper subset A such that it is both open and closed. We prove that σ(A) is a non-empty proper subset of $\overline{\mathrm{Y}}$. Since A is open then there occurs $\vartheta \in$ La(H) such that
A = Y\V(ϑ). Thus σ(A) = $\overline{\mathrm{Y}}$\V$(\overline{\vartheta :\ 1_{\mathrm{H}}})$,. If σ(A) = $\overline{\mathrm{Y}}$ then V$(\overline{\vartheta :\ 1_{\mathrm{H}}})$ = ∅, and hence
$(\overline{\vartheta :\ 1_{\mathrm{H}}})$ = χ F/ Ann(H) $\Longrightarrow$ $\vartheta = 1_{\mathrm{H}}$ $\Longrightarrow$ A=Y \ V(ϑ)= Y \ V(1_{H}) =Y,
A discrepancy, if σ(A) = ∅, then we must have V$(\overline{\vartheta :\ 1_{\mathrm{H}}})$ = $\overline{\mathrm{Y}}$, and hence
$\overline{\vartheta :\ 1_{\mathrm{H}}} = \chi\ \overline{0} \Longrightarrow \vartheta = \chi\ 0 \Longrightarrow$ A=Y \ V(χ 0) = Y \ Y = ∅, which is a discrepancy. Therefore σ(A) is a proper non-empty subset of $\overline{\mathrm{Y}}$ such that it is both open and closed, a discrepancy. Thus Y is linked.

Proposition 4.7: Suppose that H while H′ is F–modules. If Y=La-Prim(H), Y′= La-Prim(H′) and f : H → H′ be an epimorphism, then the function g : Y′ →Y is defined via g(ϑ') = f^{-1}(ϑ') be persistent.

Proof Let ϑ ∈ La(H) while V(ϑ) be a closed set in Y. For Q ∈ g^{-1}(V(ϑ)) by Proposition 3.4 (b), we have V(ϑ) = V*((ϑ: 1_{H}). 1_{H}).
Thus
Q ∈ g^{-1}(V*((ϑ: 1_{H}). 1_{H}) ⇔ g(Q′) ∈ V*((ϑ: 1_{H}). 1_{H}) ⇔ (ϑ: 1_{H}). 1_{H} ⊆ g(Q′) = f^{-1}(Q′)⇔f((ϑ: 1_{H}) . 1_{H}) ⊆ Q′ ⇔ ((ϑ: 1_{H}). $1_{\mathrm{H'}}$ ⊆ Q′ ⇔ Q′ ∈ V*((ϑ: 1_{H}). $1_{\mathrm{H'}}$)= V((ϑ: 1_{H}). $1_{\mathrm{H'}}$).

Therefore g^{-1}(V(ϑ)) = V((ϑ: 1_{H}). $1_{\mathrm{H'}}$), and hence g is persistent.

5 A basis for the Zariski topology over La-Prim(H)

Proposition 5.1 **[12]** If g is a homomorphism from **F** onto **F′**, then for each y ∈ **F** and $\boldsymbol{\alpha}$ ∈ La \ {0}; g(y $\boldsymbol{\alpha}$) = $(\mathbf{g(y)})_{\boldsymbol{\alpha}}$.

Corollary 5.2 Suppose that y ∈ **F**, then for all ideal B of **F**, and for all $\boldsymbol{\alpha}$ ∈ La \ {0}; $\overline{\boldsymbol{y}}_{\boldsymbol{\alpha}} = \overline{(\boldsymbol{y}_{\boldsymbol{\alpha}})}$, where $\overline{\boldsymbol{y}}_{\boldsymbol{\alpha}}$ be an La-point of **F** / B
For each **F**-module H, we suppose the collection C={D($\boldsymbol{y}_{\boldsymbol{\alpha}}$. $\mathbf{1_H}$)| y ∈ **F** , $\boldsymbol{\alpha}$ ∈ La \ {0}} such that D($\boldsymbol{y}_{\boldsymbol{\alpha}}$. $\mathbf{1_H}$)=Y \ **V**($\boldsymbol{y}_{\boldsymbol{\alpha}}$. $\mathbf{1_H}$). We assumption that if the lattice La is a chain then C formation a basis for Zarski topology on Y=La-Prim(H).
We suppose the following states:
(1) If $\boldsymbol{\alpha}$=1 while y = 0, D(0_1.1_{H}) = Y\V (0_1.1_{H}) = Y \V (0_{H}) = ∅.
(2) If $\boldsymbol{\alpha}$ = 1 while y = 1, D(1_1.1_{H}) = Y\V (1_1.1_{H}) = Y \V (1_{H}) = Y.

Notation In the complement for $\gamma \in$ LaI (F) we put E(γ) =La-Prim(F)\ V(γ).

Proposition 5.3 If $\sigma : Y \to \overline{Y}$ is standard function, then

(a) $\sigma^{-1}(E(\overline{y_\alpha})) = D(y_\alpha . 1_H)$;

(b) $\sigma(D(y_\alpha . 1_H)) \subseteq E(\overline{y_\alpha})$. Further if σ is inclusive then the parity satisfies.

Proof For (a) we have $\sigma^{-1}(E(\overline{y_\alpha}))=\sigma^{-1}(\overline{Y} \setminus V(\overline{y_\alpha}))=Y \setminus \sigma^{-1}(V(\overline{y_\alpha}))=Y \setminus V(y_\alpha. 1_H) = D(y_\alpha. 1_H)$.

For (b) We have $\sigma(\sigma^{-1}(E(\overline{y_\alpha}))) = \sigma(D(y_\alpha.1_H))$ and $\sigma(\sigma^{-1}(E(\overline{y_\alpha}))) = \subseteq E(\overline{y_\alpha})$. Then $\sigma(D(y_\alpha .1_H)) \subseteq E(\overline{y_\alpha})$. So if σ is inclusive then we get that $\sigma(\sigma^{-1}(E(\overline{y_\alpha}))) =E(\overline{y_\alpha})$. Thus $\sigma(D(y_\alpha.1_H)=E(\overline{y_\alpha})$.

Proposition 5.4 If a, b $\in$ F and $\alpha, \alpha' \in La\setminus\{0\}$, then $D(a_\alpha .1_H) \cap D(b_{\alpha'} .1_H)=D((ab)_{\alpha\wedge\alpha'} .1_H)$. _ Proof We have $D(a_\alpha . 1_H) \cap D(b_{\alpha'}.1_H) = \sigma^{-1}(E(\overline{a_\alpha})) \cap \sigma^{-1}(E(\overline{b_{\alpha'}})) = \sigma^{-1}(E(\overline{a_\alpha}) \cap E(\overline{b_{\alpha'}}))$

$= \sigma^{-1}(E(\overline{(ab)_{\alpha\wedge\alpha'}}) = D((ab)_{\alpha\wedge\alpha'} .1_H)$.

In the complement, we suppose that the lattice La is a chain.

Theorem 5.5 For each F-module H, the collection $C=\{D(a_\alpha .1_H) | x \in F, \alpha \in La\setminus\{0\}\}$ formation a basis for Zariski topology over Y=La-Prim(H).

Proof Let W be an arbitrary open set in Y. Then W= $D(\vartheta)=Y \setminus V(\vartheta)$ for several $\vartheta \in La(H)$.

Via Proposition 3.4, $V(\vartheta)=V((\vartheta:1_H).1_H)$. By considering $\gamma = \vartheta:1_H$, then $V(\vartheta)=V(\gamma.1_H)$

As we aforesaid in the basic concepts, we can write $\gamma=\bigcup_{c\in\gamma(F)} c\gamma_c$. Obviously we have $c\gamma_c=\bigcup_{y\in\gamma c} \gamma_c$. Thus we get that

$$V(\gamma.1_H) =V((\bigcup_{c\in\gamma(F)}(\bigcup_{y\in\gamma c} y_c)).1_H) =V((\bigcup_{c\in\gamma(F),\ y\in\gamma c} y_c).1_H)$$

$$= V(\bigcup_{c\in\gamma(F),\ y\in\gamma c} y_c. 1_H) \quad \text{(since La is a chain)}$$

$$= \bigcap_{c\in\gamma(F),\ y\in\gamma c} V(y_c. 1_H)$$

Thus

$$D(\vartheta)=Y \setminus V(\vartheta) = Y \setminus \bigcap_{c\in\gamma(F),\ y\in\gamma c} V(y_c. 1_H)$$

$$=\bigcup_{c\in\gamma(F),\ y\in\gamma c}(Y \setminus V(y_c. 1_H))=\bigcup_{c\in\gamma(F),\ y\in\gamma c} D(y_c. 1_H)$$

This proves that C is a basis for the Zariski topology over Y .

Proposition 5.6 Suppose that H is an F-module. If the standard function σ is inclusive, then Y=La-Prim(H) is compact. Proof: Suppose that $Y=\bigcup\{D(y\alpha. 1_H) | y \in F, \alpha \in La\setminus \{0\}\}$. Then $\overline{Y}= \sigma(Y)= \sigma(\bigcup\{D(y_\alpha. 1_H)| y \in F, \alpha \in La\setminus \{0\}\}) =\bigcup\{\sigma(D(y_\alpha. 1_H))| y \in F, \alpha \in La\setminus \{0\}\}$

$= \bigcup\{\overline{y_\alpha} | y \in F, \alpha \in La\setminus \{0\}\}$ (since σ is inclusive).

Also, since $\bar{Y}$ is compact, we can write $\bar{Y}=\bigcup_{j=1}^{n} \overline{yj_{\alpha j}}$, and hence $\sigma^{-1}(\bar{Y}) = \sigma^{-1}(\bigcup_{j=1}^{n}\overline{((yj)_{\alpha j}}$).
Thus $Y=\bigcup_{j=1}^{n} \sigma^{-1}\ \overline{((yj)_{\alpha j})}$, and so $\sigma^{-1}\ \overline{((yj)_{\alpha j})} = ((yj)_{\alpha j}.1_H)$. Therefore Y is compact.

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