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Purely co-Hopfian Modules

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Abstract

Let R be an associative ring with identity and M a non – zero unitary R-module.In this paper we introduce the definition of purely co-Hopfian module, where an R-module M is said to be purely co-Hopfian if for any monomorphism $f \in End(M)$, Imf is pure in M and we give some properties of this kind of modules.

Keywords: co-Hopfian module, semi co-Hopfian module, purely co-Hopfian module

Introduction and Preliminaries

Let R be an associative ring with identity and M a non – zero unitary R – module, Recall that a module M is called co-Hopfian if any injective endomorphism of M is an isomorphism [1].A module M is called semi co-Hopfian if any injective endomorphism of M has a direct summand image that means any injective endomorphism of M splits [1].A ring R is semi co-Hopfian if R is semi co-Hopfian R - module. Clearly, any co-Hopfian is semi co-Hopfian but the converse is not true in general as, for example $M = Q^N = Q \oplus Q \oplus \ldots$,as Z-module is semi co-Hopfian but it is not co-Hopfian [1]. A submodule N of M is called pure if IM∩N=IN for each ideal of R,[8].It is well–known every direct summand of a module M is pure submodule but the converse is not true in general [2].This leads us to introduce the following concept, namely purely co-Hopfian module.

Definition 1.1

An R- module M is called purely co-Hopfian if for any monomorphism $f \in End(M)$, Imf is pure in M.

Remarks and examples 1.2

1. Every semi co-Hopfian module is purely co-Hopfian.
2. Every F- regular module M is purely co-Hopfian, where M is F- regular if every submodule of M is pure,[3].
3. Every semi simple R-module is purely co-Hopfian.
4. If M is pure simple (that means M has only two pure submodules 0 ,M) [2], then M is purely co-Hopfian.

lcmma 1.3

The following are equivalent for an R-module M:

1. M is purely co-Hopfian.
2. Any submodule N of M such that $N \cong M$, N is pure in M.

Proof (1)→(2)

Let $N \leq M$, $N \cong M$. Then there exists $\alpha : M \to N$, α is an isomorphism. Hence $M \xrightarrow{\alpha} N \xrightarrow{i} M$ where i : N → M is the inclusion map , and this implies $i \circ \alpha \in End(M)$, $i \circ \alpha$ is monomorphism. So (i o α) (M) is pure in M . Thus i (α (M)) = i (N) = N is pure in M.

(2)→(1): let $f \in End(M)$, f is monomorphism. Hence $f(M) \cong M$ and so by (2), f (M) is pure in M.

Proposition 1.4

The following are equivalent for a ring R

1. R is purely co-Hopfian.
2. R is semi co-Hopfian.

Proof (1)→(2)

Let f: R → R, f is R-monomorphism. Hence f (R) = < a > for some $a \neq 0 \in$ R. Since R is purely co-Hopfian, < a > is pure ideal at R , hence < a > = < a^2 > (since < a > $\in$ <a > = < a > < a >) .Thus a = ra^2 for some r $\in$ R, this implies ra is idempotent and< a > = < ra >. It follows that < a > is a direct summand .

The proof of the part (2)→(1) is clear.

By combining proposition 1.4 and proposition 2.3 from [1] we get the following result.

Corollary 1.5

The following are equivalent for any a ring R :

1. R is purely co-Hopfian.
2. R is semi co-Hopfian.
3. ann (a) = 0 , a $\in$ R then < a > is a direct summand .
4. If ann (a) = 0 , a $\in$ R then < a > = R .
5. Every R – isomorphism < a > → R, a $\in$ R, extends to R.

Proof

(1) ↔ (2): see proposition 1.4

(2) ↔(3) ↔(4) ↔(5): (see proposition 2.3), [1].

Corollary 1.6

If R is a ring with two idempotent 0,1 then the following statement are equivalent : -

1. R is co-Hopfian.
2. R is semi co-Hopfian.
3. R is purely co-Hopfian.

Proof

(1) → (2): it is clear

(2) ↔ (3) :by proposition 1.4

(3)→(2): Let f : R → R , f is monomorphism then f (R) = < a > for some a $\in$ R , $a \neq 0$, but I = < a > is a direct summand of R (since R is Semi co-Hopfian) then < a > is generated by idempotent. Since $a \neq 0$, hence a = 1 and < a > = R.Thus f is onto and we get R is co-Hopfian.

Recall that module M has C2 if for any submodule N of M which is isomorphic to a direct summand of M, is a direct summand of M [4].

Corollary 1.7

If R is a ring only idempotent 0 and 1 the following equivalent:

1. R has C_2.
2. R is co-Hopfian.
3. R is purely co-Hopfian.
4. R is semi co-Hopfian.

Proof (1) → (2)

let f: R → R be monomorphism. To prove that R is co-Hopfian, we must prove f is an isomorphism.Since f is monomorphism,f (R) $\cong$ R . But R is C_2 by (1) and R is direct summand of R, hence f(R) is direct summand of R.It follows that f (R) is generated by idempotent.Since R has only 2 – idempotent namely 0 , 1 and f (R) $\neq$ 0 ,then f (R) = < 1 > thus f (R) = R and so that f is an isomorphism .

(2) → (3 :It is clear.

(3) → (4):It follows by proposition (1.4) .

(4) → (1): It follows by proposition 2.4 [1] .

Corollary 1.8

Let R be an integral domain. Then the following are equivalent:

1. R is co-Hopfian.
2. R is semi co-Hopfian.
3. R is purely co-Hopfian.
4. R is field.

Proof

(1) ↔ (2) ↔ (3) :It follows by corollary 1.6

(1) → (4) :Let a $\in$ R , a $\neq$ 0 then ann (a) = 0 since R is an integral domain . By corollary 1.5, < a > = R. Hence a is an invertible element.Then R is a field.

(4) → (1) :Since R is a field, R has only two ideals namely R, (0). Hence for any f: R → R, f is R – monomorphism f (R) $\neq$ 0. Hence f (R) = R.Thus f is onto then R is co-Hopfian.

Proposition 1.9

Any direct summand of purely co-Hopfian module is purely co-Hopfian.

Proof

Let N be a direct summand of M, so M = N $\oplus$ A for some submodule A of M .Let f: N → N be monomorphism. Define g : M →M by g(n+a) = f (n)+a where n $\in$ N , a $\in$ A it is easy to see that g is monomorphism Hence g (M) = f (A) $\oplus$ N . Since M is purely co-Hopfian, g (M) is pure in M.To prove f (N) pure in N,let I be any ideal of R ,

IM ∩ g (M) = I g (M) ,

I (N $\oplus$ A) ∩ (f (N) $\oplus$ A) = I (f (N) $\oplus$ A) ,

(IN $\oplus$ I A) ∩ (f (N) $\oplus$ A) =(IN ∩ f (N)) $\oplus$ (IA ∩ A)= I f (N) $\oplus$ IA ,

(IN ∩ f (N)) $\oplus$ IA = If (N) $\oplus$ IA, IN ∩ f (N) = If (N) .

Thus f(N)is pure in N and so N is purely copfian.

Recall that a submodule N of M is a non- summand if N is not direct summand of M [1].

Proposition 1.10

Let M be an R- module such that every non summand N of M is purely co-Hopfian , if for any non – summand submodule N of M, N is purely co-Hopfian, then M is purely co-Hopfian.

Proof

Suppose M is not purely co-Hopfian then there exists N < M, N $\cong$ M, N is not pure in M by lemma (1.3). But N is not pure implies N is not summand. Hence by hypothesis N is purely co-Hopfian which implies M is purely co-Hopfian which is a contradiction.

Recall that M is fully stable if for any submodule N of M, f: N → M is then f (N) $\leq$ N [5].

Proposition 1.11

Let M = $M_1 \oplus M_2$, M is fully stable.Then M is purely co-Hopfian if and only if M_1, M_2 are purely co-Hopfian

Proof

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It follows by proposition 1.9.Conversely, Let $f: M \to M$ be monomorphism put $f_1 = f|_{M1}$,$f_2 = f|_{M2}$ .Since M is fully stable,$f_1(M_1) \leq M_1$ and $f_2(M_2) \leq M_2$. Since f is monomorphism, f_1, f_2 are monomorphism. Hence $f_1(M_1)$, $f_2(M_2)$ are pure in M_1, M_2 respectively. Hence$f_1(M_1) \oplus f_2(M_2)$ is pure in M [2]. But it is easy to see that $f(M) = f_1(M_1) \oplus f_2(M_2)$. Thus f(M)is pure in M .

Corollary 1. 12

Let $M=\oplus_{i \in I} M_i$, M is fully stable M is purely co-Hopfian if and only if M_i is purely co-Hopfian for all $i \in I$.

Recall that M is torsion free if rm = 0 then r = 0 or m = 0 for any $r \in R$, $m \in M$. Note that torsion free module needs not purely co-Hopfian, for example Z as Z-module. Now we have the following result which improves proposition 2.13 in [1].Which states that ,let R be a commutative domain and let M be a torsion free semi co-Hopfian R-module .Then M is injective .

Proposition 1.13

Let R be an integral domain and let M be a torsion free purely co-Hopfian R – module .Then M is injective R-module.

Proof

Let $a \in R$, $a \neq 0$. Define $f : M \to M$ by $f(m) = am$, for all $a \in M$. Then f is monomorphism, hence $f(M) = aM$ is pure submodule in M since M is purely co-Hopfian.Thus $IM \cap f(M) = I f(M)$ for any ideal I of R. Take $I = < a >$. Hence $(a) M \cap aM = (a). aM$ thus $aM = a^2M$ .Now for any $m \in M$, $am = a^2m_1$, so $a(m - am_1) = 0$.Hence $m-am_1=0$ since M is torsion free and so $m = am_1$. Thus we have M = aM, that is M divisible torsion free, hence M is injective.

Proposition 1.14

If M has Dcc on non pure submodule (that means has Dcc on not pure submodule), then M is purely co-Hopfian.

Proof

Suppose M is not purely co-Hopfian, then by lemma 1.3, there existsM_1 (not pure submodule of M) such that $M_1 \cong M$. Hence M_1 is not purely co-Hopfian and, so there exists M_2 submodule of M_1 which is not pure of $M_2 \cong M_1$. By repeating this argument we have strictly descending chain $M_1 \supsetneq M_2 \supsetneq \ldots$Moreover M_i is not pure in M, for all i = 1, 2,…… . To show this M_1 is not pure in M (by proof). If M_2 pure in M ,then M_1 pure in M [2,Rem.7.2(1)] , which is a contradiction . Thus M_2 is not pure in M.Similary M_i is not pure in M, for all i = 3, 4,… . Thus $M_1 \supsetneq M_2 \supsetneq \ldots$. is strictly descending chain of non pure submodule of M, which is a contradiction.Thus M is purely co-Hopfian.

Remark 1.15

The endomorphism ring of purely co-Hopfian module need not be purely co-Hopfian.

Example 1.16

The Z – module Z_{p^∞} is co-Hopfian. S = End (Z_{p^∞}) is the integral domain of P-adic integers is not co-Hopfian [6] , Then S is not purely co-Hopfian by Corollary (1.6).

Recall that an R-module M is called multiplication module if for each N≤M ,there exists ideal I of R such that N=IM. Equivalently, Mis multiplication if for each N≤M, N=(N:M)M,where (N:M)={r:$r \in R$, $rM \subseteq N$}[7].

Theorem 1.17

Let M be a faithful finitely generated multiplication R – module the following statements are equivalent:

1. M is purely co-Hopfian.
2. R is semi co-Hopfian.

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3. R is purely co-Hopfian.
4. M is co-Hopfian.
5. M is Semi co-Hopfian.

Proof

(1) →(2): Let $a \in R$, $ann_R a = 0$. Define $f : M \to M$ by $f(m) = am$ for any $m \in M$ .we can see that f is monomorphism as follows, let $m \in Kerf$ then $am = 0$ and so $m \in ann_M(a)$. But $ann_M(a) = (ann_R(a))M$ .Hence $m \in (ann_R a)M = 0$. $M = 0$, then we get $m = 0$.Now $f(M) = aM$ is pure in M. Hence $< a >$ is pure in R , since M is faithful finitely generated multiplication .Thus $< a > = < a^2 >$ so $a = ra^2$, which implies $a(1 - ra) = 0$, since $ann(a) = 0$, $1-ra = 0$, $1 = ra$,that is a is an inevitable element , so $< a > = R$.

(2) ↔(3): It follows by proposition (1.4).

(3) → (4): Let $f : M \to M$ be monomorphism , Since M is finitely generated multiplication, then M is a scalar module , there exists $a \in R$, $a \neq 0$such that , $f(m) = am$ for all $m \in M$ [8]. Since $Kerf = \{0\}$, $ann_M a = 0$.[To prove this. Since $ann_M(a) = \{ m : am = 0 \} = \{ m : f(m) = 0 \} = \{ m : m = 0 \}$] .But $ann_M a = (ann_R a)M$, so $ann_R(a) .M = 0$ .Thus $ann_R(a) \subseteq annM = 0$ .It follows that $ann_R(a) = 0$. But R is purely co-Hopfian so $< a > = R$ by corollary (1.5) .

(4) →(5): It is clear any co-Hopfian is semi co-Hopfian by [1].

(5) → (1): By [Remark and Examples 1.2]

corollary 1.18

Let M be a faithful finitely generated multiplication R-Module then the following are equivalent:

1. M is purely co-Hopfian module.
2. $End_R M$ is purely co-Hopfian ring (semi co-Hopfian ,co-Hopfian)

Proof (1) ↔ (2)

Since M is a finitely generated multiplication R-module M is a scalar module by [8,prop.1.1.10].Hence End $M \cong R$ by [9,lemma 6.1,ch.3].Thus by previous theorem we obtained the result .

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المقاسات الهوبفينيةالمضادÉ النقية

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الخلاصة

لتكن R حلقه تجميعيه ذا عنصر محايد ، M مقاسا احاديا غير صفري معرفا عليها . في هذا البحث نقدم مفهوم المقاسات الهوفينيةالمضادÉ النقية ÐÇ يقال عن مقاس M على حلقه R مقاسا هوبفينيا مضادا اذا كان لكل End (M) Î f ¡ f دالة متباينة فان Imf نقي في M . واعطينا بعض خواص هذا النوع من المقاسات .

الكلمات المفتاحية:مقاسات هوفينية مضادÉ نقية ، مقاسات شبة هوفينية مضادÉ ، مقاسات هوفينية مضادÉ