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Nearly Primary-2-Absorbing Submodules and other Related Concepts

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Abstract

Our aim in this paper is to introduce the notation of a nearly primary-2-absorbing submodule as generalization of 2-absorbing submodule where a proper submodule K of an R-module C is called nearly primary-2-absorbing submodule if whenever $abx \in K$, for $a, b, \in R$, $x \in C$ implies that either $ax \in rad_{C}(K) + J(C)$ or $bx \in rad_{C}(K) + J(C)$ or $ab \in [K + J(C):_{R} C]$. We got many basic properties, examples and characterizations of this concept. Furthermore, characterizations of nearly primary-2-absorbing submodules in some classes of modules are inserted. Moreover, the behavior of nearly primary-2-absorbing submodule under R-epimorphism is studied.

Keywords: Primary submodules, prime submodules, multiplication modules, Artirian ring, projective R-modules , Jacobin of a modules .

1. Introduction

Throughout this paper, all rings are commutative with identity and we assume all R-modules are left unitary. Prime submodulse are among the most famous concepts of modules theory, where a proper submodule H of an R-module C is said to be a prime submodule if whenever $ax \in H$, where $a \in R$, $x \in C$ implies that either $x \in H$ or $aC \subseteq H$[1]. In addition, primary submodules is introduced in [2] as a generalization of a prime submodule, where a proper submodule H of an R-module C is said to be a primary submodule if whenever $ax \in H$, for $a \in R$, $x \in C$ implies that either $x \in H$ or $a^{n}C \subseteq H$ for some $n \in Z^{+}$. The well-known generalization of prime submodules is the concept of 2-absorbing submodule, where a proper submodule H of an R-module $\complement$ is called 2-absorbing submodule if whenever $abx \in H$, for $a,b \in R$, $x \in \complement$ implies that either $ax \in H$ or $bx \in H$ or $ab\complement \subseteq H$[3] is studied extensively. There are many generalizations of the concept of 2-absorbing, for example see [4 ... 6]. Dubey in [7] introduced the concept of 2-absorbing primary submodule, where a proper submodule H of an R-module $\complement$ is called 2-absorbing primary submodule if whenever $abx \in H$, for $a,b \in R$, $x \in \complement$ implies that either $ax \in rad_{\complement}(H)$ or $bx \in rad_{\complement}(H)$ or $ab \in [H:_R \complement]$, where $rad_{\complement}(H)$ is the intersection of all prime submodule of $\complement$ containing H. Recall that an R-module $\complement$ is multiplication if every submodule H of $\complement$ is of the form $H = J\complement$ for some ideal J of R [8]. Again recall that an R-module $\complement$ is said to be faithful if $ann_R(\complement) = (0)$. And a ring R is said to be Artinian if R satisfies D.CC on ideals of R, that is $I_1 \supseteq I_2 \supseteq \cdots \supseteq \cdots$, then there exists $n \in Z^+$ such that $I_n = I_m$ for some $n > m$ [11]. Finally a ring R is called a good ring, if $J(R).\complement = J(\complement)$ where $\complement$ is an R-module [12].

2. Nearly Primary-2-Absorbing Submodules

In this paper, we will introduce the concept of nearly primary-2-absorbing submodule and give some basic results of these classes of submodules. And discuss on the relationships with class of 2-absorbing submodules and nearly primary-2-absorbing submodules.

Definition 1

A proper submodule H of an R-module $\complement$ is said to be nearly primary-2-absorbing submodule of $\complement$, if whenever $abx \in H$, for $a, b \in R$, $x \in \complement$ implies that either $ax \in rad_{\complement}(H) + J(\complement)$ or $bx \in rad_{\complement}(H) + J(\complement)$ or $ab \in [H + J(\complement):_R \complement]$. And a proper ideal J of a ring R is called nearly primary-2-absorbing ideal of R, if J is nearly primary-2-absorbing submodules of an R-module R.

Remarks and Examples 2

1. It is clear that every 2-absorbing submodule of an R-module $\complement$ is a nearly primary-2-absorbing submodule, while the reverse does not hold in general, the following example show that: Let $\complement = Z_{16}$, $R = Z$ and $K = \langle \bar{8} \rangle$. K is not 2-absorbing submodule since $2.2.\bar{2} \in K$ where $2 \in R = Z$, $\bar{2} \in Z_{16}$, then $2.\bar{2} = \bar{4} \notin K$ and $2.2 = 4 \notin [K:_R Z_{16}] = 8Z$. But K is a nearly primary-2-absorbing submodule of Z_{16}, since $J(Z_{16}) = \langle \bar{2} \rangle$ and $rad_{\complement}(K) = \langle \bar{2} \rangle$ for all $a, b \in Z$, $x \in Z_{16}$ with $abx \in K = \langle \bar{8} \rangle$ implies that either $ax \in rad_{\complement}(K) + J(Z_{16}) = \langle \bar{2} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $bx \in rad_{\complement}(K) + J(Z_{16}) = \langle \bar{2} \rangle$ or $ab \in [\langle \bar{8} \rangle + J(Z_{16}):_Z Z_{16}] = [\langle \bar{2} \rangle: Z_{16}] = 2Z$. That is $2.2.\bar{2} \in K$, implies that $2.\bar{2} = \bar{4} \in \langle \bar{2} \rangle + \langle \bar{2} \rangle = \langle \bar{2} \rangle$ or $2.2 = 4 \in [\langle \bar{8} \rangle + \langle \bar{2} \rangle:_Z Z_{16}] = 2Z$.
2. It is clear that every prime submodule of an R-module $\complement$ is a nearly primary-2-absorbing submodule, while the reverse does not hold in general, the following example show that: Let $\complement = Z_8$, $R = Z$ and $K = \langle \bar{4} \rangle$. K is not prime submodule of Z_8, since $2.\bar{2} \in K$, where $2 \in Z$, $\bar{2} \in Z_8$ implies that $\bar{2} \notin K$ and $2 \notin [K:_Z \complement] = 4Z$. But K is a nearly primary-2-absorbing submodule of $\complement$, since $2.2.\bar{1} \in K$, where $2 \in Z$, $\bar{1} \in Z_8$ 2 implies that $2.\bar{1} \in rad_{\complement}(K) + J(Z_8) = \langle 2 \rangle$ or $2.2 = 4 \in [K + J(\complement):_Z \complement] = 2Z$.
3. It is clear that every primary submodule of an R-module $\complement$ is a nearly primary-2-absorbing submodule, while the reverse does not hold in general. The following example show that: Let $\complement = Z_6$, $R = Z$ and $K = \langle \bar{0} \rangle$ is a submodule of $\complement$. K is a nearly primary-2-absorbing submodule of $\complement$ but not primary submodule, since $3 \in Z$, $\bar{2} \in Z_6$ such that $3.\bar{2} \in K$, but

$\bar{2} \notin K = \langle \bar{0} \rangle$ and $3 \notin \sqrt{[\langle \bar{0} \rangle : Z_6]} = \sqrt{6Z} = 6Z$. But K is a nearly primary-2-absorbing submodule of $\mathrm{C} = Z_6$. For all $a,b \in R, x \in \mathrm{C}$, with $abx \in K$ implies that either $ax \in rad_{\mathrm{C}}(K) + J(\mathrm{C}) = \langle \bar{0} \rangle + \langle 0 \rangle = \langle \bar{0} \rangle$ or $bx \in rad_{\mathrm{C}}(K) + J(\mathrm{C}) = \langle \bar{0} \rangle + \langle 0 \rangle = \langle \bar{0} \rangle$ or $ab \in [\langle \bar{0} \rangle + \langle \bar{0} \rangle :_Z Z_6] = 6Z$. That is $2.3.\bar{1} \in K$, where $2,3 \in R$, $\bar{1} \in Z_6$ implies that $2.\bar{1} = \bar{2} \notin \langle \bar{0} \rangle$ but $2.3 = 6 \in [\langle \bar{0} \rangle + \langle \bar{0} \rangle :_Z Z_6] = 6Z$.

The following results are characterizations of a nearly primary-2-absorbing submodules.

Proposition 3

Let C be an R-module and K is a proper submodule of C. Then K is a nearly primary-2-absorbing submodule of C if and only if for each $r, s \in R$ with $rs \notin [K + J(\mathrm{C}) :_R \mathrm{C}]$, $[K :_{\mathrm{C}} rs] \subseteq [rad_{\mathrm{C}}(K) :_{\mathrm{C}} r] \cup [rad_{\mathrm{C}}(K) :_{\mathrm{C}} s]$.

Proof:

($\Rightarrow$) Let $x \in [K :_{\mathrm{C}} rs]$, where $r, s \in R$ and $rs \notin [K + J(\mathrm{C}) :_R \mathrm{C}]$, implies that $rsx \in K$. But K is a nearly primary-2-absorbing submodule of C, and $rs \notin [K + J(\mathrm{C}) :_R \mathrm{C}]$, then $rx \in rad_{\mathrm{C}}(K) + J(\mathrm{C})$ or $sx \in rad_{\mathrm{C}}(K) + J(\mathrm{C})$. That is either $x \in [rad_{\mathrm{C}}(K) + J(\mathrm{C}) :_{\mathrm{C}} r]$ or $x \in [rad_{\mathrm{C}}(K) + J(\mathrm{C}) :_{\mathrm{C}} s]$, thus $x \in [rad_{\mathrm{C}}(K) + J(\mathrm{C}) :_{\mathrm{C}} r] \cup [rad_{\mathrm{C}}(K) + sJ(\mathrm{C}) :_{\mathrm{C}} s]$.
Hence, $[K :_{\mathrm{C}} rs] \subseteq [rad_{\mathrm{C}}(K) :_{\mathrm{C}} r] \cup [rad_{\mathrm{C}}(K) :_{\mathrm{C}} s]$.
($\Leftarrow$) Let $rsx \in K$, where $x \in \mathrm{C}$ and $r, s \in R$ with $rs \notin [K + J(\mathrm{C}) :_R \mathrm{C}]$. It follows that $x \in [K :_{\mathrm{C}} rs]$, by hypothesis $x \in [rad_{\mathrm{C}}(K) :_{\mathrm{C}} r] \cup [rad_{\mathrm{C}}(K) :_{\mathrm{C}} s]$. Hence $x \in [rad_{\mathrm{C}}(K) :_{\mathrm{C}} r]$ or $x \in [rad_{\mathrm{C}}(K) :_{\mathrm{C}} s]$. Therefore $rx \in rad_{\mathrm{C}}(K) + J(\mathrm{C})$ or $sx \in rad_{\mathrm{C}}(K) + J(\mathrm{C})$, that is K is a nearly primary-2-absorbing submodule of C.

Proposition 4

Let C be an R-module and L be a proper submodule of C. Then L is a nearly primary-2-absorbing submodule of C if and only if $abK \subseteq L$ for $a, b \in R$ and K is a submodule of C, with $ab \notin [L + J(\mathrm{C}) :_R \mathrm{C}]$, implies that $aK \subseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$ or $bK \subseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$.

Proof:

($\Rightarrow$) Let L be a nearly primary-2-absorbing submodule of C, $abK \subseteq L$ with $r, s \in R$ and K is a submodule of C with $ab \notin [L + J(\mathrm{C}) :_R \mathrm{C}]$. Assume that $aK \nsubseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$ and $bK \nsubseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$, then $ak_1 \notin rad_{\mathrm{C}}(L) + J(\mathrm{C})$ and $bk_2 \notin rad_{\mathrm{C}}(L) + J(\mathrm{C})$ for some $k_1, k_2 \in K$. Now we have $abk_1 \in L$ but L is a nearly primary-2-absorbing submodule of C and $ab \notin [L + J(\mathrm{C}) :_R \mathrm{C}]$ and $ak_1 \notin rad_{\mathrm{C}}(L) + J(\mathrm{C})$, then $bk_1 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$. Also, since $abk_2 \in L$ and $ab \notin [L + J(\mathrm{C}) :_R \mathrm{C}]$ and $bk_2 \notin rad_{\mathrm{C}}(L) + J(\mathrm{C})$, then $ak_2 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$. Again since $ab(k_1 + k_2) \in L$ and $ab \notin [L + J(\mathrm{C}) :_R \mathrm{C}]$ we have $a(k_1 + k_2) \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$ or $b(k_1 + k_2) \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$. Suppose that $a(k_1 + k_2) = ak_1 + ak_2 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$, but $ak_2 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$, it follows that $ak_1 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$ a contradiction. Suppose that $b(k_1 + k_2) = bk_1 + bk_2 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$, but $bk_1 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$, we have $bk_2 \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$ a contradiction. Hence $aK \subseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$ or $bK \subseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$.
($\Leftarrow$) Let $abx \in L$, where $x \in \mathrm{C}$ and $a, b \in R$ with $ab \notin [L + J(\mathrm{C}) :_R \mathrm{C}]$. So that $ab(x) \subseteq L$, it follows by hypothesis $a(x) \subseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$ or $b(x) \subseteq rad_{\mathrm{C}}(L) + J(\mathrm{C})$. That is $ax \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$ or $bx \in rad_{\mathrm{C}}(L) + J(\mathrm{C})$. Hence L is a nearly primary-2-absorbing submodule of C.

Proposition 5

Let C be an R-module and L be a proper submodule of C. Then L is a nearly primary-2-absorbing submodule of C if and only if $IJK \subseteq L$, where I, J are ideals of R and K is a submodule of C, implies that either $IJ \subseteq [L + J(C) :_R C]$ or $IK \subseteq rad_C(L) + J(C)$ or $JK \subseteq rad_C(L) + J(C)$.

Proof:

($\Leftarrow$) Clear.

($\Rightarrow$) Assume that L is a nearly primary-2-absorbing submodule of C, and $IJK \subseteq L$, where I, J are ideals of R and K is a submodule of C and $IJ \nsubseteq [L + J(C) :_R C]$. We must prove that $IK \subseteq rad_C(L) + J(C)$ or $JK \subseteq rad_C(L) + J(C)$. Suppose that $IK \nsubseteq rad_C(L) + J(C)$and $JK \nsubseteq rad_C(L) + J(C)$. It follows that there exists $r_1 \in I$ and $r_2 \in J$ such that $r_1K \nsubseteq rad_C(L) + J(C)$ and $r_2K \nsubseteq rad_C(L) + J(C)$. Now $r_1r_2K \subseteq L$ with $r_1K \nsubseteq rad_C(L) + J(C)$ and $r_2K \nsubseteq rad_C(L) + J(C)$ and L is a nearly primary-2-absorbing submodule of C, impling that by Proposition(4) $r_1r_2 \in [L + J(C) :_R C]$. Since $IJ \nsubseteq [L + J(C) :_R C]$. It follows that there exists $s_1 \in I$, $s_2 \in J$ such that $s_1s_2 \notin [L + J(C) :_R C]$. Since $s_1s_2K \subseteq L$, and $s_1s_2 \notin [L + J(C) :_R C]$, we have by Proposition(4) either $s_1K \subseteq rad_C(L) + J(C)$ or $s_2K \subseteq rad_C(L) + J(C)$.

Now, we discuss the following cases:

Case one: Suppose that $s_1K \subseteq rad_C(L) + J(C)$ but $s_2K \nsubseteq rad_C(L) + J(C)$ Since $r_1s_2K \subseteq L$ and L is a nearly primary-2-absorbing submodule of C with $s_2K \nsubseteq rad_C(L) + J(C)$ and $r_1K \nsubseteq rad_C(L) + J(C)$, implies that $r_1s_2 \in [L + J(C) :_R C]$ by Proposition(4). Also since $s_1K \subseteq rad_C(L) + J(C)$ but $r_1K \nsubseteq rad_C(L) + J(C)$, it follows that $(r_1 + s_1)K \nsubseteq rad_C(L) + J(C)$. Since $(r_1 + s_1)s_2K \subseteq L$ and $s_2K \nsubseteq rad_C(L) + J(C)$ and $(r_1 + s_1)K \nsubseteq rad_C(L) + J(C)$ implies that by Proposition(4) $(r_1 + s_1)s_2 \in [L + J(C) :_R C]$. That is $(r_1 + s_1)s_2 = r_1s_2 + s_1s_2 \in [L + J(C) :_R C]$ and $r_1s_2 \in [L + J(C) :_R C]$, implies that $s_1s_2 \in [L + J(C) :_R C]$ a contradiction.

Case two: Let it be $s_2K \subseteq rad_C(L) + J(C)$ but $s_1K \nsubseteq rad_C(L) + J(C)$ in similarly steps of Case one we get a contradiction.

Case three: Assume that $s_1K \subseteq rad_C(L) + J(C)$ but $s_2K \subseteq rad_C(L) + J(C)$. Now since $s_2K \subseteq rad_C(L) + J(C)$ and $r_2K \nsubseteq rad_C(L) + J(C)$, it follows that $(r_2 + s_2)K \nsubseteq rad_C(L) + J(C)$. We have $r_1(r_2 + s_2)K \subseteq L$ and $r_1K \nsubseteq rad_C(L) + J(C)$ and $(r_2 + s_2)K \nsubseteq rad_C(L) + J(C)$, by proposition(4) $r_1(r_2 + s_2) = r_1r_2 + r_1s_2 \in [L + J(C) :_R C]$.But $r_1r_2 \in [L + J(C) :_R C]$ and $r_1r_2 + r_1s_2 \in [L + J(C) :_R C]$. It follows that $r_1s_2 \in [L + J(C) :_R C]$. Now, since $s_1K \subseteq rad_C(L) + J(C)$ and $r_1K \nsubseteq rad_C(L) + J(C)$, implies that $(r_1 + s_1)K \nsubseteq rad_C(L) + J(C)$ since $(r_1 + s_1)r_2K \subseteq L$ and $r_2K \nsubseteq rad_C(L) + J(C)$ and $(r_1 + s_1)K \nsubseteq rad_C(L) + J(C)$, it follows that $(r_1 + s_1)r_2 = r_1r_2 + s_1r_2 \in [L + J(C) :_R C]$ by Proposition(4). Now, since $r_1r_2 \in [L + J(C) :_R C]$ and $r_1r_2 + s_1r_2 \in [L + J(C) :_R C]$, implies that $s_1r_2 \in [L + J(C) :_R C]$. Also , since $(r_1 + s_1)(r_2 + s_2)K \subseteq L$ and $(r_1 + s_1)K \nsubseteq rad_C(L) + J(C)$ and $(r_2 + s_2)K \nsubseteq rad_C(L) + J(C)$, it follows that $(r_1 + s_1)(r_2 + s_2) = r_1r_2 + r_1s_2 + s_1r_2 + s_1s_2 \in [L + J(C) :_R C]$ by Proposition(4). Again since $r_1r_2, r_1s_2, s_1r_2 \in [L + J(C) :_R C]$, we get that $s_1s_2 \in [L + J(C) :_R C]$ a contradiction. Thus, we have either $IK \subseteq rad_C(L) + J(C)$ or $JK \subseteq rad_C(L) + J(C)$.

Proposition 6

Let H be a proper submodule of an R-module C, with $rad_C(H)$ is a prime submodule of C. Then H is a nearly primary-2-absorbing submodule of C.

Proof:

Suppose that $abx \in H$, where $a, b \in R$, $x \in C$ and $bx \notin rad_C(H) + J(C)$. Since $H \subseteq rad_C(H)$, then $a(bx) \in rad_C(H)$, but $rad_C(H)$ is a prime submodule of C, then $aC \subseteq rad_C(H) \subseteq rad_C(H) + J(C)$. That is $ax \in rad_C(H) + J(C)$, for some $x \in C$. Thus H is a nearly primary-2-absorbing submodule of C.

Before we introduce the next result, we need to recall the following Lemma.

Lemma 7[10]

If R is a good ring, C is an R-module and N is a submodule of C, then $J(C) \cap N = J(N)$.

Proposition 8

Let R be a good ring, L and K are proper submodules of an R-module C with $L \subsetneq K$ and $J(C) \subseteq K$. If L is a nearly primary-2-absorbing submodule of C, then L is a nearly primary-2-absorbing submodule of K.

Proof:

Let $rsx \in L$, where r, $s \in R$, $x \in K \subseteq C$. Since L is a nearly primary-2-absorbing submodule of C, implies that either $rx \in rad_C(L) + J(C)$ or $sx \in rad_C(L) + J(C)$ or $rsC \subseteq L + J(C)$. That is either $rx \in (rad_C(L) + J(C)) \cap K$ or $sx \in (rad_C(L) + J(C)) \cap K$ or $rsC \subseteq (L + J(C)) \cap K$. But by modular law we have $(rad_C(L) + J(C)) \cap K = (rad_C(L) \cap K) + (J(C) \cap K) = (rad_C(L) \cap K) + J(K)$ by Lemma(7). Thus we have either $rx \in (rad_C(L) \cap K) + J(K) \subseteq rad_C(L) + J(K)$ or $sx \in\in (rad_C(L) \cap K) + J(K) \subseteq rad_C(L) + J(K)$ or $rsC \subseteq (L \cap K) + (J(C) \cap K) = (L \cap K) + J(K) \subseteq L + J(K)$. Hence L is a nearly primary-2-absorbing submodule of K.

Recall that for any submodules L, K of a multiplication R-module C with $L = IC$, $K = JC$ for some ideals I and J of R. The product $LK = IC.JC = IJC$. That is $LK = IK$, in particular $LC = ICC = IC = L$. Also for any $x \in C$ we have $Lx = Ix$[13].

The following result gives a characterization of nearly primary-2-absorbing submodules in class of multiplication modules.

Proposition 9

Let C be a multiplication R-module and L is a proper submodule of C. Then L is a nearly primary-2-absorbing submodule of C if and only if, whenever $L_1L_2L_3 \subseteq L$ for L_1, L_2, L_3 are submodules of C, impling that either $L_1L_3 \subseteq rad_C(L) + J(C)$ or $L_2L_3 \subseteq rad_C(L) + J(C)$ or $L_1L_2W \subseteq L + J(C)$.

Proof:

(⇒) Let L be a nearly primary-2-absorbing submodule of C and $L_1L_2L_3 \subseteq L$ for L_1, L_2, L_3 are submodules of C, with $L_1L_2C \nsubseteq L + J(C)$. Since C is a multiplication, then $L_1 = I_1C$ and $L_2 = I_2C$ for some ideals I_1, I_2, I_3 of R. Clearly $I_1I_2L_3 \subseteq L$ and $I_1I_2 \nsubseteq [L + J(C):_R C]$. Since L is a nearly primary-2-absorbing submodule of C, implies that either $I_1L_3 \subseteq rad_C(L) + J(C)$ or $I_2L_3 \subseteq rad_C(L) + J(C)$, it follows that either $L_1L_3 \subseteq rad_C(L) + J(C)$ or $L_2L_3 \subseteq rad_C(L) + J(C)$

(⇐) Assume that $I_1I_2K \subseteq L$, where I_1, I_2 are ideals of R, and K is a submodule of C. Since C is a multiplication, then $I_1I_2K = L_1L_2K \subseteq L$, by hypothesis either $L_1K \subseteq rad_C(L) + J(C)$ or $L_2K \subseteq rad_C(L) + J(C)$ or $L_1L_2 \subseteq [L + J(C) :_R C]$. That is either $I_1K \subseteq rad_C(L) + J(C)$or $I_2K \subseteq rad_C(L) + J(C)$ or $I_1I_2 \subseteq [L + J(C) :_R C]$. Then by Proposition (5) L is a nearly primary-2-absorbing submodule of C.

Recall that an R-epimorphism $f: C \longrightarrow \bar{C}$ is called a small epimorphism if $\ker(f)$ is a small submodule of C [10].

Lemma 10 [10, Corollary (9.1.5)]

Let C and $\bar{C}$ be an R-module and N be a proper submodule of C. If $f: C \longrightarrow \bar{C}$ is an R-homomorphism, then $f(J(C)) \subseteq J(\bar{C})$ and $J(N) \subseteq J(C)$. If f is a small epimorphism then $f\big(J(C)\big) = J(\bar{C}) = J(f(C))$ and $J(C) = f^{-1}(J(\bar{C}))$.

Lemma 11 [2]

Let $f: C \longrightarrow \bar{C}$ be an R-epimorphism and L is a submodule of $\bar{C}$ with $\ker(f) \subseteq L$, then $f(rad_C(L))) = rad_{\bar{C}}(f(L))$.

Proposition 12

Let $f: C \longrightarrow \bar{C}$ be a small R-epimorphism and $\bar{K}$ is a nearly primary-2-absorbing submodule of $\bar{C}$. Then $f^{-1}(\bar{K})$ be a nearly primary-2-absorbing submodule of C.

Proof:

Let $rsx \in f^{-1}(\bar{K})$, where $r, s \in R$, $x \in C$, impling that $rsf(x) \in \bar{K}$. Since $\bar{K}$ is a nearly primary-2-absorbing submodule of $\bar{C}$. It follows that either $rf(x) \in rad_{\bar{C}}(\bar{K}) + J(\bar{C})$ or $sf(x) \in rad_{\bar{C}}(\bar{K}) + J(\bar{C})$ or $sr\bar{C} \subseteq \bar{K} + J(\bar{C})$. Thus by Lemma (10) and by Lemma (11), we have either $rx \in f^{-1}(rad_{\bar{C}}(\bar{K}) + f^{-1}\big(J(\bar{C})\big) = rad_C\big(f^{-1}(\bar{K})\big) + J(C)$ or $sx \in f^{-1}(rad_{\bar{C}}(\bar{K}) + f^{-1}\big(J(\bar{C})\big) = rad_C\big(f^{-1}(\bar{K})\big) + J(C)$ or $rsC \subseteq f^{-1}(K) + J(C)$. Hence $f^{-1}(\bar{K})$ is a nearly primary-2-absorbing submodule of C.

Proposition 13

Let $f: C \longrightarrow \bar{C}$ be a small R-epimorphism and K is a nearly primary-2-absorbing submodule of C with $\ker(f) \subseteq K$. Then $f(K)$ is a nearly primary-2-absorbing submodule of $\bar{C}$.

Proof:

Let $rs\bar{x} \in f(K)$, where r, $s \in R$, $\bar{x} \in \bar{C}$. Since f is onto, then $f(x) = \bar{x}$ for some$x \in C$. Thus $rsf(x) \in f(K)$, implies that $rsf(x) = f(k)$ for some $k \in K$, it follows that $f(rsx - k) = o$, implies that $rsx - k \in \ker(f) \subseteq K$, then $rsx \in K$. But K is a nearly primary-2-absorbing submodule of C, then either $rx \in rad_C(K) + J(C)$ or $sx \in rad_C(K) + J(C)$ or $rsC \subseteq K + J(C)$, it follows that by Lemma(11) either $rf(x) \in f\big(rad_C(K)\big) + f\big(J(C)\big) \subseteq rad_{\bar{C}}(f(K)) + f(J(C))$ or $sf(x) \in f\big(rad_C(K)\big) + f\big(J(C)\big) \subseteq rad_{\bar{C}}(f(K)) + f(J(C))$ or $rsf(C) \subseteq f(K) + f\big(J(C)\big) \subseteq f(K) + J(\bar{C})$. Also, by Lemma (12) either $r\bar{x} \in rad_{\bar{C}}(f(K)) + J(\bar{C})$ or $s\bar{x} \in rad_{\bar{C}}(f(K)) + J(\bar{C})$ or $rs\bar{C} \subseteq f(K) + J(\bar{C})$. Hence $f(K)$ is a nearly primary-2-absorbing submodule of $\bar{C}$.

Lemma 14 [9, Theorem(2.12)]

Let R be a commutative ring with identity, L be a proper submodule of a multiplication R-module C and $J = [L:_R C]$. Then $rad_C(L) = \sqrt{J}.C = \sqrt{[L:_R C]}.C$.

Lemma 15 [2, Theorem 1, (1)]

Let C is a module over Artinian Ring R, then $J(R).C = J(C)$.

Proposition 16

Let C be a multiplication module over an Artinian ring R and L be a proper submodule of C. Then L is a nearly primary-2-absorbing submodule of C if and only if $[L:_R C]$ is a nearly primary-2-absorbing ideal of R.

Proof:

($\Rightarrow$) Let $acI \subseteq [L:_R C]$ for $a,\ c \in R$, I is an ideal of R with $ac \notin \left[\left[[L:_R C] + J(R):_R R\right]\right] = [L:_R C] + J(R)$, that is $acC \nsubseteq [L:_R C].C + J(R).C$. But C is a module over Artinian ring, then by Lemma (15) $J(R).C = J(C)$, that is $acC \nsubseteq L + J(C)$ i.e. $ac \notin [L + J(C):_R C]$. Now, since $acI \subseteq [L:_R C]$, then $ac(IC) \subseteq L$. But L is a nearly primary-2-absorbing submodule of C, then by Proposition (4) either $a(IC) \subseteq rad_C(L) + J(C)$ or $c(IC) \subseteq rad_C(L) + J(C)$. But by Lemma(14) $rad_C(L) = \sqrt{[L:_R C]}.C$ and by Lemma(15) $J(R).C = J(C)$. Thus either $aIC \subseteq \sqrt{[L:_R C]}.C + J(R).C$ or $cIC \subseteq \sqrt{[L:_R C]}.C + J(R).C$, it follows that either $aI \subseteq \sqrt{[L:_R C]}.C + J(R)$ or $cI \subseteq \sqrt{[L:_R C]}.C + J(R)$. Therefore $[L:_R C]$ is a nearly primary-2-absorbing ideal of R.

($\Leftarrow$) Assume that $[L:_R C]$ is a nearly primary-2-absorbing ideal of R, and $rsK \subseteq L$, for $r,s \in R$, K is a submodule of C with $rs \notin [L + J(C):_R C]$, it follows that $rsC \nsubseteq L + J(C)$. But C is a module over Artinian ring, then by lemma (15) $J(R).C = J(C)$. Thus $rsC \nsubseteq [L:_R C].C + J(R).C$, it follows that $rs \nsubseteq [L:_R C] + J(R) = \left[\left[[L:_R C] + J(R):_R R\right]\right]$. Now, since $rsK \subseteq L$ and C is a multiplication, then $K = IC$ for some ideal I of R. Hence $rsIC \subseteq L$, implies that $rsI \subseteq [L:_R C]$. But $[L:_R C]$ is a nearly primary-2-absorbing ideal of R and $rs \notin \left[\left[[L:_R C] + J(R):_R R\right]\right]$, then by proposition (4) we have either $rI \subseteq \sqrt{[L:_R C]} + J(R)$ or $sI \subseteq \sqrt{[L:_R C]} + J(R)$. That is $rIC \subseteq \sqrt{[L:_R C]}.C + J(R).C$ or $sIC \subseteq \sqrt{[L:_R C]}.C + J(R).C$. But C is a module over Artinian ring, then by Lemma (15) $J(R).C = J(C)$ and by Lemma(14) $rad_C(L) = \sqrt{[L:_R C]}.C$ we get either $rK \subseteq rad_C(L) + J(C)$ or $sK \subseteq rad_C(L) + J(C)$. Hence by Proposition (4) L is a nearly primary-2-absorbing submodule of C.

Lemma 17 [10, proposition (17.10)]

If P is projective R-module, then $J(R).P = J(P)$.

Proposition 18

Let C be a multiplication projective R-module and L is a proper submodule of C. Then L is a nearly primary-2-absorbing submodule of C if and only if $[L:_R C]$ is a nearly primary-2-absorbing ideal of R.

Proof:

($\Rightarrow$) Let $acI \subseteq [L:_R C]$ for $a,\ c \in R$, I is an ideal of R with $ac \notin \left[\left[[L:_R C] + J(R):_R R\right]\right] = [L:_R C] + J(R)$, that is $acC \nsubseteq [L:_R C].C + J(R).C$. But C is projective R-module, then by Lemma (17) $J(R).C = J(C)$, that is $acC \nsubseteq L + J(C)$ i.e. $ac \notin [L + J(C):_R C]$. Now, since $acI \subseteq [L:_R C]$, then $ac(IC) \subseteq L$. But L is a nearly primary-2-absorbing submodule of C, then by proposition (4) either $a(IC) \subseteq rad_C(L) + J(C)$ or $c(IC) \subseteq rad_C(L) + J(C)$. But by Lemma(14) $rad_C(L) = \sqrt{[L:_R C]}.C$ and by Lemma(17) $J(R).C = J(C)$. Thus either $aIC \subseteq \sqrt{[L:_R C]}.C + J(R).C$ or $cIC \subseteq \sqrt{[L:_R C]}.C + J(R).C$. It follows that either $aI \subseteq \sqrt{[L:_R C]}.C + J(R)$ or $cI \subseteq \sqrt{[L:_R C]}.C + J(R)$. Therefore, $[L:_R C]$ is a nearly primary-2-absorbing ideal of R.

($\Leftarrow$) Assume that $[L:_R C]$ is a nearly primary-2-absorbing ideal of R, and $rsK \subseteq L$, for $r,s \in R$, K is a submodule of C with $rs \notin [L + J(C):_R C]$, it follows that $rsC \nsubseteq L + J(C)$. But C is a project R-module, then by Lemma (17) $J(R).C = J(C)$. Thus $rsC \nsubseteq [L:_R C].C + J(R).C$. It follows that $rs \nsubseteq [L:_R C] + J(R) = \left[\left[[L:_R C] + J(R):_R R\right]\right]$. Now, since $rsK \subseteq L$ and C is a multiplication, then $K = I\complement$ for some ideal I of R. Hence $rsI\complement \subseteq L$, implies that $rsI \subseteq [L:_R \complement]$. But $[L:_R \complement]$ is a nearly primary-2-absorbing ideal of R and $rs \notin \left[\left[[L:_R \complement] + \mathrm{J(R)}:_R \mathrm{R}\right]\right]$, then by Proposition (4) we have either $rI \subseteq \sqrt{[L:_R \complement]} + J(R)$ or $sI \subseteq \sqrt{[L:_R \complement]} + J(R)$. That is $rI\complement \subseteq \sqrt{[L:_R \complement]}.\complement + J(R).\complement$ or $sI\complement \subseteq \sqrt{[L:_R \complement]}.\complement + J(R).\complement$. But $\complement$ is projective R-module, then by Lemma (17) $J(R).\complement = J(\complement)$ and by Lemma(14) $rad_\complement(L) = \sqrt{[L:_R \complement]}.\complement$ we get either $rK \subseteq rad_\complement(L) + J(\complement)$ or $sK \subseteq rad_\complement(L) + J(\complement)$. Hence by proposition (4) L is a nearly primary-2-absorbing submodule of $\complement$.

We need to invite the following finding before we study the next propositions.

Lemma 19[14, Corollary of Theorem. 9]

Let I_1 and I_2 are ideals of a ring R and $\complement$ is a finitely generated multiplication R-module. Then $I_1\complement \subseteq I_2\complement$ if and only if $I_1 \subseteq I_2 + ann_R(\complement)$.

Lemma 20[15, Proposition. (2.4)]

Let $\complement$ be a multiplication R-module and I is an ideal of R such that $ann_R(\complement) \subseteq I$, then $rad_\complement(I\complement) = \sqrt{I}\complement$.

Proposition 21

Let $\complement$ be a faithful finitely generated multiplication R-module over an Artinian ring R and I is a nearly primary-2-absorbing ideal of R and $I\complement \neq \complement$. Then $I\complement$ is a nearly primary-2-absorbing submodule of $\complement$.

Proof:

Let $acx \in I\complement$ for $a,c \in R$, $x \in \complement$, then $ac(x) \subseteq I\complement$, implies that $abJ\complement \subseteq I\complement$ for some ideal J of R since $\complement$ is a multiplication. Hence by Lemma(19) $acJ \subseteq I + ann_R(\complement)$, but $\complement$ is a faithful. It follows that $ann_R(\complement) = (0)$, that is $acJ \subseteq I$. Since I is a nearly primary-2-absorbing ideal of R, then by Proposition(4) either $aJ \subseteq \sqrt{I} + J(R)$ or $cJ \subseteq \sqrt{I} + J(R)$ or $ac \in [I + J(R): R] = I + J(R)$. It follows that $aJ\complement \subseteq \sqrt{I}\complement + J(R)\complement$ or $cJ\complement \subseteq \sqrt{I}\complement + J(R)\complement$ or $ab\complement \subseteq I\complement + J(R)\complement$. But by Lemma(15) $J(R)\complement = J(\complement)$ and by Lemma(20) $\sqrt{I}\complement = rad_\complement(I\complement)$. Thus either $ax \in rad_\complement(I\complement) + J(\complement)$ or $cx \in rad_\complement(I\complement) + J(\complement)$ or $ab\complement \subseteq I\complement + J(\complement)$. Hence $I\complement$ is a nearly primary-2-absorbing submodule of $\complement$.

Proposition 22

Let $\complement$ be a faithful finitely generated multiplication R-module over an Artinian ring R and K be a proper submodule of $\complement$. Then the next statements are equivalent.

1. L is a nearly primary-2-absorbing submodule of $\complement$.

2. $[\mathrm{L}:_R \complement]$ is a nearly primary-2-absorbing ideal of R.

3. $\mathrm{L} = \mathrm{J}\complement$ for some nearly primary-2-absorbing ideal of R.

Proof:

$1 \Leftrightarrow 2$ Via Proposition(16).

$2 \Rightarrow 3$ Since $[\mathrm{L}:_R \complement]$ is a nearly primary-2-absorbing ideal of R with $ann_R(\complement) = [0: \complement] \subseteq [\mathrm{L}:_R \complement]$ and $\mathrm{L} = [\mathrm{L}:_R \complement]\complement$, implies that $\mathrm{L} = \mathrm{I}\complement$ where $\mathrm{I} = [\mathrm{L}:_R \complement]$ is a nearly primary-2-absorbing ideal of R.

$3 \Longrightarrow 2$ Suppose that $L = J\complement$ for some nearly primary-2-absorbing ideal J of R. Since $\complement$ is multiplication, then $\mathrm{L} = [\mathrm{L}:_R \complement]\complement = \mathrm{I}\complement$. Since $\complement$ is a faithful finitely generated multiplication R-module over an Artinian ring R, then we have $[\mathrm{L}:_R \complement] = \mathrm{J}$. Thus $[\mathrm{L}:_R \complement]$ is a nearly primary-2-absorbing ideal of R.

3. Conclusion

In this article we introduce a new generalization of (prime, primary, 2-absorbing) submodules called a nearly primary-2-absorbing submodules and we explain the converse implication of the above by examples. Many characterizations of this generalization are introduced. Relationships of this generalization with other classes of modules are given.

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