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The Continuous Classical Boundary Optimal Control Vector Governing by Triple Linear Partial Differential Equations of Parabolic Type

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Abstract

In this paper, the continuous classical boundary optimal control problem (CCBOCP) for triple linear partial differential equations of parabolic type (TLPDEPAR) with initial and boundary conditions (ICs & BCs) is studied. The Galerkin method (GM) is used to prove the existence and uniqueness theorem of the state vector solution (SVS) for given continuous classical boundary control vector (CCBCV). The proof of the existence theorem of a continuous classical boundary optimal control vector (CCBOCV) associated with the TLPDEPAR is proved. The derivation of the Fréchet derivative (FrD) for the cost function (CoF) is obtained. At the end, the theorem of the necessary conditions for optimality (NCsThOP) of this problem is stated and proved.

Keywords: Continuous Classical Boundary Optimal Control, Triple Linear Partial Differential Equations, Galerkin Method, Necessary Conditions for Optimality.

1. Introduction

Different applications for real life problems take a main place in the optimal control problems, for example in medicine [1]. Robots [2]. Engineering [3]. Economic [4]. And many others fields.

In the field of mathematics, optimal control problem (OCP) usually governing either by ordinary differential equations (ODEs) or partial differential equations (PDEs), examples of OCP which are governined by parabolic, hyperbolic or elliptic PDEs are studied by [5,-7]. Respectively, while OCP which are governing by couple of PDEs (CPDEs) of Parabolic, hyperbolic or elliptic type are studied by [8-10]. On the other hand, [11-13]. Are studied boundary OCP associated with CPDEs of these three types; while [14, 15]. Are studied the OCP for triple PDEs (TPDEs) of parabolic and elliptic type respectively.

All these works push us to seek about the CCBOCV for the TLPDEPAR. This work starts with the state and prove the existence theorem of a unique solution (SVS) for the triple state equations (TSEs) of PDEs of parabolic type (TLPDEPAR) by using the GM when the

CCBCV is fixed, then we deals with the proof of the existence theorem of a CCBOCV, the solution vector of the triple adjoint equations (TAPEs) associated the TLPDEPAR is studied. The derivation of the FrD for the CoF is obtained. At the end, the NCsThOP of this OCP is sated and proved..

2. Description of the problem:

Let $\Omega \subset \mathbb{R}^2$, $x = (x_{1,}, x_2)$, $Q = [0, \mathrm{T}] \times \Omega$, $\tilde{\mathrm{I}} = [0, \mathrm{T}]$, $\Gamma = \partial\Omega$, $\Sigma = \Gamma \times \tilde{\mathrm{I}}$. The CCBOCP consists of the TSEs which are given by the following TLPDEPAR

$$y_{1t} - \Delta y_1 + y_1 - y_2 - y_3 = f_1(x,t), \text{ in } Q \quad (1)$$

$$y_{2t} - \Delta y_2 + y_2 + y_3 + y_1 = f_2(x,t), \text{ in } Q \quad (2)$$

$$y_{3t} - \Delta y_3 + y_3 + y_1 - y_2 = f_3(x,t), \text{ in } Q \quad (3)$$

with the BCs and ICs.

$$\frac{\partial y_1}{\partial n_a} = \sum_{i=1}^{2} \frac{\partial y_1}{\partial x_j} \cos(n_1, x_j) = u_1(x,t), \text{ on } \Sigma \quad (4)$$

$$\frac{\partial y_2}{\partial n_b} = \sum_{i=1}^{2} \frac{\partial y_2}{\partial x_j} \cos(n_2, x_j) = u_2(x,t), \text{ on } \Sigma \quad (5)$$

$$\frac{\partial y_3}{\partial n_c} = \sum_{i=1}^{2} \frac{\partial y_3}{\partial x_j} \cos(n_3, x_j) = u_3(x,t), \text{ on } \Sigma \quad (6)$$

$$y_1(x,0) = y_1^0(x), \text{ on } \Omega \quad (7)$$

$$y_2(x,0) = y_2^0(x), \text{ on } \Omega \quad (8)$$

$$y_3(x,0) = y_3^0(x), \text{ on } \Omega \quad (9)$$

Where n_{g}, $\forall\, g = 1,2,3$, is an outer normal vector on the boundary Σ, and (n_{g}, x_j) is the angle between n_{g} and the x_j –axis, (f_1, f_2, f_3) is a vector of a given function on Ω,$\vec{u} = (u_1, u_2, u_3) \in \left(L^2(\Sigma)\right)^3$ is the CCBCV and $\vec{y} = (y_1, y_2, y_3) \in \left(H^1(Q)\right)^3$ is their corresponding SVS. The set of admissible CCBCV is defined by

$\overrightarrow{W}_A = \{\vec{u} = (u_1, u_2, u_3) \in (L^2(\Sigma))^3 | (u_1, u_2, u_3) \in \vec{U} = U_1 \times U_2 \times U_3 \subset R^3 \text{a.e. in } \Sigma\}$,

where a.e. denotes to almost everywhere.

The CoF is defined by

$$G_0(\vec{u}) = \tfrac{1}{2}\|y_1 - y_{1d}\|_Q^2 + \tfrac{1}{2}\|y_2 - y_{2d}\|_Q^2 + \tfrac{1}{2}\|y_3 - y_{3d}\|_Q^2 + \tfrac{\beta}{2}\|u_1\|_\Sigma^2 + \tfrac{\beta}{2}\|u_2\|_\Sigma^2 + \tfrac{\beta}{2}\|u_3\|_\Sigma^2, \quad \vec{u} \in \overrightarrow{W}_A,\ \beta > 0 \quad (10)$$

Let $\vec{V} = V_1 \times V_2 \times V_3 = H^1(\Omega)$, $\vec{V} = \{\vec{v}: \vec{v} = (v_1(x), v_2(x), v_3(x)) \in \left(H^1(\Omega)\right)^3\}$.

The weak form (wf) of the boundary value problem (BVP) (1)-(9), when $\vec{y} \in \left(H^1(Q)\right)^3$ is given by

$$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) - (y_3, v_1) = (f_1, v_1) + (u_1, v_1)_\Gamma, \forall\, v_1 \in V_1 \quad (11.a)$$

$$(y_1(0), v_1) = (y_1^0, v_1), \quad \forall\, v_1 \in V_1 \quad (11.b)$$

$$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_3, v_2) + (y_1, v_2) = (f_2, v_2) + (u_2, v_2)_\Gamma, \ \forall\, v_2 \in V_2 \quad (12.a)$$

$$(y_2(0), v_2) = (y_2^0, v_2), \quad \forall\, v_2 \in V_2 \quad (12.b)$$

$$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3) = (f_3, v_3) + (u_3, v_3)_\Gamma, \forall\, v_3 \in V_3 \quad (13.a)$$

$$(y_3(0), v_3) = (y_3^0, v_3), \quad \forall\, v_3 \in V_3 \quad (13.b)$$

The following assumption is important to study the existence theorem of the SVS for the wf (11)−(13).

2.1. Assumption (A): The function f_i ($\forall\, i = 1,2,3$) is satisfied the following condition with respect to (w.r.t.) x & t, that is (i.e.) $|f_i| \leq \eta_i(x,t)$, where $(x,t) \in Q$, $\eta_i \in L^2(Q, \mathbb{R})$.

3. The Existence Solution for the wf:

Theorem 3.1: Existence of a Unique Solution for the wf of the SEs: With assumption (A), for each given CCBCV $\vec{u} \in \left(L^2(\Sigma)\right)^3$, the wf (11)–(13) of the TSEs has a unique SVS $\vec{y}$ with $\vec{y} \in \left(L^2(\bar{I}, V)\right)^3$, $\vec{y}_t = (y_{1t}, y_{2t}, y_{3t}) \in \left(L^2(\bar{I}, V^*)\right)^3$.

Proof: Let for each n, $\vec{V}_n = V_n \times V_n \times V_n \subset \vec{V}$ be the set of continuous and piecewise affine functions in Ω, let $v_{ij} \in V_{in} = V_n$, $i = 1,2,3$, and $j = 1,2,\dots,n$, be a basis of V_n, let $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n})$ be an approximate solution for the solution $\vec{y}$, then by GM:

$$y_{1n} = \sum_{j=1}^{n} c_{1j}(t)\, v_{1j}(x), \tag{14}$$

$$y_{2n} = \sum_{j=1}^{n} c_{2j}(t)\, v_{2j}(x), \tag{15}$$

$$y_{3n} = \sum_{j=1}^{n} c_{3j}(t)\, v_{3j}(x), \tag{16}$$

where $c_{ij}(t)$ are unknown functions of t, $\forall\, i = 1,2,3$, $j = 1,2,\dots,n$.

The wf $(11)-(13)$ is approximated by using $(14)-(16)$ as follows:

$$\langle y_{1nt}, v_1\rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) - (y_{3n}, v_1) = (f_1, v_1) + (u_1, v_1)_\Gamma,\ \forall\, v_1 \in V_n \tag{17.a}$$

$$(y_{1n}^0, v_1) = (y_1^0, v_1), \qquad \forall\, v_1 \in V_n \tag{17.b}$$

$$\langle y_{2nt}, v_2\rangle + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_{3n}, v_2) + (y_{1n}, v_2) = (f_2, v_2) + (u_2, v_2)_\Gamma,\ \forall\, v_2 \in V_n \tag{18.a}$$

$$(y_{2n}^0, v_2) = (y_2^0, v_2), \qquad \forall\, v_2 \in V_n \tag{18.b}$$

$$\langle y_{3nt}, v_3\rangle + (\nabla y_{3n}, \nabla v_3) + (y_{3n}, v_3) + (y_{1n}, v_3) - (y_{2n}, v_3) = (f_3, v_3) + (u_3, v_3)_\Gamma,\ \forall\, v_3 \in V_n \tag{19.a}$$

$$(y_{3n}^0, v_3) = (y_3^0, v_3), \qquad \forall\, v_3 \in V_n \tag{19.b}$$

Where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n \subset V_i \subset L^2(\Omega)$ is the projection of y_i^0, thus $(y_{in}^0, v_i) = (y_i^0, v_i),\ \forall\, v_i \in V_n \iff \| y_{in}^0 - y_i^0 \|_0 \le \| y_i^0 - v_i \|_0,\ \forall\, v_i \in V_n$.

Substituting $(14)-(16)$ in $(17)-(19)$ respectively, and then setting $v_1 = v_{1l}$, $v_2 = v_{2l}$ & $v_3 = v_{3l}$ $\forall\, l = 1,2,\dots,n$, the obtained equations are equivalent to the following linear system (LS) of 1st order ODEs with ICs (which has a unique solution), i.e.

$$AC_1'(t) + BC_1(t) - DC_2(t) - EC_3(t) = b_1, \tag{20.a}$$

$$AC_1(0) = b_1^0, \tag{20.b}$$

$$FC_2'(t) + GC_2(t) + HC_3(t) + KC_1(t) = b_2, \tag{21.a}$$

$$FC_2(0) = b_2^0, \tag{21.b}$$

$$MC_3'(t) + NC_3(t) + RC_1(t) - WC_2(t) = b_3, \tag{22.a}$$

$$MC_3(0) = b_3^0, \tag{22.b}$$

Where

$A = (a_{lj})_{n\times n}$, $a_{lj} = (v_{1j}, v_{1l})$, $B = (b_{lj})_{n\times n}$, $b_{lj} = (\nabla v_{1j}, \nabla v_{1l}) + (v_{1j}, v_{1l})$, $D = (d_{lj})_{n\times n}$, $d_{lj} = (v_{2j}, v_{1l})$, $E = (e_{lj})_{n\times n}$, $e_{lj} = v_{3j}, v_{1l})$, $F = (f_{lj})_{n\times n}$, $f_{lj} = (v_{2j}, v_{2l})$, $G = (g_{lj})_{n\times n}$, $g_{lj} = (\nabla v_{2j}, \nabla v_{2l}) + (v_{2j}, v_{2l})$, $H = (h_{lj})_{n\times n}$, $h_{lj} = (v_{3j}, v_{2l})$, $K = (k_{lj})_{n\times n}$, $k_{lj} = (v_{1j}, v_{2l})$, $M = (m_{lj})_{n\times n}$, $m_{lj} = (v_{3j}, v_{3l})$, $N = (n_{lj})_{n\times n}$, $n_{lj} = (\nabla v_{3j}, \nabla v_{3l}) + (v_{3j}, v_{3l})$, $R = (r_{lj})_{n\times n}$, $r_{lj} = (v_{1j}, v_{3l})$, $W = (w_{lj})_{n\times n}$, $w_{lj} = (v_{2j}, v_{3l})$, $b_i^0 = (b_{il}^0)$, $b_{il}^0 = (y_i^0, v_{il})$, $b_i = (b_{il})_{n\times 1}$, $b_{il} = (f_i, v_{il}) + (u_i, v_{il})_\Gamma$, $C_i'(t) = (c_{ij}'(t)_{n\times 1})$, $C_i(t) = (c_{ij}(t)_{n\times 1})$, $C_i(0) = (c_{ij}(0)_{n\times 1})$, $\forall\, l = 1,2,3\dots n$, $i = 1,2,3$.

To show the norm $\left\|\vec{y}_n^0\right\|_0$ is bounded:

Since $y_1^0 \in L^2(\Omega)$, there exists a sequence $\{v_{1n}^0\}$ with $v_{1n}^0 \in V_n$, such that $v_{1n}^0 \longrightarrow y_1^0$ strongly in $L^2(\Omega)$, then from the projection Theorem [16]. And (17.b),

$\|y_{1n}^0 - y_1^0\|_0 \le \|y_1^0 - v_1\|_0, \forall\, v_1 \in V$, and then

$\|y_{1n}^0 - y_1^0\|_0 \le \|y_1^0 - v_{1n}^0\|_0,\ \forall\, v_{1n}^0 \in V_n \subset V,\ \forall n$

$\Rightarrow y_{1n}^0 \longrightarrow y_1^0$, strongly in $L^2(\Omega)$, implies to

$\|y_{1n}^0\|_0 \le b_1$, similarly $\|y_{2n}^0\|_0 \le b_2$ & $\|y_{3n}^0\|_0 \le b_3$, thus $\left\|\overrightarrow{y_n^0}\right\|_{\mathbf{0}}$ is bounded in $\left(L^2(\Omega)\right)^3$.

The norms $\|\vec{y}_n(t)\|_{L^\infty(\bar{I},L^2(\Omega))}$ and $\|\vec{y}_n(t)\|_{Q}$ are bounded:

Setting $v_1 = y_{1n}$, $v_2 = y_{2n}$ and $v_3 = y_{3n}$ in (17), (18) & (19) respectively, integrating w.r.t. t from 0 to T, and then adding the obtained three equations, one gets

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle dt + \int_0^T \|\vec{y}_n\|_1^2 dt =$$
$$\int_0^T [(f_1, y_{1n}) + (u_1, y_{1n})_\Gamma + (f_2, y_{2n}) + (u_2, y_{2n})_\Gamma + (f_3, y_{3n}) + (u_3, y_{3n})_\Gamma] dt, \qquad (23)$$

Using Lemma (1.2) in [11]. For the 1st term in the L.H.S. of (23), and since the 2nd term is positive, taking $T = t \in [0,T]$. Finally, using assumption (A) for the R.H.S. of (23), it yields to

$$\frac{1}{2}\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_0^2\, dt \le$$
$$\frac{1}{2}[\int_0^t \int_\Omega (\eta_1^2 + |y_{1n}|^2) dx dt + \int_0^t \int_\Gamma (|u_1|^2 + |y_{1n}|^2) d\gamma dt + \int_0^t \int_\Omega (\eta_2^2 + |y_{2n}|^2) dx dt +$$
$$\int_0^t \int_\Gamma (|u_2|^2 + |y_{2n}|^2) d\gamma dt + \int_0^t \int_\Omega (\eta_3^2 + |y_{3n}|^2) dx dt + \int_0^t \int_\Gamma (|u_3|^2 + |y_{3n}|^2) d\gamma dt\,].$$

Using the Trace Theorem [17]. Of the R.H.S., to get

$$\|\vec{y}_n(t)\|_0^2 - \|\vec{y}_n(0)\|_0^2 \le$$
$$\|\eta_1\|_Q^2 + \|\eta_2\|_Q^2 + \|\eta_3\|_Q^2 + \|u_1\|_\Sigma^2 + \|u_2\|_\Sigma^2 + \|u_3\|_\Sigma^2 + \int_0^t \|y_{1n}\|_0^2\, dt + c_1^2 \int_0^t \|y_{1n}\|_0^2\, dt +$$
$$\int_0^t \|y_{2n}\|_0^2\, dt + c_2^2 \int_0^t \|y_{2n}\|_0^2\, dt + \int_0^t \|y_{3n}\|_0^2\, dt + c_3^2 \int_0^t \|y_{3n}\|_0^2\, dt.$$

Since $\|\eta_i\|_Q^2 \le \acute{b}_i$, $\|u_i\|_\Sigma^2 \le \acute{c}_i$, $\forall\, i = 1,2,3$, $\|\vec{y}_n(0)\|_0^2 \le b$, then

$$\|\vec{y}_n(t)\|_0^2 \le c_1^* + \hbar \int_0^t \|\vec{y}_n\|_0^2\, dt,$$

where $c_1^* = \acute{b}_1 + \acute{b}_2 + \acute{b}_3 + \acute{c}_1 + \acute{c}_2 + \acute{c}_3 + b$ and $\hbar = \max(\hbar_1^*, \hbar_2^*, \hbar_3^*)$; $\hbar_1^* = 1 + c_1^2$, $\hbar_2^* = 1 + c_2^2$, $\hbar_3^* = 1 + c_3^2$.

Using the continuous Bellman Gronwall Inequality (BGI), one gets

$\|\vec{y}_n(t)\|_0^2 \le c_1^* e^{\hbar T} = b^2(c)$, $\forall\, t \in [0,T]$ or $\|\vec{y}_n(t)\|_{L^\infty(\bar{I},L^2(\Omega))} \le b(c) \Rightarrow \|\vec{y}_n(t)\|_{Q} \le b_1(c)$.

***The norm $\|\vec{y}_n(t)\|_{L^2(\bar{I},V)}$ is bounded*:**

Again for (23) by using Lemma (1.2) in [11]. For the L.H.S. the same results may be obtained (from the above steps) and since $\|\vec{y}_n(0)\|_0^2$ is bounded, equation (23) with $t = T$, becomes

$$\|\vec{y}_n(t)\|_0^2 - \|\vec{y}_n(0)\|_0^2 + 2\int_0^T \|\vec{y}_n\|_1^2\, dt \le \|\eta_1\|_Q^2 + \|\eta_2\|_Q^2 + \|\eta_3\|_Q^2 + \|u_1\|_\Sigma^2 + \|u_2\|_\Sigma^2 +$$
$$\|u_3\|_\Sigma^2 + \hbar\|\vec{y}_n\|_{Q}^2.$$

Which gives

$\int_0^T \|\vec{y}_n\|_1^2\, dt \le b_3^2(c)$, with $b_3^2(c) = \frac{\left(\acute{b}_1 + \acute{b}_2 + \acute{b}_3 + \acute{c}_1 + \acute{c}_2 + \acute{c}_3 + b + \hbar b_1(c)\right)}{2}$, thus $\|\vec{y}_n\|_{L^2(\bar{I},V)} \le b_3(c)$.

The convergence of the solution:

Let $\{\vec{V}_n\}_{n=1}^\infty$ be a sequence of subspaces of $\vec{V}$, s.t. $\forall\ \vec{v} \in \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n \in \vec{V}_n$, $\forall\, n$ and $\vec{v}_n \longrightarrow \vec{v}$, strongly in $\vec{V} \Rightarrow \vec{v}_n \longrightarrow \vec{v}$, strongly in $\left(L^2(\Omega)\right)^3$, since for each n, with $\vec{V}_n \subset \vec{V}$, $(17) - (19)$ has a unique solution $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n})$, hence corresponding to the sequence of subspaces $\{\vec{V}_n\}_{n=1}^\infty$, there exists a sequence of (approximation) problems like $(17) - (19)$, thus by substituting $\vec{v} = \vec{v}_n = (v_{1n}, v_{2n}, v_{3n})$, one has for $n = 1,2,\ldots$

$$\langle y_{1nt}, v_{1n}\rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) - (y_{3n}, v_{1n}) =$$
$$(f_1, v_{1n}) + (u_1, v_{1n})_\Gamma, \qquad \forall\ v_{1n} \in V_n \qquad (24.a)$$
$$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n}), \qquad \forall\ v_{1n} \in V_n \qquad (24.b)$$

$\langle y_{2nt}, v_{2n}\rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n}) + (y_{3n}, v_{2n}) + (y_{1n}, v_{2n}) =$
$(f_2, v_{2n}) + (u_2, v_{2n})_\Gamma, \quad \forall\ v_{2n} \in V_n$ (25.a)
$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n}), \quad \forall\ v_{2n} \in V_n$ (25.b)
$\langle y_{3nt}, v_{3n}\rangle + (\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n}) + (y_{1n}, v_{3n}) - (y_{2n}, v_{3n}) =$
$(f_3, v_{3n}) + (u_3, v_{3n})_\Gamma, \quad \forall\ v_{3n} \in V_n$ (26.a)
$(y_{3n}^0, v_{3n}) = (y_3^0, v_{3n}), \quad \forall\ v_{3n} \in V_n$ (26.b)

which has a sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$, with $\vec{y}_n$, but from the above steps we have $\|\vec{y}_n\|_{L^2(Q)}$ and $\|\vec{y}_n\|_{L^2(\tilde{I},V)}$ are bounded, then by Alaoglu's Theorem (ATh), there exists a subsequence of $\{\vec{y}_n\}_{n\in N}$, say again $\{\vec{y}_n\}_{n\in N}$ s.t. $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^3$ and in $\left(L^2(\tilde{I},V)\right)^3$. Multiplying both sides of (24.a), (25.a) & (26.a) by $\varphi_i(t) \in C^1[0,T]$, $\forall i = 1,2,3$, respectively, s.t. $\varphi_i(T) = 0$, $\varphi_i(0) \neq 0$, integrating both sides w.r.t. t on $[0,T]$, then integrating by parts the 1st terms in the L.H.S. of each one obtained equation, one gets

$$-\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n})]\varphi_1(t)dt - \int_0^T (y_{2n}, v_{1n})\varphi_1(t)\,dt - \int_0^T (y_{3n}, v_{1n})\varphi_1(t)\,dt = \int_0^T (f_1, v_{1n})\varphi_1(t)dt + \int_0^T (u_1, v_{1n})_\Gamma \varphi_1(t)dt + (y_{1n}^0, v_{1n})\varphi_1(0), \tag{27}$$

$$-\int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n})]\varphi_2(t)dt + \int_0^T (y_{3n}, v_{2n})\varphi_2(t)\,dt + \int_0^T (y_{1n}, v_{2n})\varphi_2(t)\,dt = \int_0^T (f_2, v_{2n})\varphi_2(t)dt + \int_0^T (u_2, v_{2n})_\Gamma \varphi_2(t)dt + (y_{2n}^0, v_{2n})\varphi_2(0), \tag{28}$$

$$-\int_0^T (y_{3n}, v_{3n})\varphi_3'(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n})]\varphi_3(t)dt + \int_0^T (y_{1n}, v_{3n})\varphi_3(t)\,dt - \int_0^T (y_{2n}, v_{3n})\varphi_3(t)\,dt = \int_0^T (f_3, v_{3n})\varphi_3(t)dt + \int_0^T (u_3, v_{3n})_\Gamma \varphi_3(t)dt + (y_{3n}^0, v_{3n})\varphi_3(0), \tag{29}$$

Since $\left.\begin{array}{l} v_{in} \longrightarrow v_i \text{ strongly in } L^2(\Omega) \\ v_{in} \longrightarrow v_i \text{ strongly in } V \end{array}\right\} \Rightarrow \left\{\begin{array}{l} v_{in}\varphi_i' \longrightarrow v_i\varphi_i' \text{ strongly in } L^2(Q) \\ v_{in}\varphi_i \longrightarrow v_i\varphi_i \text{ strongly in } L^2(\tilde{I},V) \end{array}\right.$,

Also, since
$y_{in} \longrightarrow y_i$ weakly in $L^2(Q)$, and
$y_{in}^0 \longrightarrow y_i^0$ weakly in $L^2(Q)$, $\forall\ i = 1,2,3$, then

$$\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n})]\varphi_1(t)dt - \int_0^T (y_{2n}, v_{1n})\varphi_1(t)\,dt - \int_0^T (y_{3n}, v_{1n})\varphi_1(t)\,dt \longrightarrow \int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt - \int_0^T (y_3, v_1)\varphi_1(t)\,dt, \tag{30.a}$$

$$\int_0^T (y_{2n}, v_{2n})\varphi_2'(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n})]\varphi_2(t)dt + \int_0^T (y_{3n}, v_{2n})\varphi_2(t)\,dt + \int_0^T (y_{1n}, v_{2n})\varphi_2(t)\,dt \longrightarrow \int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\varphi_2(t)\,dt + \int_0^T (y_1, v_2)\varphi_2(t)\,dt, \tag{31.a}$$

$$\int_0^T (y_{3n}, v_{3n})\varphi_3'(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n})]\varphi_3(t)dt + \int_0^T (y_{1n}, v_{3n})\varphi_3(t)\,dt - \int_0^T (y_{2n}, v_{3n})\varphi_3(t)\,dt \longrightarrow \int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\varphi_3(t)\,dt - \int_0^T (y_2, v_3)\varphi_3(t)\,dt, \tag{32.a}$$

$$(y_{1n}^0, v_{1n})\varphi_1(0) \longrightarrow (y_1^0, v_1)\varphi_1(0), \tag{30.b}$$
$$(y_{2n}^0, v_{2n})\varphi_2(0) \longrightarrow (y_2^0, v_2)\varphi_2(0), \tag{31.b}$$
$$(y_{3n}^0, v_{3n})\varphi_3(0) \longrightarrow (y_3^0, v_3)\varphi_3(0), \tag{32.b}$$

since $v_{in} \longrightarrow v_i$, weakly in $L^2(\Omega)$, then

$\int_0^T [(f_1, v_{1n}) + (u_1, v_{1n})_\Gamma]\varphi_1(t)dt \longrightarrow \int_0^T [(f_1, v_1) + (u_1, v_1)_\Gamma]\varphi_1(t)dt,$ (30.c)

$\int_0^T [(f_2, v_{2n}) + (u_2, v_{2n})_\Gamma]\varphi_2(t)dt \longrightarrow \int_0^T [(f_2, v_2) + (u_2, v_2)_\Gamma]\varphi_2(t)dt,$ (31.c)

$\int_0^T [(f_3, v_{3n}) + (u_3, v_{3n})_\Gamma]\varphi_3(t)dt \longrightarrow \int_0^T [(f_3, v_3) + (u_3, v_3)_\Gamma]\varphi_3(t)dt,$ (32.c)

which means (30) $-$ (32), converge to (33$-$35), respectively

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt -$
$\int_0^T (y_3, v_1)\varphi_1(t)\,dt = \int_0^T (f_1, v_1)\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0),$ (33)

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\varphi_2(t)\,dt +$
$\int_0^T (y_1, v_2)\varphi_2(t)\,dt = \int_0^T (f_2, v_2)\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0),$ (34)

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\varphi_3(t)\,dt -$
$\int_0^T (y_2, v_3)\varphi_3(t)\,dt = \int_0^T (f_3, v_3)\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0),$ (35)

Case1: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall i = 1,2,3$. Substituting in (33)$-$(35), using integration by parts for the 1st terms in the L.H.S. of each one of the obtained equations, one has

$\int_0^T \langle y_{1t}, v_1\rangle\varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt -$
$\int_0^T (y_3, v_1)\varphi_1(t)\,dt = \int_0^T (f_1, v_1)\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt\,,$ (36)

$\int_0^T \langle y_{2t}, v_2\rangle\varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\varphi_2(t)\,dt +$
$\int_0^T (y_1, v_2)\varphi_2(t)\,dt = \int_0^T (f_2, v_2)\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt\,,$ (37)

$\int_0^T \langle y_{3t}, v_3\rangle\varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\varphi_3(t)\,dt -$
$\int_0^T (y_2, v_3)\varphi_3(t)\,dt = \int_0^T (f_3, v_3)\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt\,,$ (38)

i.e. $\vec{y}$ is a solution of the wf (11) $-$ (13) .

Case 2: Choose $\varphi_i \in C^1[0,T]$, s.t. $\varphi_i(T) = 0$ & $\varphi_i(0) \neq 0$, $\forall i = 1,2,3$. Using integration by parts for 1st term in the L.H.S. of (36), one gets

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt -$
$\int_0^T (y_3, v_1)\varphi_1(t)\,dt = \int_0^T (f_1, v_1)\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0),$ (39)

Subtracting (33) from (39), one obtains that

$(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0)\,, \varphi_1(0) \neq 0, \forall\, \varphi_1 \in C^1[0,T] \Rightarrow (y_1^0, v_1) = (y_1(0), v_1),$
i.e. the ICs (11.b) holds. By the same above way one can show that $(y_2^0, v_2) = (y_2(0), v_2)$, $(y_3^0, v_3) = (y_3(0), v_3)$ which means the ICs (12.b) & (13.b) are holds.

The strongly convergence for $\vec{\mathbf{y}}_\mathbf{n}$:

Substituting $v_1 = y_{1n}$,$v_2 = y_{2n}$ and $v_3 = y_{3n}$ in (17.a), (18.a)&(19.a) respectively, adding the three obtained equations together, and then integrating the obtained equation from 0 to T, on the other hand substituting $v_1 = y_1$,$v_2 = y_2$ & $v_3 = y_3$ in (11.a), (12.a) &(13.a) respectively, adding them together, integrating the obtained equations from 0 to T, one gets

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle\,dt + \int_0^T a(\vec{y}_n, \vec{y}_n)dt =$
$\int_0^T [(f_1, y_{1n}) + (u_1, y_{1n})_\Gamma + (f_2, y_{2n}) + (u_2, y_{2n})_\Gamma + (f_3, y_{3n}) + (u_3, y_{3n})_\Gamma]dt,$ (40.a)

$\int_0^T \langle \vec{y}_t, \vec{y}\rangle\,dt + \int_0^T a(\vec{y}\,, \vec{y}\,)dt =$
$\int_0^T [(f_1, y_1) + (u_1, y_1)_\Gamma + (f_2, y_2) + (u_2, y_2)_\Gamma + (f_3, y_3) + (u_3, y_3)_\Gamma]dt,$ (40.b)

using Lemma(1.2) in [11] for the 1st terms in the L.H.S. of (40.a&b), they become

$\frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a(\vec{y}_n, \vec{y}_n)dt =$
$\int_0^T [(f_1, y_{1n}) + (u_1, y_{1n})_\Gamma + (f_2, y_{2n}) + (u_2, y_{2n})_\Gamma + (f_3, y_{3n}) + (u_3, y_{3n})_\Gamma]dt,$ (41.a)

$\frac{1}{2}\|\vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T a(\vec{y},\vec{y})dt = \int_0^T[(f_1,y_1) + (u_1,y_1)_\Gamma + (f_2,y_2) + (u_2,y_2)_\Gamma + (f_3,y_3) + (u_3,y_3)_\Gamma]dt,$ (41.b)

since

$\frac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|_0^2 + \int_0^T a(\vec{y}_n - \vec{y},\vec{y}_n - \vec{y})\,dt = A_1 - B_1 - C_1,$ (42)

where

$A_1 = \frac{1}{2}\|\vec{y}_n(T)\|_0^2 - \frac{1}{2}\|\vec{y}_n(0)\|_0^2 + \int_0^T a\big(\vec{y}_n(t),\vec{y}_n(t)\big)dt$

$B_1 = \frac{1}{2}\big(\vec{y}_n(T),\vec{y}(T)\big) - \frac{1}{2}\big(\vec{y}_n(0),\vec{y}(0)\big) + \int_0^T a\big(\vec{y}_n(t),\vec{y}(t)\big)\,dt,$

$C_1 = \frac{1}{2}\big(\vec{y}(T),\vec{y}_n(T) - \vec{y}(T)\big) - \frac{1}{2}\big(\vec{y}(0),\vec{y}_n(0) - \vec{y}(0)\big) + \int_0^T a\big(\vec{y}(t),\vec{y}_n(t) - \vec{y}(t)\big)dt,$

Since

$\vec{y}_n^0 = \vec{y}_n(0) \longrightarrow \vec{y}^0 = \vec{y}(0)$ strongly in $\big(L^2(\Omega)\big)^3,$ (42.a)

$\vec{y}_n(T) \longrightarrow \vec{y}(T)$ strongly in $\big(L^2(\Omega)\big)^3,$ (42.b)

Then

$\begin{cases} \big(\vec{y}(0),\vec{y}_n(0) - \vec{y}(0)\big) \longrightarrow 0 \\ \big(\vec{y}(T),\vec{y}_n(T) - \vec{y}(T)\big) \longrightarrow 0 \end{cases},$ (42.c)

$\begin{cases} \|\vec{y}_n(0) - \vec{y}(0)\|_0^2 \longrightarrow 0 \\ \|\vec{y}_n(T) - \vec{y}(T)\|_0^2 \longrightarrow 0 \end{cases},$ (42.d)

Since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\big(L^2(\text{Ĩ},V)\big)^3$, then $\int_0^T a\big(\vec{y}(t),\vec{y}_n(t) - \vec{y}(t)\big)dt \longrightarrow 0,$ (42.e)

As well as, since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\big(L^2(Q)\big)^3$, then

$\int_0^T[(f_1,y_{1n}) + (u_1,y_{1n})_\Gamma + (f_2,y_{2n}) + (u_2,y_{2n})_\Gamma + (f_3,y_{3n}) + (u_3,y_{3n})_\Gamma\,]dt \longrightarrow$

$\int_0^T[\,(f_1,y_1) + (u_1,y_1)_\Gamma + (f_2,y_2) + (u_2,y_2)_\Gamma + (f_3,y_3) + (u_3,y_3)_\Gamma]dt,$ (42.f)

i.e. when $n \to \infty$ in both sides of (42), one has the following results:

(1) The first two terms in the L.H.S. of (42) are tending to zero from (42.d).

(2) From (41.a)

Eq. $A_1 = \int_0^T[\,(f_1,y_{1n}) + (u_1,y_{1n})_\Gamma + (f_2,y_{2n}) + (u_2,y_{2n})_\Gamma + (f_3,y_{3n}) + (u_3,y_3)_\Gamma]dt$

from

$\longrightarrow \int_0^T[\,(f_1,y_1) + (u_1,y_1)_\Gamma + (f_2,y_2) + (u_2,y_2)_\Gamma + (f_3,y_3) + (u_3,y_3)_\Gamma]dt\,,$

(42.f)

(3) Eq. $B_1 \longrightarrow$ L.H.S. of (3.41.b) $= \int_0^T[\,(f_1,y_1) + (u_1,y_1)_\Gamma + (f_2,y_2) + (u_2,y_2)_\Gamma + (f_3,y_3) + (u_3,y_3)_\Gamma]dt,$

(4) The 1st two terms in Eq. C_1 are tending to zero from (42.c), and the last one term also tend to zero from (42.e), from these results (42) gives when $n \to \infty$

$\int_0^T \|\vec{y}_n - \vec{y}\|_1^2\,dt = \int_0^T a(\vec{y}_n - \vec{y},\vec{y}_n - \vec{y})dt \longrightarrow 0 \Rightarrow \vec{y}_n \longrightarrow \vec{y}$ strongly in $\big(L^2(\text{Ĩ},V)\big)^3.$

Uniqueness of the solution: Let $\vec{y}$, $\vec{\bar{y}}$ are two solutions of the wf (11)−(13) i.e. y_1 and $\bar{y}_1$ are satisfied (11.a), or

$\langle y_{1t},v_1\rangle + a_1(y_1,v_1) - (y_2,v_1) - (y_3,v_1) = (f_1,v_1) + (u_1,v_1)_\Gamma\,, \quad \forall\, v_1 \in V_1$

$\langle \bar{y}_{1t},v_1\rangle + a_1(\bar{y}_1,v_1) - (\bar{y}_2,v_1) - (\bar{y}_3,v_1) = (f_1,v_1) + (u_1,v_1)_\Gamma\,, \quad \forall\, v_1 \in V_1$

Subtracting the 2^{nd} equation from the 1st one and substituting $v_1 = y_1 - \bar{y}_1$ in the obtained equation, one gets that

$\langle (y_1 - \bar{y}_1)_t\,, y_1 - \bar{y}_1\rangle + a_1(y_1 - \bar{y}_1, y_1 - \bar{y}_1) - (y_2 - \bar{y}_2, y_1 - \bar{y}_1) - (y_3 - \bar{y}_3, y_1 - \bar{y}_1) = 0,$ (43)

by the same way, one gets

$\langle (y_2 - \bar{y}_2)_t\,, y_2 - \bar{y}_2\rangle + a_2(y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (y_3 - \bar{y}_3, y_2 - \bar{y}_2) + (y_1 - \bar{y}_1, y_2 - \bar{y}_2) = 0,$ (44)

$\langle (y_3-\bar{y}_3)_t\,, y_3-\bar{y}_3\rangle + a_3(y_3-\bar{y}_3, y_3-\bar{y}_3) + (y_1-\bar{y}_1, y_3-\bar{y}_3) - (y_2-\bar{y}_2, y_3-\bar{y}_3) = 0,$ (45)

adding (43) − (45), using Lemma(1.2) in [11]. In the 1st term of the obtained equations, to get

$\frac{1}{2}\frac{d}{dt}\left\|\vec{y}-\vec{\bar{y}}\right\|_0^2 + \left\|\vec{y}-\vec{\bar{y}}\right\|_1^2 = 0,$ (46)

The 2nd term of the L.H.S. of (46) is positive, integrating both sides of (46) w.r.t. t from 0 to t, one gets

$\int_0^t \frac{d}{dt}\|\vec{y}-\vec{\bar{y}}\|_0^2\, dt \le 0 \;\Rightarrow\; \left\|(\vec{y}-\vec{\bar{y}})(t)\right\|_0^2 \le 0 \;\Rightarrow\; \left\|\vec{y}-\vec{\bar{y}}\right\|_0^2 = 0 \quad \forall t \in \tilde{I},$

integrating both sides of (46) from 0 to T, using the given ICs, one has

$\int_0^T \left\|\vec{y}-\vec{\bar{y}}\right\|_1^2 dt = 0 \Rightarrow \left\|\vec{y}-\vec{\bar{y}}\right\|_{L^2(\tilde{I},V)} = 0 \;\Rightarrow\; \vec{y} = \vec{\bar{y}}\,.$

4. Existence of a CCBOCP:

Theorem 4.1: In addition to assumptions (A), assume that $\vec{y}$ and $\vec{y}+\delta\vec{y}$ are the SVS corresponding to the CVS $\vec{u}$ and $\vec{u}+\delta\vec{u}$ respectively with $\vec{u}$ and $\delta\vec{u}$ are bounded in $(L^2(\Sigma))^3$, then

$\|\delta\vec{y}\,\|_{L^\infty(\tilde{I},L^2(\Omega))} \le L\|\delta\vec{u}\|_\Sigma\,,\quad L \in \mathbb{R}^+$

$\|\delta\vec{y}\|_{L^2(Q)} \le \bar{L}\|\delta\vec{u}\|_\Sigma\,,\qquad \bar{L} \in \mathbb{R}^+$

$\|\delta\vec{y}\|_{L^2(\tilde{I},V)} \le \bar{L}_1\|\delta\vec{u}\|_\Sigma\,,\qquad \bar{L}_1 \in \mathbb{R}^+$

Proof: Let $\vec{u}=(u_1,u_2,u_3)\in(L^2(\Sigma))^3$ be given, then by Theorem3.1, there exists $\vec{y} = (y_1=y_{u_1}, y_2=y_{u_2}, y_3=y_{u_3})$ which is satisfied (11)−(13) and also let $\vec{\bar{y}}=(\bar{y}_1,\bar{y}_2,\bar{y}_3)$ be the solution of (11) − (13), corresponds to the CV $\vec{\bar{u}}=(\bar{u}_1,\bar{u}_2,\bar{u}_3)\in(L^2(\Sigma))^3$ i.e.

$\langle \bar{y}_{1t}, v_1\rangle + (\nabla\bar{y}_1,\nabla v_1) + (\bar{y}_1,v_1) - (\bar{y}_2,v_1) - (\bar{y}_3,v_1) = (f_1,v_1) + (\bar{u}_1,v_1)_\Gamma,$ (47.a)

$(\bar{y}_1(0)\,, v_1) = (y_1^0, v_1)\,,$ (47.b)

$\langle \bar{y}_{2t}, v_2\rangle + (\nabla\bar{y}_2,\nabla v_2) + (\bar{y}_2,v_2) + (\bar{y}_3,v_2) + (\bar{y}_1,v_2) = (f_2,v_2) + (\bar{u}_2,v_2)_\Gamma,$ (48.a)

$(\bar{y}_2(0)\,, v_2) = (y_2^0, v_2)\,,$ (48.b)

$\langle \bar{y}_{3t}, v_3\rangle + (\nabla\bar{y}_3,\nabla v_3) + (\bar{y}_3,v_3) + (\bar{y}_1,v_3) - (\bar{y}_2,v_3) = (f_3,v_3) + (\bar{u}_3,v_3)_\Gamma,$ (49.a)

$(\bar{y}_3(0)\,, v_3) = (y_3^0, v_3\,),$ (49.b)

Subtracting (11.a&b) from (47.a&b), (12.a&b) from (48.a&b) ,and (13.a&b) from (49.a&b) and setting $\delta y_1 = \bar{y}_1 - y_1$, $\delta y_2 = \bar{y}_2 - y_2$, $\delta y_3 = \bar{y}_3 - y_3$, $\delta u_1 = \bar{u}_1 - u_1$, $\delta u_2 = \bar{u}_2 - u_2$ and $\delta u_3 = \bar{u}_3 - u_3$ in the obtained equations, they give

$\langle \delta y_{1t}, v_1\rangle + (\nabla\delta y_1,\nabla v_1) + (\delta y_1,v_1) - (\delta y_2,v_1) - (\delta y_3,v_1) = (\delta u_1,v_1)_\Gamma,$ (50.a)

$(\delta y_1(0), v_1) = 0,$ (50.b)

$\langle \delta y_{2t}, v_2\rangle + (\nabla\delta y_2,\nabla v_2) + (\delta y_2,v_2) + (\delta y_3,v_2) + (\delta y_1,v_2) = (\delta u_2,v_2)_\Gamma,$ (51.a)

$(\delta y_2(0), v_2) = 0,$ (51.b)

$\langle \delta y_{3t}, v_3\rangle + (\nabla\delta y_3,\nabla v_3) + (\delta y_3,v_3) + (\delta y_1,v_3) - (\delta y_2,v_3) = (\delta u_3,v_3)_\Gamma,$ (52.a)

$(\delta y_3(0), v_3) = 0,$ (52.b)

substituting $v_1 = \delta y_1$, $v_2 = \delta y_2$ & $v_3 = \delta y_3$ in (3.50), (3.51) & (3.52) respectively, adding the obtained equations, using Lemma(1.2) in[11]. They give

$\frac{1}{2}\frac{d}{dt}\|\delta\vec{y}\|_0^2 + \|\delta\vec{y}\|_1^2 = (\delta u_1,\delta y_1)_\Gamma + (\delta u_2,\delta y_2)_\Gamma + (\delta u_3,\delta y_3)_\Gamma\,,$ (53)

Since the 2nd term of (53) is positive, integrating w.r.t. t from 0 to t, and then using the Cauchy Schwartz inequality (CSI), it becomes

$\int_0^t \frac{d}{dt}\|\delta\vec{y}\|_0^2 dt \le \|\delta u_1\|_\Sigma^2 + \int_0^t \|\delta y_1\|_\Gamma^2\, dt + \|\delta u_2\|_\Sigma^2 + \int_0^t \|\delta y_2\|_\Gamma^2\, dt + \|\delta u_3\|_\Sigma^2 + \int_0^t \|\delta y_3\|_\Gamma^2\, dt,$

which gives by using the Trace Theorem [17].

$\|\delta\vec{y}(t)\|_0^2 \le \|\delta\vec{u}\|_{\Sigma}^2 + c^2\int_0^t \|\delta\vec{y}\|_0^2\, dt, \forall\; t \in [0,T]$ using the BGI, one gets

$\|\delta\vec{y}(t)\|_0^2 \le e^{Tc^2}\|\delta\vec{u}\|_{\Sigma}^2 = L^2\|\delta\vec{u}\|_{\Sigma}^2\,,\; e^{Tc^2} = L^2\,, L > 0$

or $\|\delta\vec{y}\|_{L^\infty(\bar{I},L^2(\Omega))} \le L\|\delta\vec{u}\|_{\Sigma}$

Since $\|\delta\vec{y}\|^2_{L^2(Q)} \le TL^2\|\delta\vec{u}\|^2_{\Sigma}$, thus $\|\delta\vec{y}\|_{L^2(Q)} \le \bar{L}\|\delta\vec{u}\|_{\Sigma}$, $TL^2 = \bar{L}^2$

Using a similar way which is used in the above steps, gives

$\int_0^T \frac{d}{dt}\|\delta\vec{y}\|_0^2 + 2\int_0^T \|\delta\vec{y}\|_1^2 \, dt \le \|\delta\vec{u}\|^2_{\Sigma} + c^2\int_0^T \|\delta\vec{y}\|_0^2 \, dt$

$\Rightarrow \|\delta\vec{y}\|^2_{L^2(\bar{I},V)} \le \bar{L}_1^2\|\delta\vec{u}\|^2_{\Sigma}$ where $\bar{L}_1^2 = (1+\bar{L}^2c^2)/2$

or $\|\delta\vec{y}\|_{L^2(\bar{I},V)} \le \bar{L}_1\|\delta\vec{u}\|_{\Sigma}$.

Theorem 4.2: With assumption (A), the operator $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is continuous from $(L^2(\Sigma))^3$ in to $(L^\infty{}_{(\bar{I},L^2(\Omega))})^3$ or in to $(L^2(\bar{I},V))^3$ or in to $(L^2(Q))^3$.

Proof: Let $\delta\vec{u} = \vec{\bar{u}} - \vec{u}$ and $\vec{y} = \vec{\bar{y}} - \vec{y}$, where $\vec{\bar{y}}$ and $\vec{y}$ are the corresponding SVS to the CVS $\vec{\bar{u}}$ and $\vec{u}$ respectively, using the first result in Theorem4.1, we get

$\vec{\bar{y}} \xrightarrow[L^\infty(\bar{I},L^2(\Omega))]{} \vec{y}$ if $\vec{\bar{u}} \xrightarrow[L^2(\Sigma)]{} \vec{u}$,

i.e. the operator $\vec{u} \mapsto \vec{y}_{\vec{u}}$ is Lipschitz continuous (LC) from $L^2(\Sigma)$ in to $L^\infty\left(\bar{I}, L^2(\Omega)\right)$. Easily, one can get this operator is also LC from $L^2(\Sigma)$ into $L^2(Q)$ and into $L^2(\bar{I},V)$.

Lemma 4.1: [**10**]. The norm $\|.\|_0$ is weakly lower semi continuous (W.L.S.C.).

Lemma 4.2: The CoF which is given by (10) is W.L.S.C.

Proof: From Lemma(4.1), we got that the norm $\|\vec{u}\|_{\Sigma}$ is W.L.S.C., $\vec{u}_k \to \vec{u}$ weakly in $L^2(\Sigma)$, then by (Theorem 4.2) $\vec{y}_k \to \vec{y} = \vec{y}_{\vec{u}}$ is weakly in $L^2(\Sigma)$, which gives that the norm $\|\vec{y} - \vec{y}_d\|_{\Sigma}$ is W.L.S.C. (by Lemma 4.1), hence $G_0(\vec{u})$ is W.L.S.C. .

Theorem 4.3: Consider the cost function (10), if $G_0(\vec{u})$is coercive, then there exists a CCBOCV.

Proof: Since $G_0(\vec{u}) \ge 0$ and $G_0(\vec{u})$ is coercive, then there exists a minimizing sequence $\{\vec{u}_k\} = \{(u_{1k}, u_{2k}, u_{3k})\} \in \vec{W}_A$, $\forall k$ such that $\lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}\in\vec{W}_A} G_0(\vec{\bar{u}})$, then $\|\vec{u}_k\|_{\Sigma} \le \hat{C}_1$, $\hat{C}_1 > 0$, then by ATh there exists a subsequence of $\{\vec{u}_k\}$, for simplicity say again $\{\vec{u}_k\}$ s.t. $\vec{u}_k \longrightarrow \vec{u}$ weakly in $\left(L^2(\Sigma)\right)^3$, as $k \to \infty$, from Theorem 3.1, corresponding to the sequence of controls $\{\vec{u}_k\}$, there exists a sequence of solutions $\{\vec{y}_k\}$, but the norms $\|\vec{y}_k\|_{L^\infty(\bar{I},L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ & $\|\vec{y}_k\|_{L^2(\bar{I},V)}$ are bounded, then by ATh there exists a subsequence of $\{\vec{y}_k\}$, for simplicity, say again $\{\vec{y}_k\}$, such that

$\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^\infty\left(\bar{I}, L^2(\Omega)\right)\right)^3$, in $\left(L^2(Q)\right)^3$, and in $\left(L^2(\bar{I},V)\right)^3$,

Suppose that (17.a), (18.a) & (19.a) can be rewritten as

$\langle y_{1kt}, v_1\rangle = -(\nabla y_{1k}, \nabla v_1) - (y_{1k}, v_1) + (y_{2k}, v_1) + (y_{3k}, v_1) + (f_1, v_1) + (u_{1k}, v_1)_\Gamma$,

$\langle y_{2kt}, v_2\rangle = -(\nabla y_{2k}, \nabla v_2) - (y_{2k}, v_2) - (y_{3k}, v_2) - (y_{1k}, v_2) + (f_2, v_2) + (u_{2k}, v_2)_\Gamma$,

$\langle y_{3kt}, v_3\rangle = -(\nabla y_{3k}, \nabla v_3) - (y_{3k}, v_3) - (y_{1k}, v_3) + (y_{2k}, v_3) + (f_1, v_1) + (u_{3k}, v_1)_\Gamma$,

Adding the above three equations and integrating both sides of the obtained equation from 0 to T, taking the absolute value, then using CSI. Finally, using assumption (A), it yields

$$\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right| \le \|\nabla y_{1k}\|_Q\|\nabla v_1\|_Q + \|y_{1k}\|_Q\|v_1\|_Q + \|y_{2k}\|_Q\|v_1\|_Q + \|y_{3k}\|_Q\|v_1\|_Q + \|\nabla y_{2k}\|_Q\|\nabla v_2\|_Q + \|y_{2k}\|_Q\|v_2\|_Q + \|y_{3k}\|_Q\|v_2\|_Q + \|y_{1k}\|_Q\|v_2\|_Q + \|\nabla y_{3k}\|_Q\|\nabla v_3\|_Q + \|y_{3k}\|_Q\|v_3\|_Q + \|y_{1k}\|_Q\|v_3\|_Q + \|y_{2k}\|_Q\|v_3\|_Q + \|\eta_1\|_Q\|v_1\|_Q + \|\eta_2\|_Q\|v_2\|_Q + \|\eta_3\|_Q\|v_3\|_Q + \|u_{1k}\|_\Sigma\|v_1\|_\Sigma + \|u_{2k}\|_\Sigma\|v_2\|_\Sigma + \|u_{3k}\|_\Sigma\|v_3\|_\Sigma .$$

Since for each $i = 1,2,3$, the following inequalities are satisfied

$\|\nabla y_{ik}\|_Q \le \|\nabla\vec{y}_k\|_{\boldsymbol{Q}} \le \|\vec{y}_k\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$, $\|\nabla v_i\|_Q \le \|\nabla\vec{v}\|_{\boldsymbol{Q}} \le \|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$, $\|y_{ik}\|_Q \le \|\vec{y}_k\|_{\boldsymbol{Q}} \le \|\vec{y}_k\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$, $\|v_i\|_Q \le \|\vec{v}\|_{\boldsymbol{Q}} \le \|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$, $\|u_{ik}\|_\Sigma \le \|\vec{u}_k\|_{\boldsymbol{\Sigma}} \le \hat{\mathrm{C}}_1$, $\|v_i\|_\Sigma \le \|\vec{v}\|_{\boldsymbol{\Sigma}} \le \hat{\mathrm{h}}_1\|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$, $\|\eta_i\|_Q \le b_i'$, then

$\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle\,dt\right| \le 12\|\vec{y}_k\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}\|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})} + (b_1' + b_2' + b_3' + 3\hat{\mathrm{C}}_1\hat{\mathrm{h}}_1)\|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$

Or $\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle dt\right| \le \left(12b_2(c) + b'(c)\right)\|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}$

With $\|\vec{y}_k\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})} \le b_2(c)$&$b'(c) = b_1' + b_2' + b_3' + 3\hat{\mathrm{C}}_1\hat{\mathrm{h}}_1$

$\Rightarrow \dfrac{\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle dt\right|}{\|\vec{v}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V})}} \le b_3(c)$, with $b_3(c) = 12b_2(c) + b'(c) \Rightarrow \|\vec{y}_{kt}\|_{\boldsymbol{L^2}(\tilde{\mathrm{I}},\boldsymbol{V^*})} \le b_3(c)$

Since for each k, $\vec{y}_k$ is a solution of the TSEs $(1)-(9)$, then

$$\langle y_{1kt},v_1\rangle + (\nabla y_{1k},\nabla v_1) + (y_{1k},v_1) - (y_{2k},v_1) - (y_{3k},v_1) = (f_1,v_1) + (u_{1k},v_1)_\Gamma\,, \quad (54)$$
$$\langle y_{2kt},v_2\rangle + (\nabla y_{2k},\nabla v_2) + (y_{2k},v_2) + (y_{3k},v_2) + (y_{1k},v_2) = (f_2,v_2) + (u_{2k},v_2)_\Gamma\,, \quad (55)$$
$$\langle y_{3kt},v_3\rangle + (\nabla y_{3k},\nabla v_3) + (y_{3k},v_3) + (y_{1k},v_3) - (y_{2k},v_3) = (f_3,v_3) + (u_{3k},v_3)_\Gamma\,, \quad (56)$$

Let $\varphi_i \in C^1[0,T]$, s.t. $\varphi_i(T) = 0$, $\varphi_i(0) \ne 0$, $\forall i = 1,2,3$, rewriting the 1st terms in the L.H.S. of $(54)-(56)$ multiplying their both sides by $\varphi_i(t)$, $\forall i = 1,2,3$, respectively, integrating both sides w.r.t. t from 0 to T, and integration by parts for the 1st terms in the L.H.S. of each obtained equations, one gets that

$$-\int_0^T(y_{1k},v_1)\varphi_1'(t)dt + \int_0^T[(\nabla y_{1k},\nabla v_1) + (y_{1k},v_1)]\varphi_1(t)dt - \int_0^T(y_{2k},v_1)\,\varphi_1(t)dt - \int_0^T(y_{3k},v_1)\,\varphi_1(t)dt = \int_0^T(f_1,v_1)\varphi_1(t)dt + \int_0^T(u_{1k},v_1)_\Gamma\varphi_1(t)dt + (y_{1k}(0),v_1)\varphi_1(0), \quad (57)$$
$$-\int_0^T(y_{2k},v_2)\varphi_2'(t)dt + \int_0^T[(\nabla y_{2k},\nabla v_2) + (y_{2k},v_2)]\varphi_2(t)dt + \int_0^T(y_{3k},v_2)\,\varphi_2(t)dt + \int_0^T(y_{1k},v_2)\,\varphi_2(t)dt = \int_0^T(f_2,v_2)\varphi_2(t)dt + \int_0^T(u_{2k},v_2)_\Gamma\varphi_2(t)dt + (y_{2k}(0),v_2)\varphi_2(0), \quad (58)$$
$$-\int_0^T(y_{3k},v_3)\varphi_3'(t)dt + \int_0^T[(\nabla y_{3k},\nabla v_3) + (y_{3k},v_3)]\varphi_3(t)dt + \int_0^T(y_{1k},v_3)\,\varphi_3(t)dt - \int_0^T(y_{2k},v_3)\,\varphi_3(t)dt = \int_0^T(f_3,v_3)\varphi_3(t)dt + \int_0^T(u_{3k},v_3)_\Gamma\varphi_3(t)dt + (y_{3k}(0),v_3)\varphi_3(0), \quad (59)$$

Since $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(Q)\right)^3$ and in $\left(L^2(\tilde{\mathrm{I}},V)\right)^3$, then the following convergences are held

$$-\int_0^T(y_{1k},v_1)\varphi_1'(t)dt + \int_0^T[(\nabla y_{1k},\nabla v_1) + (y_{1k},v_1)]\varphi_1(t)dt - \int_0^T(y_{2k},v_1)\,\varphi_1(t)dt - \int_0^T(y_{3k},v_1)\,\varphi_1(t)dt \longrightarrow -\int_0^T(y_1,v_1)\varphi_1'(t)dt + \int_0^T[(\nabla y_1,\nabla v_1) + (y_1,v_1)]\varphi_1(t)dt - \int_0^T(y_2,v_1)\,\varphi_1(t)dt - \int_0^T(y_3,v_1)\,\varphi_1(t)dt\,, \quad (60)$$
$$-\int_0^T(y_{2k},v_2)\varphi_2'(t)dt + \int_0^T[(\nabla y_{2k},\nabla v_2) + (y_{2k},v_2)]\varphi_2(t)dt + \int_0^T(y_{3k},v_2)\,\varphi_2(t)dt + \int_0^T(y_{1k},v_2)\,\varphi_2(t)dt \longrightarrow -\int_0^T(y_2,v_2)\varphi_2'(t)dt + \int_0^T[(\nabla y_2,\nabla v_2) + (y_2,v_2)]\varphi_2(t)dt + \int_0^T(y_3,v_2)\,\varphi_2(t)dt + \int_0^T(y_1,v_2)\,\varphi_2(t)dt, \quad (61)$$
$$-\int_0^T(y_{3k},v_3)\varphi_3'(t)dt + \int_0^T[(\nabla y_{3k},\nabla v_3) + (y_{3k},v_3)]\varphi_3(t)dt + \int_0^T(y_{1k},v_3)\,\varphi_3(t)dt - \int_0^T(y_{2k},v_3)\,\varphi_3(t)dt \longrightarrow -\int_0^T(y_3,v_3)\varphi_3'(t)dt + \int_0^T[(\nabla y_3,\nabla v_3) + (y_3,v_3)]\varphi_3(t)dt + \int_0^T(y_1,v_3)\,\varphi_3(t)dt - \int_0^T(y_2,v_3)\,\varphi_3(t)dt\,, \quad (62)$$

Since $y_{ik}(0)$ is bounded in $L^2(\Omega)$ $\forall\, i = 1,2,3$, then

$$(y_{1k}^0,v_1)\varphi_1(0) \longrightarrow (y_1^0,v_1)\varphi_1(0), \quad (63)$$
$$(y_{2k}^0,v_2)\varphi_2(0) \longrightarrow (y_2^0,v_2)\varphi_2(0), \quad (64)$$
$$(y_{3k}^0,v_3)\varphi_3(0) \longrightarrow (y_3^0,v_3)\varphi_3(0), \quad (65)$$

and since $\vec{u}_k \longrightarrow \vec{u}$ weakly in $(L^2(\Sigma))^3$, then

$\int_0^T (f_1, v_1)dt + \int_0^T (u_{1k}, v_1)_\Gamma dt \to \int_0^T (f_1, v_1)dt + \int_0^T (u_1, v_1)_\Gamma dt$, (66)

$\int_0^T (f_2, v_2)dt + \int_0^T (u_{2k}, v_2)_\Gamma dt \to \int_0^T (f_2, v_2)dt + \int_0^T (u_2, v_2)_\Gamma dt$, (67)

$\int_0^T (f_3, v_3)dt + \int_0^T (u_{3k}, v_3)_\Gamma dt \to \int_0^T (f_3, v_3)dt + \int_0^T (u_3, v_3)_\Gamma dt$, (68)

Finally, using (60) −(62),(63) − (65),(66) − (68) in (57) − (59), one gets

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\,\varphi_1(t)dt -$
$\int_0^T (y_3, v_1)\,\varphi_1(t)dt = \int_0^T (f_1, v_1)\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0),$ (69)

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\,\varphi_2(t)dt +$
$\int_0^T (y_1, v_2)\,\varphi_2(t)dt = \int_0^T (f_2, v_2)\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0),$ (70)

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\,\varphi_3(t)dt -$
$\int_0^T (y_2, v_3)\,\varphi_3(t)dt = \int_0^T (f_3, v_3)\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0),$ (71)

Case1: We choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall i = 1,2,3$, now by using integration by parts for the 1^{st} terms in the L.H.S. of (69−71), one gets

$\int_0^T \langle y_{1t}, v_1\rangle\varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)\,dt -$
$\int_0^T (y_3, v_1)\varphi_1(t)\,dt = \int_0^T (f_1, v_1)\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt$, $\forall v_1 \in V, \forall \varphi_1 \in D[0,T]$ (72)

$\int_0^T \langle y_{2t}, v_2\rangle\varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\varphi_2(t)\,dt +$
$\int_0^T (y_1, v_2)\varphi_2(t)\,dt = \int_0^T (f_2, v_2)\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt$, $\forall v_2 \in V, \forall \varphi_2 \in D[0,T]$ (73)

$\int_0^T \langle y_{3t}, v_3\rangle\varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\varphi_3(t)\,dt -$
$\int_0^T (y_2, v_3)\varphi_3(t)\,dt = \int_0^T (f_3, v_3)\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt, \forall v_3 \in V, \forall \varphi_3 \in D[0,T]$ (74)

Then

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) - (y_3, v_1) = (f_1, v_1) + (u_1, v_1)_\Gamma,$
$\forall v_1 \in V$, a.e. on $\tilde{I}$

$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_3, v_2) + (y_1, v_2) = (f_2, v_2) + (u_2, v_2)_\Gamma,$
$\forall v_2 \in V$, a.e. on $\tilde{I}$

$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) + (y_3, v_3) + (y_1, v_3) - (y_2, v_3) = (f_3, v_3) + (u_3, v_3)_\Gamma,$
$\forall v_3 \in V$, a.e. on $\tilde{I}$

i.e.$\vec{y}$ is satisfied the wf of the TSEs .

Case2: We choose $\varphi_i \in C^1[\tilde{I}]$, s.t. $\varphi_i(T) = 0$ & $\varphi_i(0) \neq 0$, $\forall i = 1,2,3$, by using integration by parts for the 1^{st} terms in the L.H.S. of (72−74), one has

$-\int_0^T (y_1, v_1)\varphi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\varphi_1(t)dt - \int_0^T (y_2, v_1)\,\varphi_1(t)dt -$
$\int_0^T (y_3, v_1)\,\varphi_1(t)dt = \int_0^T (f_1, v_1)\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0),$ (75)

$-\int_0^T (y_2, v_2)\varphi_2'(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\varphi_2(t)dt + \int_0^T (y_3, v_2)\,\varphi_2(t)dt +$
$\int_0^T (y_1, v_2)\,\varphi_2(t)dt = \int_0^T (f_2, v_2)\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2(0), v_2)\varphi_2(0),$ (76)

$-\int_0^T (y_3, v_3)\varphi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\varphi_3(t)dt + \int_0^T (y_1, v_3)\,\varphi_3(t)dt -$
$\int_0^T (y_2, v_3)\,\varphi_3(t)dt = \int_0^T (f_3, v_3)\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y_3(0), v_3)\varphi_3(0),$ (77)

And subtracting (75) − (77) from (69) − (71) respectively, one gets

$(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0)$, $\varphi_1(0) \neq 0, \forall\, \varphi_1 \in D[0,T] \Rightarrow y_1^0 = y_1(0) = y_1^0(x)$,
by the same above way one can show that $y_2^0 = y_2(0) = y_2^0(x)$ and $y_3^0 = y_3(0) = y_3^0(x)$.
Then $\vec{y}$ is a solutions of the wf of the TSEs, since $G_0(\vec{u})$ is W.L.S.C. from Lemma 4.2 and $\vec{u}_k \longrightarrow \vec{\vec{u}}$ weakly in $(L^2(\Sigma))^3$, then

$G_0(\vec{u}) \leq \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\vec{u}} \in \vec{W}_A} G_0(\vec{\vec{u}})$

$\Rightarrow G_0(\vec{u}) \leq \inf_{\vec{\vec{u}} \in \vec{W}_A} G_0(\vec{\vec{u}}) = \min_{\vec{\vec{u}} \in \vec{W}_A} G_0(\vec{\vec{u}})$, then $\vec{u}$ is a CCBOCV .

5. NCsThOP:

In order to state the NCcThOP for CCBOCV, the FrD of the CoF is derived and the NCcThOP is proved .

Theorem 5.1: Consider CoF (10), then the TAPEs of the TSEs are given by

$-z_{1t} - \Delta z_1 + z_1 + z_2 + z_3 = (y_1 - y_{1d})$, in Q (78)

$-z_{2t} - \Delta z_2 + z_2 - z_1 - z_3 = (y_2 - y_{2d})$, in Q (79)

$-z_{3t} - \Delta z_3 + z_3 - z_1 + z_2 = (y_3 - y_{3d})$, in Q (80)

$z_1(x,T) = 0$, in Ω (81)

$z_2(x,T) = 0$, in Ω (82)

$z_3(x,T) = 0$, in Ω (83)

$\frac{\partial z_1}{\partial n_a} = \sum_{l=1}^{2} \frac{\partial z_1}{\partial x_l} \cos(n_1, x_l) = 0$, on Σ (84)

$\frac{\partial z_2}{\partial n_b} = \sum_{l=1}^{2} \frac{\partial z_2}{\partial x_l} \cos(n_2, x_l) = 0$, on Σ (85)

$\frac{\partial z_3}{\partial n_c} = \sum_{l=1}^{2} \frac{\partial z_3}{\partial x_l} \cos(n_3, x_l) = 0$, on Σ (86)

Then $(u_1, u_2, u_3) \in \vec{W}_A$ and the FrD of the CoF is given by $\left(G_0'(\vec{u}), \delta\vec{u}\right)_{\Sigma} = (\vec{z} + \beta\vec{u}, \delta\vec{u})_{\Sigma}$

Proof: The wf of (78) − (86) for $v_i \in V_i$, $\forall\, i = 1,2,3$, is given by

$-\langle z_{1t}, v_1\rangle + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) + (z_3, v_1) = (y_1 - y_{1d}, v_1)$, (87)

$-\langle z_{2t}, v_2\rangle + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) - (z_3, v_2) = (y_2 - y_{2d}, v_2)$, (88)

$-\langle z_{3t}, v_3\rangle + (\nabla z_3, \nabla v_3) + (z_3, v_3) - (z_1, v_3) + (z_2, v_3) = (y_3 - y_{3d}, v_3)$, (89)

The existence of a unique solution of (87−89) can be proved by the same manner which is used in the proof of Theorem 3.1.

Now substituting $v_1 = z_1, v_2 = z_2$ and $v_3 = z_3$ in (50.a), (51.a) and (52.a) respectively, to get

$\langle \delta y_{1t}, z_1\rangle + (\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) - (\delta y_3, z_1) = (\delta u_1, z_1)_\Gamma$, (90.a)

$\langle \delta y_{2t}, z_2\rangle + (\nabla\delta y_2, \nabla z_2) + (\delta y_2, z_2) + (\delta y_3, z_2) + (\delta y_1, z_2) = (\delta u_2, z_2)_\Gamma$, (90.b)

$\langle \delta y_{3t}, z_3\rangle + (\nabla\delta y_3, \nabla z_3) + (\delta y_3, z_3) + (\delta y_1, z_3) - (\delta y_2, z_3) = (\delta u_3, z_3)_\Gamma$, (90.c)

Also, substituting $v_1 = \delta y_1, v_2 = \delta y_2$ and $v_3 = \delta y_3$ in (87),(88)&(89) respectively, to get

$-\langle z_{1t}, \delta y_1\rangle + (\nabla z_1, \nabla\delta y_1) + (z_1, \delta y_1) + (z_2, \delta y_1) + (z_3, \delta y_1) = (y_1 - y_{1d}, \delta y_1)$, (91.a)

$-\langle z_{2t}, \delta y_2\rangle + (\nabla z_2, \nabla\delta y_2) + (z_2, \delta y_2) - (z_1, \delta y_2) - (z_3, \delta y_2) = (y_2 - y_{2d}, \delta y_2)$, (91.b)

$-\langle z_{3t}, \delta y_3\rangle + (\nabla z_3, \nabla\delta y_3) + (z_3, \delta y_3) - (z_1, \delta y_3) + (z_2, \delta y_3) = (y_3 - y_{3d}, \delta y_3)$, (91.c)

Integrating both sides of equations (90.a,b & c) and (91.a,b & c), w.r.t. t from 0 to T, using integration by parts for the 1st terms of the L.H.S. of each of the obtained equations from (91.a), (91.b) & (91.c), then subtracting each one of the obtained equations from its corresponding equation (90.a,b & c), and adding all the obtained equations, give

$\int_0^T [(\delta u_1, z_1)_\Gamma + (\delta u_2, z_2)_\Gamma + (\delta u_3, z_3)_\Gamma]\ dt = \int_0^T [(y_1 - y_{1d}, \delta y_1) + (y_2 - y_{2d}, \delta y_2) +$

$(y_3 - y_{3d}, \delta y_3)]\ dt,$ (92)

Now, adding together each of the pair of equations (11.a&50.a), (12.a&51.a) and (13.a&52.a), one has

$\langle (y_1 + \delta y_1)_t, v_1\rangle + (\nabla(y_1 + \delta y_1), \nabla v_1) + (y_1 + \delta y_1, v_1) - (y_2 + \delta y_2, v_1) - (y_3 + \delta y_3, v_1)$

$= (f_1, v_1) + (u_1 + \delta u_1, v_1)_\Gamma$, (93)

$\langle (y_2 + \delta y_2)_t, v_2\rangle + (\nabla(y_2 + \delta y_2), \nabla v_2) + (y_2 + \delta y_2, v_2) + (y_3 + \delta y_3, v_2) + (y_1 + \delta y_1, v_2)$

$= (f_2, v_2) + (u_2 + \delta u_2, v_2)_\Gamma$, (94)

$\langle (y_3+\delta y_3)_t, v_3\rangle + (\nabla(y_3+\delta y_3), \nabla v_3) + (y_3+\delta y_3, v_3) + (y_1+\delta y_1, v_3) - (y_2+\delta y_2, v_3)$

$= (f_3, v_3) + (u_3+\delta u_3, v_3)_\Gamma,$ (95)

Which means the CV $(u_1+\delta u_1, u_2+\delta u_2, u_3+\delta u_3)$ gives that the solution $(y_1+\delta y_1, y_2+\delta y_2, y_3+\delta y_3)$ of (93) – (95).

On the other hand, from (92) and the CoF, we have

$$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = (\delta u_1, z_1)_\Sigma + (\delta u_2, z_2)_\Sigma + (\delta u_3, z_3)_\Sigma + (\beta u_1, \delta u_1)_\Sigma + (\beta u_2, \delta u_2)_\Sigma + (\beta u_3, \delta u_3)_\Sigma + \tfrac{1}{2}\|\delta\vec{y}\|_Q^2 + \tfrac{\beta}{2}\|\delta\vec{u}\|_\Sigma^2 = (\delta\vec{u}, \vec{z})_\Sigma + (\beta\vec{u}, \delta\vec{u})_\Sigma + \tfrac{1}{2}\|\delta\vec{y}\|_Q^2 + \tfrac{\beta}{2}\|\delta\vec{u}\|_\Sigma^2$$

Or

$$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = (\vec{z}+\beta\vec{u}, \delta\vec{u})_{\boldsymbol{\Sigma}} + \tfrac{1}{2}\|\delta\vec{y}\|_{\boldsymbol{Q}}^2 + \tfrac{\beta}{2}\|\delta\vec{u}\|_{\boldsymbol{\Sigma}}^2 .$$

From Theorem 4.1, we have

$\frac{1}{2}\|\delta\vec{y}\|_{\boldsymbol{Q}}^2 = \varepsilon_1(\delta\vec{u})\|\delta\vec{u}\|_{\boldsymbol{\Sigma}}$ and $\frac{\beta}{2}\|\delta\vec{u}\|_{\boldsymbol{\Sigma}}^2 = \varepsilon_2(\delta\vec{u})\|\delta\vec{u}\|_{\boldsymbol{\Sigma}}$

With $\varepsilon_1(\delta\vec{u}) = \frac{1}{2}\bar{M}^2\|\delta\vec{u}\|_{\boldsymbol{\Sigma}}$, where $\varepsilon_1(\delta\vec{u}), \varepsilon_2(\delta\vec{u}) \longrightarrow 0$ as $\|\delta\vec{u}\|_{\boldsymbol{\Sigma}} \longrightarrow 0$

Then

$$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = (\vec{z}+\beta\vec{u}, \delta\vec{u})_{\boldsymbol{\Sigma}} + \varepsilon(\delta\vec{u})\|\delta\vec{u}\|_{\boldsymbol{\Sigma}}$$

Where $\varepsilon_1(\delta\vec{u}) + \varepsilon_2(\delta\vec{u}) = \varepsilon(\delta\vec{u}) \longrightarrow 0$ as $\|\delta\vec{u}\|_\Sigma \longrightarrow 0$

Using the definition of FrD of G_0, one has

$$\left(G_0'(\vec{u}), \delta\vec{u}\right)_{\boldsymbol{\Sigma}} = (\vec{z}+\beta\vec{u}, \delta\vec{u})_{\boldsymbol{\Sigma}} .$$

Theorem 5.2: The NCsThOP for the CCBOCV of the above problem is $G_0'(\vec{u}) = \vec{z}+\beta\vec{u} = 0$ with $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{z} = \vec{z}_{\vec{u}}$.

Proof: If $\vec{u}$ is an CCBOCV of the problem, then

$$G_0(\vec{\vec{u}}) = \min_{\vec{u}\in\vec{W}_A} G_0(\vec{u}) \quad \forall\vec{u} \in \left(L^2(\Sigma)\right)^3,$$

i.e. $G_0'(\vec{\vec{u}}) = 0 \Rightarrow \quad \vec{z}+\beta\vec{\vec{u}} = 0$

From Theorem 5.1 $\left(\vec{z}+\beta\vec{\vec{u}}, \delta\vec{u}\right)_\Sigma \geq 0$ with $\delta\vec{u} = \vec{w} - \vec{\vec{u}}$

$\Rightarrow \left(\vec{z}+\beta\vec{\vec{u}}, \vec{w}\right)_{\boldsymbol{\Sigma}} \geq \left(\vec{z}+\beta\vec{\vec{u}}, \vec{\vec{u}}\right)_{\boldsymbol{\Sigma}}, \ \forall\,\vec{w} \in \left(L^2(\Sigma)\right)^3.$

6. Conclusions

In this paper, The GM is employed to prove the existence theorem of a unique solution for a SVS of the TLPDEPAR when the CCBOCV is fixed. The existence theorem of a CCBOCV governed by the TLPDEPAR is developed and proved. The existence and uniqueness of a solution for the TAPEs associated with the TLPDEPAR is studied, the FrD for the CoF is obtained. At the end, the NCsThOP of the CCBOCV problem is stated and proved.

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