Solution of 2nd Order Nonlinear Three-Point Boundary Value Problems By Semi-Analytic Technique

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Abstract

In this paper, we present new algorithm for the solution of the second order nonlinear three-point boundary value problem with suitable multi boundary conditions. The algorithm is based on the semi-analytic technique and the solutions which are calculated in the form of a rapid convergent series. It is observed that the method gives more realistic series solution that converges very rapidly in physical problems. Illustrative examples are provided to demonstrate the efficiency and simplicity of the proposed method in solving this type of three point boundary value problems.

Keywords: Differential Equation, Multi-point Boundary Value Problem, Approximate Solution.

Introduction

Some problems which have wide classes of application in science and engineering have usually been solved by perturbation methods. These methods have some limitations, e.g., the approximate solution involves a series of small parameters which poses difficulty since the majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters do lead to ideal solution while in most other cases, unsuitable choices lead to serious effects in the solutions [1]. The semi-analytic technique employed here, is a new approach for finding the approximate solution that does not require small parameters, thus over-coming the limitations of the traditional perturbation techniques. The method was first proposed by Grundy (2003) and successfully applied by other researchers like Grundy (2003- 2007) who examined the feasibility of using two points Hermite interpolation as a systematic tool in the analysis of initial-boundary value problems for nonlinear diffusion equations. In 2005 Grundy analyzed initial - boundary value problems involving nonlocal nonlinearities using two points Hermite interpolation[1], also, in 2006 he showed how two-points Hermite interpolation can be used to construct polynomial representations of solutions to some initial-boundary value problems for the inviscid Proudman-Johnson equation. In 2008, Maqbool [2] used a Semi-analytical Method to Model Effective SINR Spatial Distribution in WiMAX Networks. Also ,in 2008, Debabrata[3] studied Elasto-plastic strain analysis by a semi-analytical method .In 2009, Mohammed [4] investigated the feasibility of using osculatory interpolation to solve two points second order boundary value problems .In 2011, Samaher[5] used semi-analytic technique for solving High order ordinary two point BVPs.

The existence of positive solutions for multi-point boundary value problems is one of the key areas of research these days owing to its wide application in engineering like in the modeling of physical problems involving vibrations occurring in a wire of uniform cross section and composed of material with different densities, in the theory of elastic stability and also its applications in fluid flow through porous media.

Kwong [6], studied of multiple solutions of Two and multi-point BVPs of nonlinear second order ODE as fixed points of a cone mapping.

Thompson [7] established existence results to three-point BVPs for nonlinear second order ODE with nonlinear boundary conditions.

Castelani [8] studied the existence of solution of second order nonlinear three-point BVPs using Fixed Point Theorems.

In this paper we use two-point osculatory interpolation, essentially this is a generalization of interpolation using Taylor polynomials. The idea is to approximate a function y by a polynomial P in which values of y and any number of its derivatives at given points are fitted by the corresponding values and derivatives of P.

We are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0,1] where a useful and succinct way of writing osculatory interpolation P_{2n+1} of degree 2n + 1 was given for example by Phillips [9] as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ y^{(j)}(0)\, q_j(x) + (-1)^j\, y^{(j)}(1)\, q_j(1-x) \} \tag{1}$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j! \tag{2}$$

so that (1) with (2) satisfies :

$$y^{(j)}(0) = P_{2n+1}^{(j)}(0)\ ,\quad y^{(j)}(1) = P_{2n+1}^{(j)}(1)\ ,\quad j = 0, 1, 2,\ldots, n\ .$$

Implying that P_{2n+1} agrees with the appropriately truncated Taylor series for y about x = 0 and x = 1. We observe that (1) can be written directly in terms of the Taylor coefficients a_i and b_i about x = 0 and x = 1 respectively, as:

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{a_j Q_j(x) + (-1)^j b_j Q_j(1-x)\} \qquad (3)$$

2. Solution of Three-Point 2nd Order Nonlinear BVP's for ODE

A general form of 2nd - order BVP's is:-

$$y''(x) = f(x, y, y')\ ,\qquad 0 \le x \le 1 \qquad (4a)$$

Subject to the boundary conditions:

$$y(0) = Ay\ ,\quad y(1) = B\,y(\eta)\ ,\ \text{where } \eta \in (0, 1)\ ,\ B \in R \qquad (4b)$$

The simple idea of semi - analytic method is to use a two - point polynomial interpolation to replace y in (4) by a P_{2n+1} which enables any unknown derivatives of y to be computed, the first step therefore is to construct the P_{2n+1}. To do this we need to evaluate Taylor coefficients of y about x =0:

$$y = \sum_{i=0}^{\infty} a_i\, x\, x^i \qquad \ni\ a_i = y^{(i)}(0)/i!\ , \qquad (5a)$$

Then insert the series form (5a) into (4a) and put x= 0 and equate the coefficients of powers of x to obtain a_i , $i \ge 2$. Also, evaluate Taylor coefficients of y about $x = \eta$:

$$y = \sum_{i=0}^{\infty} b_i\, (x - c_i(x-\eta)^i \qquad \ni\ b_i = c_i = y^{(i)}(\eta)/i!\ , \qquad (5b)$$

Then insert the series form (5b) into (4a) and put x = η and equate coefficients of powers of (x- η) , to obtain c_i , $i \ge 2$. Also, evaluate Taylor coefficients of y about x =1 **:**

$$y = \sum_{i=0}^{\infty} b_i\, (x-1)^i \qquad \ni\ b_i = y^{(i)}(1)/i!\ , \qquad (5c)$$

Then insert the series form (5c) into (4a) and put x = 1 and equate coefficients of powers of (x-1) **,** to obtain b_i , $i \ge 2$,then derive equation (4a) with respect to x to obtain new form of equation say (6) then, insert the series form (5a) into (6) and put x = 0 and equate coefficients of powers of x**,** to obtain a_3 , also insert the series form (5b) into (6) and put x = η and equate coefficients of powers of x**,** to obtain c_3 , also insert the series form (5c) into (6) and put x = 1 and equate coefficients of powers of x**,** to obtain b_3 , now iterate the above process many times to obtain a_4, c_4, b_4, then a_5, c_5, b_5 and so on, that is ,we can get a_i , c_i and b_i for all $i \ge 2$, the resulting equations can be solved using MATLAB to obtain a_i ,c_i and b_i for all $i \ge 2$, the notation implies that the coefficients depend only on the indicated unknowns a_0 , a_1, c_0 , c_1 , b_0 , b_1, and we get a_0 , b_0 defined by c_0 , by the boundary conditions. Now, divided the domain [0,1] by η into two subinterval [0,η] and [η,1] then construct a $P_{2n+1}(x)$ for each subinterval from these coefficients (a_iś , c_iś and b_iś) by the following :

$$p_{2n+1}(x) = \sum_{i=0}^{n} \{a_i Q_i(x) + (-1)^i c_i Q_i(\eta - x)\} + \sum_{i=0}^{n} \{c_i Q_i(x-\eta) + (-1)^i b_i Q_i(1-x)\} \ldots \quad (7a)$$

$$\text{Where}\quad Q_j(x)/j! = (x^j/j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s\ , \qquad (7b)$$

We see that (7a) have 4 unknown coefficients a_1, c_1, b_1 and $c_0 = b_0$**.**

Now**,** to evaluate the remainder coefficients integrate equation (4a) on [0, x] to obtain:

$$y'(x) - y'(0) - \int_0^x f(x, y, y')\, dx = 0$$

i.e. $$y'(x) - a_1 - \int_0^x f(x, y, y')\, dx = 0 \qquad (8a)$$

and again integrate equation (8a) on [0, x] to obtain:

$$y(x) - y(0) - a1\, x - \int_0^x (1 - x) f(x, y, y')\, dx = 0$$

i.e. $$y(x) - a_0 - a_1 x - \int_0^x (1 - x) f(x, y, y')\, dx = 0 \qquad (8b)$$

use P_{2n+1} as a replacement of y, y' in (8) and putting x= η in all above integration . Again integrate equation (4a) on [η , x] to obtain :

$$y'(x) - c_1 - \int_\eta^x f(x, y, y')\, dx = 0 \qquad (9a)$$

and again integrate equation (9a) on [η , x] to obtain:

$$y(x) - c_0 - c_1 x - \int_\eta^x (1 - x) f(x, y, y')\, dx = 0 \qquad (9b)$$

Use P_{2n+1} as a replacement of y, y' in (9) and putting x= 1 in all above integration.

We have system of 4 equations (8), (9) with 4 unknown coefficients which can be solved using the MATLAB package, version 7.9**,** to get the unknown coefficients**,** thus insert it into (7a), thus (7a) represent the solution of (4) **.**

Now, we introduce many examples of 2nd order three-point BVP's for ODE to illustrate suggested method. Accuracy and efficiency of the suggested method is established through comparison with other methods.

Example 1

Consider the following nonlinear, 2nd order, 3point BVP's:

$$y'' + 2 = 0 \quad , \quad 0 \le x \le 1 \,,$$

Subject to the BC: y(0)= 0 , y(1) = 3 y(0.5)
The exact solution for this problem is: $y(x) = -0.5\, x - x^2$

Now, we solve this equation using semi-analytic method from equation (7) we have: $P_3 = -x^2 - 0.5\, x$

Liu[10] constructed a two-stage Lie-group shooting method for finding solution of this example, Figure(1a) compared the numerical result of [10] with exact solution. It can be seen that the numerical error of x is of order of 10^{-5} as shown in Figure (1b) and Figure(2) illustrates the accuracy of suggested method P_3

Example 2

Consider the following linear, 2nd order, 3 point BVP's:

$$y'' + 6x = 0 \quad , \quad 0 \le x \le 1$$

With BC: $y(0) = 0$, $y(1) = y(0.12) / 2$,

The exact solution for this problem is $y(x) = 31223\, x / 29375 - x^3$.

Now, we solve this equation using semi - analytic method from equations (7) we have:
$P_3 = 1.06291063829787234042553191489 36\, x - x^3$

Figure (3) illustrate the comparison between the exact and suggested method P_3.

E. V. Castelani [11], solved this example using iterative method with mesh size

$h = 0.1$and $h = 0.05$ the maximum absolute error in the k-th iteration are given in Table1, but the maximum absolute error of suggested method P_3 is 0.111022302462516e-015.

Example 3

Consider the linear, 2nd order, 3 point BVP's.

$$y'' + \cos x = 0 \quad , \quad 0 \le x \le 1$$

Subject to the BC: $y(0) = 0$, $y'(1) = -3y(1/3) / 2$
With exact solution is: $y = (2x\sin 1) /3 - x\cos(1/3) + x + \cos x - 1$
Now, we solve this equation using semi-analytic method from equation (7), if n = 7, we have:

$$P_{15} = 2.18915110^{-13}\, x^{15} - 1.1627290310^{-11}\, x^{14} - 1.0514471510^{-12} x^{13} + 0.00000000209094434 x^{12} - 4.4374915110^{-12} x^{11} - 0.000000275569894\, x^{10} - 1.3142893210^{-12}\, x^9 + 0.0000248015875\, x^8 - 0.00138888889\, x^6 + 0.0416666667\, x^4 - 0.5\, x^2 + 0.61602371\, x$$

For more details, Table (2) gives the results for different nodes in the domain, for n=7, i.e. P_{15} and errors obtained by comparing it with the exact solution. Figure (5) illustrate the comparison between the exact and suggested method P_{15}.

Liu [10] construct a two-stage Lie-group shooting method for finding solution of this example, figure(4a) compared the numerical result of [10]
with exact solution. It can be seen that the numerical error of x is of order of 10^{-5} as shown in Figure (4b).

3. Conditioning of BVP's

In particular, BVP's for *which* a small change to the ODE or boundary conditions results in a small change to the solution must be considered, a BVP's that has this property is said to be well-conditioned. Otherwise, the BVP's is said to be ill-conditioned[12] . To be useful in applications, a BVP's should be well posed. This means that given an input to the problem there exists a unique solution, which depends continuously on the input. Consider the following 2nd order BVP's

$$y''(x) = f(x, y(x), y'(x)) \quad , \quad x \in [0, 1] \qquad (10a)$$

With BC: $y(0) = A$, $y(1) = B\, y(\eta)$, where $\eta \in (0, 1)$ (10b)

For a well-posed problem we now make the following assumptions:
1. Equation (10) has an approximate solution $P \in C^n[0, 1]$, with this solution and $\rho > 0$, we associate the spheres:

$S\rho(P(x)) = \{ y \in IR^n : | P(x) - y(x) | \leq \rho \}$

2. f(x, P(x), P'(x)) is continuously differentiable with respect to P, and $\partial f / \partial P$ is continuous.

This property is important due to the error associated with approximate solutions to BVP's, depending on the semi-analytic technique, approximate solution $\check{y}$ to the linear 2nd order BVP's (10) may exactly satisfy the perturbed ODE :

$$\check{y}'' = d(x)\, \check{y}' + q(x)\, \check{y} + r(x) \quad , \ 0 < x < 1 \tag{11a}$$

Where r: $R \rightarrow R^m$, and the linear BC:

$$B_0\, \check{y}(0) + B_1\, \check{y}(1) = \beta + \propto \tag{11b}$$

Where $\beta + \propto = \sigma$, $\sigma \in R^m$ and $\{\propto, \beta, \sigma\}$ are constants. If $\check{y}$ is a reasonably good approximate solution to (10), then $\|r(x)\|$ and $\|\sigma\|$ are small. However, this may not imply that $\check{y}$ is close to the exact solution y. A measure of conditioning for linear BVP's that relates both $\|r(x)\|$ and $\|\sigma\|$ to the error in the approximate solution can be determined. The following discussion can be extended to nonlinear BVP's by considering the variational problem on small sub domains of the nonlinear BVP's [13].

Letting: $e(x) = |\check{y}(x) - y(x)|$; then subtracting the original BVP's (10) from the perturbed BVP's (11) results in:

$$e''(x) = \check{y}''(x) - y''(x) \tag{12a}$$

$$e''(x) = d(x)\, e'(x) + q(x)\, e(x) + r(x) \quad ; \ 0 < x < 1 \tag{12b}$$

With BC: $B_0\, e(0) + B_1\, e(1) = \sigma$ (12c)

However, the form of the solution can be furthered simplified by letting:
$\Theta(x) = Y(x)\, Q^{-1}$; where Y is the fundamental solution and Q is defined in (7b). Then the general solution can be written as:

$$e(x) = \Theta(x)\, \sigma + \int_0^1 G(x, t)\, r(t)\, dt \tag{13}$$

Where G(x, t) is Green's function [14], taking norms of both sides of (13) and using the Cauchy - Schwartz inequality [14] results in :

$$\|e(x)\|_\infty \leq k_1 \|\sigma\|_\infty + k_2 \|r(x)\|_\infty \tag{14}$$

Where $k_1 = \|Y(x)Q^{-1}\|_\infty$; and $k_2 = \sup\limits_{0 \leq x \leq 1} \int_0^1 \|G(x, t)\|_\infty\, dt$,

In (14), the L_∞ norm, sometimes called a maximum norm, is used due to the common use of this norm in numerical BVP's software. For any vector $v \in R^N$, the L_∞ norm is defined as: $\| v \|_\infty = \max\limits_{1 \leq i \leq N} | v_i |$.

The measure of conditioning is called the conditioning constant k, and it is given by:

$$k = \max (k_1, k_2) \tag{15}$$

When the conditioning constant is of moderate size, then the BVP's is said to be well-conditioned.

Referring again to (14), the constant k thus provides an upper bound for the norm of the error associated with the perturbed solution,

$$\|e(x)\|_\infty \leq k\, [\|\sigma\|_\infty + \|r(x)\|_\infty] \tag{16}$$

It is important to note that the conditioning constant only depends on the original BVP's and not the perturbed BVP's. As a result, the conditioning constant provides a good measure of conditioning that is independent of any numerical technique that may cause such

perturbations. The well conditioned nature of a BVP's and the local uniqueness of its desired solution are assumed in order to numerically solving of the problem.

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Table No. (1): Maximum absolute error of iterative method in [11]

Iteration	maximum error when h=0.1	maximum error when h=0.05
1	.629106e-1	.629106e-1
2	.377465e-2	.377463e-2
3	.226494e-3	.226478e-3
10	.165500e-7	.300000e-9
20	.165500e-7	.300000e-9
30	.165500e-7	.300000e-9

Table No.(2): the comparison between exact solution & P_{15}

x_i	Suggested method P_{15}	Exact y(x)	Error $\|y(x) - P_{15}\|$
0	0	0	0
0.1	0.056606536300412	0.056606536300412	0.006938893903907e-015
0.2	0.103271319886014	0.103271319886014	0.138777878078145e-015
0.3	0.140143602192764	0.140143602192764	0
0.4	0.167470478092429	0.167470478092429	0.138777878078145e-015
0.5	0.185594417002303	0.185594417002303	0
0.6	0.194949841043994	0.194949841043994	0.083266726846887e-015
0.7	0.196058784441190	0.196058784441191	0.055511151231258e-015
0.8	0.189525677526254	0.189525677526253	0.027755575615629e-015
0.9	0.176031307472139	0.176031307472138	0.083266726846887e-015
1	0.156326016092000	0.156326016092000	0.083266726846887e-015
		S.S.E.	6.321865042451102e-032

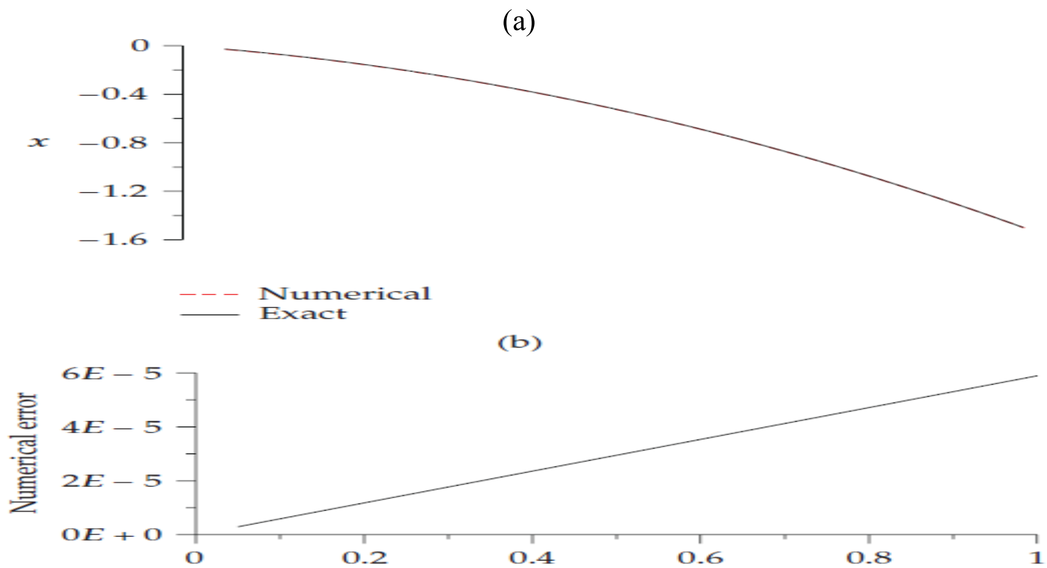

Figure No.(1) : solution of example1given in[10] :
(a)comparing numerical and exact solutions, (b) displaying the error

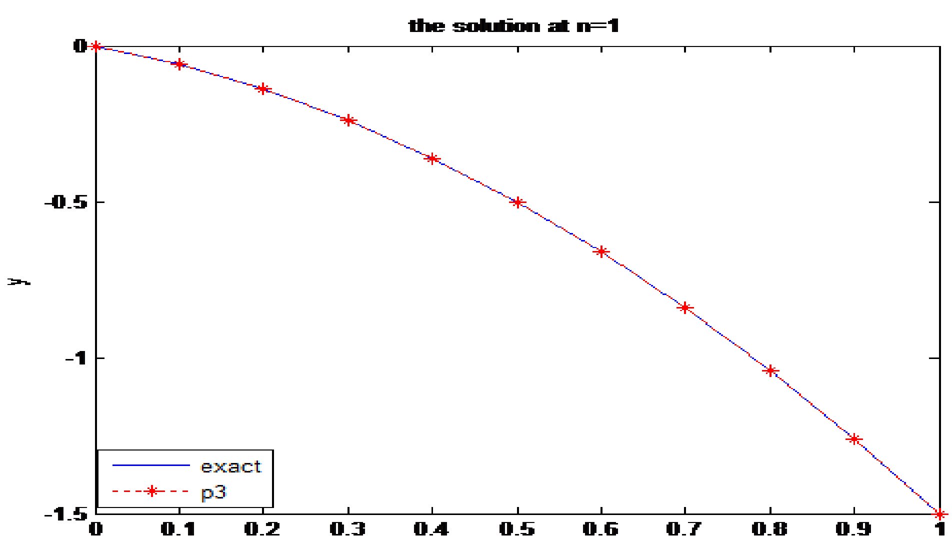

Figure No.(2) : A comparison between exact & P_3 of example1

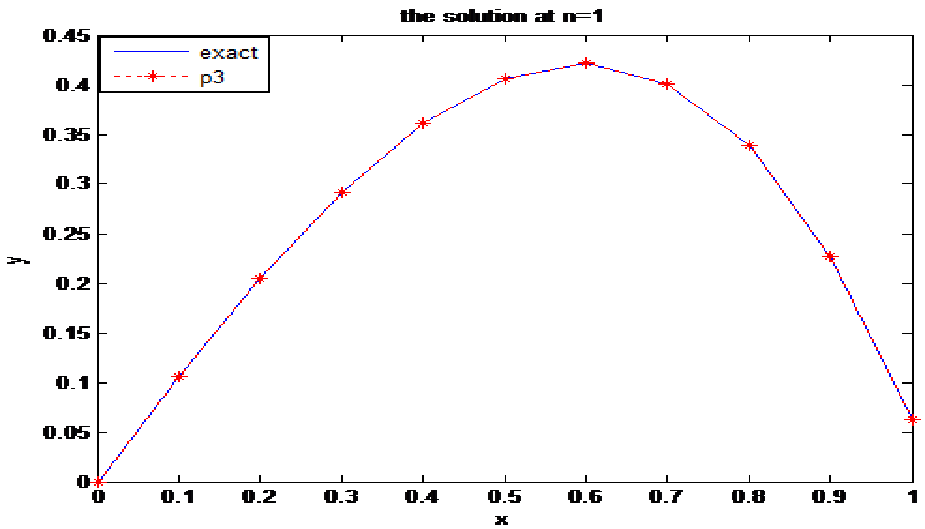

Figure No.(3) : A comparison between exact & P_3 of example2

(a)

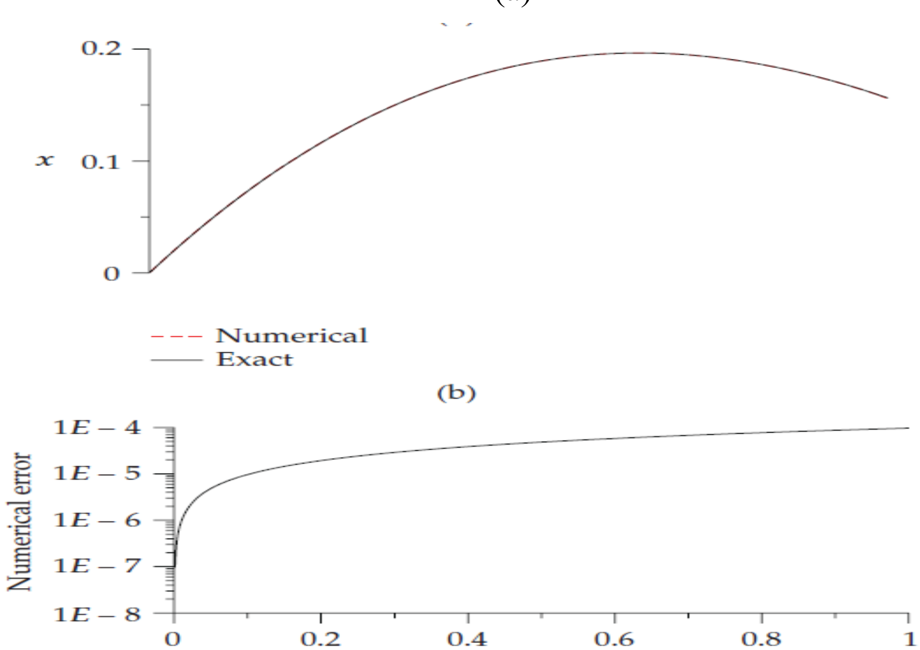

Figure No.(4) : solution of example 3 given in [10] :
(a) comparing numerical and exact solutions, (b) displaying the error.

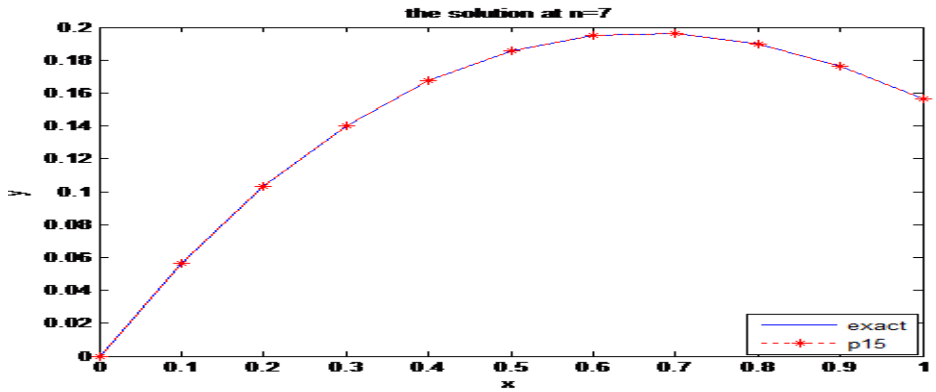

Figure No.(5) : A comparison between exact & P_{15} of example3

حل مسائل القيم الحدودية ذو ثلاث نقاط من الرتبة الثانية باستخدام التقنية شبه التحليلية

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الخلاصة

في هذا البحث نعرض خوارزمية جديدة لحل معادلات تفاضلية اعتيادية من الرتبة الثانية ذات الشروط الحدودية عند ثلاث نقاط ,الخوارزمية تعمل على أساس التقنية شبه التحليلية والحل حسب بصيغة متسلسلة سريعة التقارب وهذا يتضح أكثر في المسائل الفيزيائية ,و ناقشنا بعض الأمثلة لتوضيح الدقة و الكفاءة وسهولة أداء الطريقة المقترحة في حل هذا النوع من المسائل الحدودية متعددة النقاط .

الكلمات المفتاحية: المعادلات التفاضلية, مسائل قيم حدودية متعددة النقاط , الحل التقريبي