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Strongly Pseudo Nearly Semei-2-Absorbing submodule(II)

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Abstract

Let $\mathcal{H}$ be a moduleover a commutativering R with identity.Before studying the notion of Strongly Pseudo Nearly Semi-2-Absorbing submodule, where a propersubmodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is called to be Strongly Pseudo Nearly Semi-2-Absorbingsubmodule of $\mathcal{H}$ if $\forall\ u^2\varkappa \in \mathcal{F}$, for $u \in R$, $\varkappa \in \mathcal{H}$ it follows that either $u\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}):_R \mathcal{H}]$, we need to mention the ideal $[\mathcal{F}:_R\mathcal{H}] = \{r \in R: r\mathcal{H} \subseteq \mathcal{F}\}$ and the basics that you need to study the notion of Strongly Pseudo Nearly Semi-2-Absorbing submodule. Also we introduce several characterzations of Strongly Pseudo Nearly Semi-2-Absorbing submodule in classes of multiplication modules and other types of modules. We also had no luck the ideal $[\mathcal{F}:_R\mathcal{H}] = \{r \in R: r\mathcal{H} \subseteq \mathcal{F}\}$ is not Strongly Pseudo Nearly Semi-2-Absorbing ideal. Also noted that $[\mathcal{F}:_R\mathcal{H}]$ is Strongly Pseudo Nearly Semi-2-Absorbing ideal under several conditions. Also we introduce the characterization of the notion of Strongly Pseudo Nearly Semi-2-Absorbing ideals by special kind of submodules.

Keywords: STPNS-Asubmodules, STPNS-Aideal, faithful module, projective module, Z-regular module.

1. Introduction

In this part we note that the ideal $[\mathcal{F}:_R\mathcal{H}] = \{r \in R: r\mathcal{H} \subseteq \mathcal{F}\}$ is not STPNS-Aideals and we gave an example of that. We also noted that $[\mathcal{F}:_R\mathcal{H}]$ is STPNS-Aideals under several conditions, which was previously by several researchers an is the first condition faithful module submitted by researcher kach in 1982. The second condition projective module was also presented by the same researcher. The third condition is two conditions combined together Z-regular module and content module, which was presented in (1973, 1989). The quarter condition is two conditions combined together non-singular module and content module was presented in (1976, 1989). Also in this part we introduce the characterization of the concept of STPNS-Aideals by special kind of submodules.

2. Basic Properties

In this part we introduce the basic properties of the concept Strongly Pseudo Nearly Semei-2-Absorbing submodules.

Definition 2.1 [1].

$[\mathcal{F}:_R\mathcal{H}] = \{r \in R: r\mathcal{H} \subseteq \mathcal{F}\}$ where $\mathcal{F}$ is a submodule of an R-module $\mathcal{H}$.

Lemma 2.2 [2, prop.(3.3)].

Let $\mathcal{H}$ an $\mathcal{R}$-module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is a STPNS-2-A submodule of $\mathcal{H}$ ifandonlyif $I^2\mathcal{L} \subseteq \mathcal{F}$ for I is an ideal of R and $\mathcal{L} \subseteq \mathcal{H}$ it means that either $I\mathcal{L} \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $I^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_{\mathcal{R}} \mathcal{H}]$.

Definition 2.3 [3].

AnR–module $\mathcal{H}$ is called to befaithful if $Ann_R(\mathcal{H}) = (0)$ $.Where\ Ann(\mathcal{H}) = \{r \in R : r\mathcal{H} = (0)\}$.

Definition 2.4 [4].

AnR–module $\mathcal{H}$ is called to bemultiplication , if any submodule $\mathcal{F}$ of $\mathcal{H}$ is of theform $\mathcal{F} = I\mathcal{H}$ for some ideal I of R . Equivalent to $\mathcal{F} = [\mathcal{F}:_R\mathcal{H}]\mathcal{H}$.

Lemma 2.5 [5 , coro. (2 .1. 14) (i)].

Let $\mathcal{H}$ be a faithfulmultiplication R–module , then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Lemma 2.6 [6].

Let $\mathcal{H}$ be a faithfulmultiplication R–module , then $J(\mathcal{H}) = J(R)\mathcal{H}$.

Lemma 2.7 [2, coro.(3.4)].

Let $\mathcal{H}$ an R-module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNS-2-A submodule of $\mathcal{H}$ ifandonlyif $u^2\mathcal{L} \subseteq \mathcal{F}$ for $u \in R$ and $\mathcal{L} \subseteq \mathcal{H}$ it means that either $u\mathcal{L} \subseteq \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}):_{\mathcal{R}} \mathcal{H}]$.

Definition 2.8 [3].

An R–module $\mathcal{H}$ isaprojective if for any R–epimorphism $f : \mu \to \mu'$ and every R –homomorphism $g : \mathcal{H} \to \mu'$, there exists an R–homomorphism $h : \mathcal{H} \to \mu$ such that the followingdiagram is commutethat is $foh = g$.

Lemma 2.9 [6 , prop. (3 . 24)]

Let $\mathcal{H}$ beaprojective R–module, then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Lemma 2.10 [3 , Theo. (9 . 2.1) (a)]

For anyprojective R–module $\mathcal{H}$, we have $J(\mathcal{H}) = J(R)\mathcal{H}$.

Definition 2.11 [7]

An R–module $\mathcal{H}$ is calledZ–regular if for any $x \in \mathcal{H}$ $\exists$ $g \in \mathcal{H} = Hom_R(\mathcal{H}, R)$ such that $x = g(x)x$.

Lemma 2.12 [6 , proposition (3 . 25)]

Let $\mathcal{H}$ be a Z – regularR–module then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Definition 2.13 [8].

An R-module $\mathcal{H}$ is said to be content module if $(\cap_{i\in I} A_i)\mathcal{H} = \cap_{i\in I} A_i\mathcal{H}$ for each family of ideals A_i in R.

Lemma 2.14 [6 , proposition (1 . 11)].
If $\mathcal{H}$ is content module, then $J(\mathcal{H}) = J(R)\mathcal{H}$.

Definition 2.15 [9].
$Z(\mathcal{H}) = \{x \in \mathcal{H}: ann(x) essential\ ideal\ in\ R\}$ is called singular submodule of $\mathcal{H}$. If $Z(\mathcal{H}) = \mathcal{H}$, then $\mathcal{H}$ is called the singular module . If $Z(\mathcal{H}) = 0$, then $\mathcal{H}$ is called the non-singular module.

Lemma 2.16 [9 , coro. (1 . 26)].
Let $\mathcal{H}$ be a non-singularR–module , then $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$.

Definition 2.17 [3].
An R – module $\mathcal{H}$ is finitely generated if $\mathcal{H} = \langle z_1, z_2, z_3, \dots, z_n\rangle = Rz_1, Rz_2, Rz_3, \dots Rz_n$, where $z_1, z_2, z_3, \dots, z_n \in \mathcal{H}$.

Lemma 2.18 [10 ,coro. of theo. (9)]
Let $\mathcal{H}$ be a finitelygenerated multiplicationR–module and I , J ideals of R . Then $I\mathcal{H} \subseteq J\mathcal{H}$ ifandonlyif $I \subseteq J + ann_R(\mathcal{H})$.
Definition 2.19 [11].
AnR–module $\mathcal{H}$ is called to becancellation ifwhenever $I\mathcal{H} = J\mathcal{H}$ for any ideals I, J of R, implies that $I = J$.
Lemma 2.20 [11 , prop. (3 . 1)].
If $\mathcal{H}$ is a multiplicationR–module, then $\mathcal{H}$ iscancellation ifandonlyif $\mathcal{H}$ is a faithfulfinitely generated.

3. The Results
Inthis section we introduce the definition of Strongly Pseudo Nearly Semi-2-Absorbingsubmodule and we introduce several characterzations of STPNS-Asubmodules in classes of multiplicationmodules and other types of modules:
Definition 3.1[2]
A propersubmodule $\mathcal{F}$ of an R-module $\mathcal{H}$ is called to be Strongly Pseudo Nearly Semi-2-Absorbing submodule of $\mathcal{H}$ (for short STPNS) if whenever $u^2\varkappa \in \mathcal{F}$, for $u \in R$, $\varkappa \in \mathcal{H}$ implies that either $u\varkappa \in \mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H})$ or $u^2 \in [\mathcal{F} + J(\mathcal{H}) \cap soc(\mathcal{H}):_R \mathcal{H}]$.
The following proposition gives characterization of STPNS-Asubmodules in classes of multiplication modules.
Proposition 3.2
A propersubmodule F of a multiplication$R$-module H is STPNS-Asubmodule of H if and only if $A^2K \subseteq F$ for A, K are submodules of H, implies that either $AK \subseteq F + (J(H) \cap soc(H))$ or $A^2 \subseteq F + (J(H) \cap soc(H))$. .
Proof
($\Longrightarrow$) Let $r^2K \subseteq F$ for $r \in R$, $K \subseteq H$. But H is a multiplicationmodule, then $K = IH$ forsome ideal I of R, it follows that $r^2IH \subseteq F$, it follows by hypothesis either $rIH \subseteq F + (J(H) \cap soc(H))$or $r^2 \in [F + (J(H) \cap soc(H)) :_R H]$. That is either $rK \subseteq F + (J(H) \cap soc(H))$ or $r^2 \in [F + (J(H) \cap soc(H)) :_R H]$. Hence F is STPNS-Asubmodule of H.
($\Longleftarrow$)Let $A^2K \subseteq F$ for A, K are submodules of a multiplication module H, it follows that $(IH)^2(JH) = I^2JH \subseteq F$ for some ideals I, J of R. Since F is STPNS-Asubmodule of H, then by

lemma 2.2 we have either $IJH \subseteq F + J(H) \cap soc(H)$ or $I^2 \subseteq [F + J(H) \cap soc(H) \ :_R H]$, that is either $AK \subseteq F + (J(H) \cap soc(H))$ or $A^2 \subseteq F + (J(H) \cap soc(H))$.

As a directresult of proposition 3.2 we have the followingcorollaries.

Corollary 3.3

A propersubmodule F of a multiplication R-module H is STPNS-Asubmodule of H if and only if $A^2 h \subseteq F$ for A is a submodules of H and $h \in H$, it means that either $Ah \subseteq F + (J(H) \cap soc(H))$ or $A^2 \subseteq F + (J(H) \cap soc(H))$.

Corollary 3.4

A propersubmodule F of a multiplication$R$-module H is STPNS-Asubmodule of H if and only if $h^2 K \subseteq F$ for $h \in H$ and $K \subseteq H$, it means that either $hK \subseteq F + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $h^2 \subseteq F + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.

Remaek 3.5

If $\mathcal{F}$ is an STPNS-Asubmodule of an R_module $\mathcal{H}$, then $[\mathcal{F}_R \mathcal{H}]$ it doesn't have to be an STPNS-Aideal of R, the followingexample explainthat:

Considerthe Z-module Z_{36}, the submodule $\mathcal{F} = \langle \overline{12} \rangle$ is an STPNS-Asubmodule of the Z_module Z_{36}, since $2^2 . \overline{3} \in \langle \overline{12} \rangle$, implies that $2. \overline{3} \in \langle \overline{12} \rangle + \big(J(Z_{36}) \cap soc(Z_{36})\big) = \langle \overline{12} \rangle + (\langle \overline{6} \rangle \cap \langle \overline{6} \rangle) = \langle \overline{6} \rangle$, but $[\langle \overline{12} \rangle_R Z_{36}] = 12Z$ is not to be an STPNS-Aideal of Z, because $2^2 . 3 \in 12Z$,but $2.3 \notin 12Z + \big(soc(Z) \cap J(Z)\big) = 12Z$ and$2^2 \notin \left[12Z + \big(J(Z) \cap soc(Z)\big)_R : Z\right] = 12Z$.

The above remark satsfay under certain conditions.

Proposition 3.6

Let $\mathcal{H}$ a faithful multiplication R-module and $\mathcal{F} \subset \mathcal{H}$.Then $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$ if and only if $[\mathcal{F} :_R \mathcal{H}]$ is STPNS-Aideal of R.

Proof

($\Longrightarrow$) Let $I^2 J \subseteq [\mathcal{F} :_R \mathcal{H}]$ forsome ideals I, J of R, hence $I^2(J\mathcal{H}) \subseteq \mathcal{F}$. But $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$, then by lemma 2.2 either $I(J\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $I^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}]$. Since $\mathcal{H}$ is multiplication, then $\mathcal{F} = [\mathcal{F} :_R \mathcal{H}]\mathcal{H}$ and since $\mathcal{H}$ is faithful multiplication, then by lemmas 2.5, 2.6 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $I(J\mathcal{H}) \subseteq [\mathcal{F} :_R \mathcal{H}]\mathcal{H} + J(R)\mathcal{H} \cap soc(R)\mathcal{H}$ or $I^2\mathcal{H} \subseteq [\mathcal{F} :_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$, thus either $IJ \subseteq [\mathcal{F} :_R \mathcal{H}] + (J(R) \cap soc(R))$ or $I^2 \subseteq [\mathcal{F} :_R \mathcal{H}] + (J(R) \cap soc(R)) = \big[[\mathcal{F} :_R \mathcal{H}] + (J(R) \cap soc(R)) :_R R\big]$. Hence $[\mathcal{F} :_R \mathcal{H}]$ is STPNS-Aideal of R.

($\Longleftarrow$) Let $A^2 L \subseteq \mathcal{F}$ for A, L aresubmodules of $\mathcal{H}$. Since $\mathcal{H}$ isamultiplication, then $A = I\mathcal{H}$and $L = J\mathcal{H}$ forsome ideals I, J of R, that is $(I\mathcal{H})^2 J\mathcal{H} \subseteq \mathcal{F}$, implies that $I^2 J \subseteq [\mathcal{F} :_R \mathcal{H}]$, but $[\mathcal{F} :_R \mathcal{H}]$ is STPNS-Aideal of R, theneither $IJ \subseteq [\mathcal{F} :_R \mathcal{H}] + (J(R) \cap soc(R))$ or $I^2 \subseteq \big[[\mathcal{F} :_R \mathcal{H}] + (J(R) \cap soc(R)) :_R R\big] = [\mathcal{F} :_R \mathcal{H}] + (J(R) \cap soc(R))$,thus either $IJ\mathcal{H} \subseteq [\mathcal{F} :_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $I^2\mathcal{H} \subseteq [\mathcal{F} :_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Hence by lemmaies 2.5, 2.6 either $AL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $A^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})) :_R \mathcal{H}]$. Thus by proposition 3.2 $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proposition 3.7

Let $\mathcal{H}$ be a multiplication projectiveR–module and $\mathcal{F} \subset \mathcal{H}$. Then $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$ if and only if $[\mathcal{F} :_R \mathcal{H}]$is STPNS-Aideal of R.

Proof

($\Longrightarrow$) Let $u^2 J \subseteq [\mathcal{F} :_R \mathcal{H}]$ for some $u \in R$ and J is an ideals of R, hence $u^2(J\mathcal{H}) \subseteq \mathcal{F}$. But $\mathcal{F}$ is STPNS -2-Absorbing submodule of $\mathcal{H}$, then by lemma 2.7 either $u(J\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap$

$soc(\mathcal{H})$)or $u^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Since $\mathcal{H}$ is multiplication, then $\mathcal{F} = [\mathcal{F}:_R \mathcal{H}]\mathcal{H}$ and since $\mathcal{H}$ is projective, then by lemmaies 2.9, 2.10 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $u(J\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + J(R)\mathcal{H} \cap soc(R)\mathcal{H}$ or $u^2\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$, thus either $uJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$ or $u^2 \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)) = \big[[\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)):_R R\big]$. Hence $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R.

($\Leftarrow$) Let $A^2h \subseteq \mathcal{F}$ for A is a submodules of $\mathcal{H}$ and $h \in \mathcal{H}$. But $\mathcal{H}$ is a multiplication, then $A = I\mathcal{H}$ and $h = J\mathcal{H}$ for some ideals I, J of R, that is $(I\mathcal{H})^2J\mathcal{H} \subseteq \mathcal{F}$, implies that $I^2J \subseteq [\mathcal{F}:_R \mathcal{H}]$, but $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R, then either $IJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$ or $I^2 \subseteq \big[[\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)) :_R R\big] = [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$, thus either $IJ\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $I^2\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Hence by lemmaies 2.9, 2.10 either $Ah \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $A^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Thus by corollary 3.3 $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proposition 3.8

Let $\mathcal{H}$ be a multiplicationZ-Regular content R–module and $\mathcal{F}$ be a propersubmdule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$ if and only if $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .

Proof

($\Rightarrow$) Let $a^2b \in [\mathcal{F}:_R \mathcal{H}]$ for some $a, b \in R$, hence $a^2(b\mathcal{H}) \subseteq \mathcal{F}$. But $\mathcal{F}$ is STPNS -2-Absorbing submodule of $\mathcal{H}$, then by lemma 2.7 either $a(b\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $a^2 \in [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Since $\mathcal{H}$ is multiplication, then $\mathcal{F} = [\mathcal{F}:_R \mathcal{H}]\mathcal{H}$ and since $\mathcal{H}$ is Z-Regular content R–module, then by lemmas 2.12, 2.14 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $a(b\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + J(R)\mathcal{H} \cap soc(R)\mathcal{H}$ or $a^2\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$, thus either $ab \in [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$ or $a^2 \in [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)) = \big[[\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)):_R R\big]$. Hence $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R.

($\Leftarrow$) Let $h^2A \subseteq \mathcal{F}$ for $h \in \mathcal{H}$ and A is a submodules of $\mathcal{H}$. But $\mathcal{H}$ is a multiplication, then $h = I\mathcal{H}$ and $A = J\mathcal{H}$ forsome ideals I, J of R, that is $(I\mathcal{H})^2J\mathcal{H} \subseteq \mathcal{F}$, implies that $I^2J \subseteq [\mathcal{F}:_R \mathcal{H}]$, but $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R, then either $IJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$ or $I^2 \subseteq \big[[\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)) :_R R\big] = [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$, thus either $IJ\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $I^2\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Hence lemmas 2.12, 2.14 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$ either $hA \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $h^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Thus by corollary 3.4 $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proposition 3.9

Let $\mathcal{H}$ be a multiplicationnon-singular contentR–module and $\mathcal{F}$ be a propersubmdule of $\mathcal{H}$. Then $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$ ifandonlyif $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R.

Proof

($\Rightarrow$) Let $I^2J \subseteq [\mathcal{F}:_R \mathcal{H}]$ for some ideals I, J of R, hence $I^2(J\mathcal{H}) \subseteq \mathcal{F}$. But $\mathcal{F}$ is STPNS -2-Absorbing submodule of $\mathcal{H}$, then by lemma 2.2 either $I(J\mathcal{H}) \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$or $I^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Since $\mathcal{H}$ is multiplication, then $\mathcal{F} = [\mathcal{F}:_R \mathcal{H}]\mathcal{H}$ and since $\mathcal{H}$ non-singular content R–module, then by lemmas 2.16, 2.14 $soc(\mathcal{H}) = soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $I(J\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + J(R)\mathcal{H} \cap soc(R)\mathcal{H}$ or $I^2\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$, thus either $IJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$ or $I^2 \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)) = \big[[\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)):_R R\big]$. Hence $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R.

($\Leftarrow$) Let $A^2L \subseteq \mathcal{F}$ for A, L are submodules of $\mathcal{H}$. Since $\mathcal{H}$ is a multiplication, then $A = I\mathcal{H}$and $L = J\mathcal{H}$ for some ideals I, J of R, that is $(I\mathcal{H})^2J\mathcal{H} \subseteq \mathcal{F}$, implies that $I^2J \subseteq [\mathcal{F}:_R \mathcal{H}]$, but $[\mathcal{F}:_R \mathcal{H}]$

is STPNS-Aideal of R, then either $IJ \subseteq [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$ or $I^2 \subseteq \left[[\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R)) :_R R\right] = [\mathcal{F}:_R \mathcal{H}] + (J(R) \cap soc(R))$, thus either $IJ\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $I^2\mathcal{H} \subseteq [\mathcal{F}:_R \mathcal{H}]\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Hence by lemmas2.16, 2.14 either $AL \subseteq \mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $A^2 \subseteq [\mathcal{F} + (J(\mathcal{H}) \cap soc(\mathcal{H})):_R \mathcal{H}]$. Thus by proposition 3.2 $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.

4. Characterization of STPNS-Aideals by special kind of submodules

In this section we introduce the characterization of the notion of STPNS-Aideals by special kind of submodules.

The following proposition gives characterization of the notion of STPNS-Aideals.

Proposition 4.1

Let $\mathcal{H}$ be a finitelygenerated Z–regularmultiplication content-R–module and I is a STPNS-Aideal of R with $\text{ann}_R(\mathcal{H}) \subseteq$ I if and only if I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proof

(⇒) Let $B^2L \subseteq \text{I}\,\mathcal{H}$ for B is an ideal of $R$and $L \subseteq \mathcal{H}$. But $\mathcal{H}$ is a multiplicationR–module , then $L = \text{J}\mathcal{H}$ forsome ideal J of R , that is $B^2L = B^2\,\text{J}\mathcal{H} \subseteq \text{I}\,\mathcal{H}$. But $\mathcal{H}$ is a finitely generatedmultiplication, then by lemma 2.18 we have $B^2\text{J} \subseteq \text{I} + \text{ann}_R(\mathcal{H})$. Since $\text{ann}_R(\mathcal{H}) \subseteq \text{I}$ then $\text{I} + \text{ann}_R(\mathcal{H}) = \text{I}$, that $B^2\text{j} \subseteq \text{I}$. But I is STPNS-Aideal of R then either $B\text{j} \subseteq \text{I} + J(R) \cap soc(R)$ or $B^2 \in \left[\text{I} + \left(J(R) \cap soc(R)\right):_R R\right] = \text{I} + \left(J(R) \cap soc(R)\right)$. It means that either $B\text{j}\mathcal{H} \subseteq \text{I}\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $B^2\mathcal{H} \subseteq \text{I}\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Since $\mathcal{H}$ is a Z – regular and content R-module then by lemmas 2.12, 2.14 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. So that either $\text{BL} \subseteq \text{I}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq \text{I}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore by lemma 2.5 I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

(⇐) Let $u^2\text{J} \subseteq I$ for $u \in R$ and J is an ideal of R , it means that $u^2\text{J}\,\mathcal{H} \subseteq I\mathcal{H}$. But $\mathcal{H}$ is multiplication, then $u^2J\mathcal{H} = u^2L \subseteq I\mathcal{H}$. But $I\mathcal{H}$ is a STPNS-Athen by lemma 2.7 either $uL \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $\mathcal{H}$ is Z-Regular content R–module, then by lemmas 2.12, 2.14 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$, and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $uJ\mathcal{H} \subseteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $u^2\mathcal{H} \subseteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. That is either $uJ \subseteq I + (J(R) \cap soc(R))$ or $u^2 \in I + \left(J(R) \cap soc(R)\right) = \left[I + \left(J(R) \cap soc(R)\right):_R R\right]$. Therefore by lemma 2.7 I is STPNS-Aideal of R.

Proposition 4.2

Let $\mathcal{H}$ be a finitelygenerated multiplicationprojective R–module and I is a STPNS-Aideal of R with $\text{ann}_R(\mathcal{H}) \subseteq$ I if and only if I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proof

(⇒) Let $A^2h \subseteq \text{I}\mathcal{H}$ for A is a submodule of $\mathcal{H}$ and $\text{h} \in \mathcal{H}$. Since $\mathcal{H}$ isamultiplication R–module , then $A = B\mathcal{H}$ and $h = J\mathcal{H}$ forsome ideals B, J of R , that is $A^2h = (B\mathcal{H})^2\,\text{J}\mathcal{H} \subseteq \text{I}\,\mathcal{H}$. But $\mathcal{H}$ is a finitelygenerated multiplication , then by lemma 2.18 we have $B^2\text{J} \subseteq \text{I} + \text{ann}_R(\mathcal{H})$. Since $\text{ann}_R(\mathcal{H}) \subseteq \text{I}$ then $\text{I} + \text{ann}_R(\mathcal{H}) = \text{I}$, that $B^2\text{J} \subseteq \text{I}$. But I is STPNS-Aideal of R, then either $B\text{J} \subseteq \text{I} + J(R) \cap soc(R)$ or $B^2 \in \left[\text{I} + \left(J(R) \cap soc(R)\right):_R R\right] = \text{I} + \left(J(R) \cap soc(R)\right)$. It means that either $B\text{j}\mathcal{H} \subseteq \text{I}\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $B^2\mathcal{H} \subseteq \text{I}\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Since $\mathcal{H}$ is a projective R–module then by lemmas 2.9, 2.10 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. So that either $\text{Ah} \subseteq \text{I}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $h^2 \subseteq \text{I}\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore by corollary 3.3 I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

(⇐) Let u^2J $\subseteq I$ for $u \in R$ and J is an ideal of R , it means that u^2J $\mathcal{H} \subseteq I\mathcal{H}$. But $\mathcal{H}$ is multiplication, then $u^2 J\mathcal{H} = u^2 L \subseteq I\mathcal{H}$. But $I\mathcal{H}$ is a STPNS-Athen by lemma 2.7 either $uL \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.But $\mathcal{H}$ is projective R–module, then by lemmas 2.9, 2.10 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $uJ\mathcal{H} \subseteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $u^2\mathcal{H} \subseteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. That is either $uJ \subseteq I + (J(R) \cap soc(R))$ or $u^2 \in I + \big(J(R) \cap soc(R)\big) = \big[I + \big(J(R) \cap soc(R)\big):_R R\big]$. Therefore by lemma 2.17 I is STPNS-Aideal of R.

Proposition 4.3

Let $\mathcal{H}$ be a finitelygenerated multiplicationnon-singularcontentR–module and I is a STPNS-Aideal of R with $\text{ann}_\text{R}(\mathcal{H}) \subseteq$ I if and only if I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proof

(⇒) Let $B^2 L \subseteq$ I $\mathcal{H}$ for B is an ideal of R and $L \subseteq \mathcal{H}$. Since $\mathcal{H}$ is a multiplication R–module , then $L =$ J$\mathcal{H}$ forsome ideal J of R , that is $B^2 L = B^2$ J$\mathcal{H} \subseteq$ I $\mathcal{H}$. But $\mathcal{H}$ is a finitelygenerated multiplication , then by lemma 2.18 we have B^2J$\subseteq$ I + $\text{ann}_\text{R}(\mathcal{H})$. Since $\text{ann}_\text{R}(\mathcal{H}) \subseteq$ I then I + $\text{ann}_\text{R}(\mathcal{H})$ = I, that B^2j$\subseteq$ I. But I is STPNS-Aideal of R, then either Bj $\subseteq$ I + $J(R) \cap soc(R)$ or $B^2 \in \big[\text{I} + \big(J(R) \cap soc(R)\big):_R R\big] = \text{I} + \big(J(R) \cap soc(R)\big)$. It means that either Bj$\mathcal{H} \subseteq$ I$\mathcal{H}$ + $(J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $B^2\mathcal{H} \subseteq$ I$\mathcal{H}$ + $(J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Since $\mathcal{H}$ is a non-singular content-R–module then by lemmaies 2.16, 2.14 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. So that either BL $\subseteq$I$\mathcal{H}$ + $(J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq$I$\mathcal{H}$ + $(J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore by lemma 2.2 I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

(⇐) Let u^2v $\subseteq I$ for $u, v \in R$, it means that u^2v $\mathcal{H} \subseteq I\mathcal{H}$. But $I\mathcal{H}$ is a STPNS-Athen by lemma 2.7 either $uv\mathcal{H} \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2 \in I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$. But $\mathcal{H}$ is a non-singular content-R–module then by lemmas 2.16, 2.14 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $uv\mathcal{H} \subseteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $u^2\mathcal{H} \subseteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. That is either $uv \in I + (J(R) \cap soc(R))$ or $u^2 \in I + \big(J(R) \cap soc(R)\big) = \big[I + \big(J(R) \cap soc(R)\big):_R R\big]$. Therefore I is STPNS-A ideal of R.

Proposition 4.4

Let $\mathcal{H}$ be a faithfulfinitely generatedmultiplication R–moduleand I is a STPNS-Aideal of R with $\text{ann}_\text{R}(\mathcal{H}) \subseteq$ I if and only if Then I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

Proof

(⇒) Let $h^2 K \subseteq$ I$\mathcal{H}$ for h $\in \mathcal{H}$ and $K \subseteq \mathcal{H}$. Since $\mathcal{H}$ is a multiplicationR–module , then $h = B\mathcal{H}$and $K =$ J$\mathcal{H}$ forsome ideals B, J of R , that is $h^2 K = (B\mathcal{H})^2$ J$\mathcal{H} \subseteq$ I $\mathcal{H}$. But $\mathcal{H}$ is a finitelygenerated multiplication , then by lemma 2.18 we have B^2J$\subseteq$ I + $\text{ann}_\text{R}(\mathcal{H})$. Since $\text{ann}_\text{R}(\mathcal{H}) \subseteq$ I then I + $\text{ann}_\text{R}(\mathcal{H})$ = I, that B^2J$\subseteq$ I. But I is STPNS-Aideal of R, then either BJ $\subseteq$ I + $J(R) \cap soc(R)$ or $B^2 \in \big[\text{I} + \big(J(R) \cap soc(R)\big):_R R\big] = \text{I} + \big(J(R) \cap soc(R)\big)$. It means that either Bj$\mathcal{H} \subseteq$ I$\mathcal{H}$ + $(J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $B^2\mathcal{H} \subseteq$ I$\mathcal{H}$ + $(J(R)\mathcal{H} \cap soc(R)\mathcal{H})$. Since $\mathcal{H}$ is faithful then by lemmas 2.8, 2.9 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. So that either hK $\subseteq$I$\mathcal{H}$ + $(J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $h^2 \subseteq$I$\mathcal{H}$ + $(J(\mathcal{H}) \cap soc(\mathcal{H}))$. Therefore by corollary 3.4 I$\mathcal{H}$ is STPNS-Asubmodule of $\mathcal{H}$.

(⇐) Let u^2J $\subseteq I$ for $u \in R$ and J is an ideal of R , it means that u^2J $\mathcal{H} \subseteq I\mathcal{H}$. But $\mathcal{H}$ is multiplication, then $u^2 J\mathcal{H} = u^2 L \subseteq I\mathcal{H}$. But $I\mathcal{H}$ is a STPNS-Athen by lemma 2.7 either $uL \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$ or $u^2\mathcal{H} \subseteq I\mathcal{H} + (J(\mathcal{H}) \cap soc(\mathcal{H}))$.But $\mathcal{H}$ is faithful, then by lemmas 2.5, 2.6 $Soc(\mathcal{H}) = Soc(R)\mathcal{H}$ and $J(\mathcal{H}) = J(R)\mathcal{H}$. Thus either $uJ\mathcal{H} \trianglelefteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ or $u^2\mathcal{H} \trianglelefteq I\mathcal{H} + (J(R)\mathcal{H} \cap soc(R)\mathcal{H})$ .That is either $uJ \subseteq I + (J(R) \cap soc(R))$ or $u^2 \in I + (J(R) \cap soc(R)) = [I + (J(R) \cap soc(R)):_R R]$. Therefore by lemma 2.7 I is STPNS-Aideal of R.

Proposition 4.5

Let $\mathcal{H}$ be a faithfulfinitely generatedmultiplicationR–module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$. Then the followingstatements are valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R .

Proof

(1) ⇔ (2) Follows by proposition 4.4.

(2) ⇒ (3) Suppose that $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R. But $\mathcal{H}$ is a multiplication , then $\mathcal{F} = [\mathcal{F}:_R \mathcal{H}]\mathcal{H}$. Put $\mathrm{I} = [\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R and $\mathcal{F} = \mathrm{I}\mathcal{H}$.

(3) ⇒ (2) Suppose that $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal of R. Since $\mathcal{H}$ is a multiplication , $\mathcal{F} = [\mathcal{F}:_R \mathcal{H}]\mathcal{H} = \mathrm{I}\mathcal{H}$. But $\mathcal{H}$ is a faithfulfinitely generated , then by lemma 2.18 $\mathcal{H}$ is cancellation , therefore $[\mathcal{F}:_R \mathcal{H}] = \mathrm{I}$ is STPNS-Aideal of R.

Proposition 4.6

Let $\mathcal{H}$ be a finitelygenerated multiplicationZ–regular content R – module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$ with$\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the followingstatements are valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R with $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq I$.

Proof

(1) ⇔ (2) Follows byproposition 4.1.

(2) ⇒ (3) Let $\mathcal{F} \subseteq \mathcal{H}$, then $\mathcal{F} = [\mathcal{F}:_\mathrm{R} \mathcal{H}]\mathcal{H}$ (for $\mathcal{H}$ isamultiplication) . Put $[\mathcal{F}:_\mathrm{R} \mathcal{H}] = \mathrm{I}$, impliesthat I is STPNS-Aideal of R with $\mathrm{ann}_\mathrm{R}(\mathcal{H}) = [0:_\mathrm{R} \mathcal{H}] \subseteq [\mathcal{F}:_\mathrm{R} \mathcal{H}] = \mathrm{I}$, that is $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq \mathrm{I}$.

(3) ⇒ (2) Supposethat $\mathcal{F} = \mathrm{I}\mathcal{H}$ forsome STPNS-Aideal of R with $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq \mathrm{I}$. But $\mathcal{H}$ isamultiplication , then $\mathcal{F} = [\mathcal{F}:_\mathrm{R} \mathcal{H}]\mathcal{H} = \mathrm{I}\mathcal{H}$, and since $\mathcal{H}$ is a finitelygenerated then by lemma 2.20 $\mathcal{H}$ is a weakcancellation , it means that $[\mathcal{F}:_\mathrm{R} \mathcal{H}] + \mathrm{ann}_\mathrm{R}(\mathcal{H}) = \mathrm{I} + \mathrm{ann}_\mathrm{R}(\mathcal{H})$. But $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq \mathrm{I}$, it means that $\mathrm{I} + \mathrm{ann}_\mathrm{R}(\mathcal{H}) = \mathrm{I}$ and $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq [\mathcal{F}:_\mathrm{R} \mathcal{H}]$, it means that $[\mathcal{F}:_\mathrm{R} \mathcal{H}] + \mathrm{ann}_\mathrm{R}(\mathcal{H}) = [\mathcal{F}:_\mathrm{R} \mathcal{H}]$. Thus $[\mathcal{F}:_\mathrm{R} \mathcal{H}] = \mathrm{I}$, but I is STPNS–2-A ideal of R, itfollows that $[\mathcal{F}:_\mathrm{R} \mathcal{H}]$ is STPNS-Aideal of R.

Proposition 4.7

Let $\mathcal{H}$ be a finitelygenerated multiplicationprojective R–module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$ with $\mathrm{ann}_\mathrm{R}(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the followingstatements are valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ forsome STPNS-Aideal I of R withann$_\mathrm{R}(\mathcal{H}) \subseteq I$.

Proof

(1) ⇔ (2) Followsby proposition 4.2.

(2) ⇔ (3) Followssimilarly as inproposition 4.6.

Proposition 4.8

Let $\mathcal{H}$ be a finitelygenerated multiplicationnon-singular content R–module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$ with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the following statementsare valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R withann$_\mathrm{R}$ $(\mathcal{H}) \subseteq I$.

Proof

$(1) \Leftrightarrow (2)$ Follows by proposition 4.3.

$(2) \Leftrightarrow (3)$ Follows similarly as in proposition 4.6.

5. Conclusion

We will present the most important propositions in this research:

. Let $\mathcal{H}$ be a faithfulfinitely generatedmultiplication R–module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$. Then thefollowing statementsare valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R .

. Let $\mathcal{H}$ be a finitelygenerated multiplicationZ–regular content R – module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$ with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the followingstatements are valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq I$.

. Let $\mathcal{H}$ be a finitelygenerated multiplicationprojective R–module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$ with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then the followingstatements are valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq I$.

. Let $\mathcal{H}$ be a finitelygenerated multiplicationnon-singular content R–module, and $\mathcal{F}$ is a propersubmodule of $\mathcal{H}$ with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq [\mathcal{F}:_R \mathcal{H}]$. Then thefollowing statements are valent :

1. $\mathcal{F}$ is STPNS-Asubmodule of $\mathcal{H}$.
2. $[\mathcal{F}:_R \mathcal{H}]$ is STPNS-Aideal of R .
3. $\mathcal{F} = \mathrm{I}\mathcal{H}$ for some STPNS-Aideal I of R with $\mathrm{ann_R}\,(\mathcal{H}) \subseteq I$.

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