Existence of Positive Solution for Boundary Value Problems

S. M. Hussein
Department of Mathematics, Education College For Pure Sciences, University of Anbar

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Abstract

This paper studies the existence of positive solutions for the following boundary value problem :-

$$-y'' = \lambda\, g(t)\, f(y) \qquad a < t < b$$
$$\alpha\, y(a) - \beta\, y'(a) = 0$$
$$y(b) = 0$$

The solution procedure follows using the Fixed point theorem and obtains that this problem has at least one positive solution .Also,it determines (λ) Eigenvalue which would be needed to find the positive solution .

Keywords: Positive Solution , Boundary Value Problem , Fixed Point Theorem .

Introduction

In this paper we shall consider the second - order boundary value problem (BVP)

$$\left.\begin{aligned} &-y'' = \lambda\, g(t)\, f(y) \qquad a < t < b \\ &\alpha\, y(a) - \beta\, y'(a) = 0 \\ &y(b) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(1.1)$$

The following conditions will be assumed throughout :-

A- $f : [0 , \infty) \to [0 , \infty)$ is continuous ,

B- $g : [0 , 1] \to [0 , \infty)$ is continuous and does not vanish identically on any subinterval ,

C- $f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}$ and $f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}$ exist ,

D- α , β such that α and β are not both zero and $Z = \alpha + \beta > 0$, and

E- $a \geq 0 , b \leq 1$.

The boundary value problem (1.1) arises in the applied mathematical sciences such as nonlinear diffusion generated by nonlinear sources , thermal ignition of gases and chemical concentrations in biological problems ; for example see [1] , [2] , [3] . When λ=1 and f is either superlinear that is ($f_0 = 0$ and $f_\infty = \infty$) or f is sublinear that is ($f_0 = \infty$ and $f_\infty = 0$) , Erbe and Wang [5] obtained solutions that are positive with respect to a cone which lies in an annular type region .The methods of [5] were then extended to higher order BVP in [4] .

For the case α =1,β = 0,γ =1, δ = 0, Johnny Henderson and Haiyan Wang [7] obtained solutions that are positive for an open interval of eigenvalues. Not required in this work that f would be either superlinear or sublinear , yet, as in [4] , [5] but as in [7] , the arguments presented here for obtaining solutions of(1.1)for certainλ involve concavity properties of solutions, which are employed in defining a cone on which a positive integral operator is defined . A Krasnosel'skii fixed point theorem [8] is applied to yield positive solutions of (1.1) , for λ belongs to an open interval.

Section 2 , presents some properties of Green's functions that are used in defining a positive operator , also states the Krasnosel'skii fixed point theorem .

Section 3 , gives an appropriate Banach space and constructs a cone to which we apply the fixed point theorem yielding solutions of 1 .1 , for an open interval of eigenvalues .

2- Some Preliminaries

In this section , we state the above mentioned Krasnosel'skii fixed point theorem. We will apply this fixed point theorem to completely continuous integral operator , whose kernal , G (t , s) , is the Green's function for

$$- y'' = 0$$

$$\alpha\, y(a) - \beta\, y'(a) = 0$$

$$y(b)= 0$$

Is

$$G(t,s) = \begin{cases} \frac{1}{Z}(\alpha t + \beta)\,(1-s) & a \le t \le s \le b \\ \frac{1}{Z}(\alpha s + \beta)\,(1-t) & a \le s \le t \le b \end{cases} \quad \ldots\ldots\ldots..(2.1)$$

from which

$$G(t\,,s) > 0 \quad \text{on } (0\,,1) \times (0\,,1)\,, \quad \ldots\ldots\ldots(2.2)$$

$$G(t\,,s) \le G(s\,,s) = \frac{1}{Z}(\alpha s + \beta)\,(1-s) \quad , \; a \le t \le b\,, a \le s \le b\,, \quad \ldots\ldots(2.3)$$

and it is shown in [5] that :-

$$G(t\,,s) \ge M\, G(s\,,s) = M\, \frac{1}{Z}(\alpha s+\beta)\,(1-s) \quad , \frac{2a+1}{4} \le t \le \frac{2b+1}{4}\,, a \le s \le b\,, \ldots(2.4)$$

$$\text{Where } M = \min\left\{\frac{1}{4}, \frac{\alpha + 4\beta}{4(\alpha+\beta)}\right\}$$

We shall apply the following fixed point theorem to obtain solutions of (1.1) , for certain λ □

Theorem 1 [8]. Let B a Banach space , and let P be a cone in B . Assume N , K are be $0 \in N \subset \overline{N} \subset K$, and let $T : P \cap (\overline{K} \setminus N) \to P$ open subsets of B with a completely continuous operator such that , either

1- $\| Tu \| \le \| u \|$, $u \in P \cap \partial N$, and $\| Tu \| \ge \| u \|$, $u \in P \cap \partial K$, or

2- $\| Tu \| \ge \| u \|$, $u \in P \cap \partial N$, and $\| Tu \| \le \| u \|$, $u \in P \cap \partial K$

. $P \cap (\overline{K} \setminus N)$ Then T has a fixed point in

3. Solutions in The Cone

In this section , apply Theorem 1 to the eigenvalue problem (1.1) . Note that y(t) is a solution of (1.1) if , and only if ,

$$y(t) = \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds \quad , \quad a \le t \le b$$

For our construction , let B = C[a , b] , with norm , $\|x\| = \sup_{a \le t \le b} |x(t)|$

Define a cone P by :

$$P = \left\{ x \in B : x(t) \ge 0 \text{ on } [a,b] \ , \min_{\frac{2a+1}{4} \le t \le \frac{2b+1}{4}} x(t) \ge M\|x\| \right\}$$

$$M = \min\left\{ \frac{1}{4}, \frac{\alpha + 4\beta}{4(\alpha+\beta)} \right\}$$ Where

Also , let the number $h \in [a,b]$ be defined by

$$\int_{\frac{2a+1}{4}}^{\frac{2b+1}{4}} G(h,s)\, g(s)\, ds = \max \int_{\frac{2a+1}{4}}^{\frac{2b+1}{4}} G(t,s)\, g(s)\, ds \qquad \text{.............(3.1)}$$

Theorem 2. Assume that conditions (A),(B),(C) and (D) are satisfied .Then , for each λ satisfying

$$\text{... (3.2)......} \qquad \frac{4}{\left(M \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right) f_\infty} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right) f_0}$$

there exists at least one solution of (1.1) in P .

Proof. Let λ be given as in (3.2) . Now , let $\varepsilon > 0$ be chosen such that

$$\frac{4}{\left(M \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right)(f_\infty - \varepsilon)} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right)(f_0 + \varepsilon)} \qquad \text{...........(3.3)}$$

Define an integral operator $T : P \to B$ by

$$Ty(t) = \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds \quad , \quad y \in P \qquad \text{.........(3.4)}$$

We seek a fixed point of T in the cone P.

From (2.2), we note that , for $y \in P$, $Ty(t) \ge 0$ on [a,b] . Also , for $y \in P$, we have from (2.3) that

$$Ty(t) = \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds$$

$$||Ty|| \le \lambda \int_a^b G(s,s)\, g(s)\, f(y(s))ds \qquad \text{.........(3.5)}$$

Now , if $y \in P$, we have by (2.4) and (3.5) ,

$$\min_{\frac{2a+1}{4} \le t \le \frac{2b+1}{4}} Ty(t) = \min_{\frac{2a+1}{4} \le t \le \frac{2b+1}{4}} \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds$$

$$\ge M \lambda \int_a^b G(s,s)\, g(s)\, f(y(s))ds$$

$$\ge M\|Ty\|$$

$\rightarrow p$. In addition , standard arguments show that T is As a consequence , $T : p$ completely continuous.

Now, turning to f_0 , there exist an $K_1 > 0$ such that $f(x) \le (f_0 + \varepsilon)\, x$, for $0 < x \le K_1$.

$y \in P$ such that $||\, y\, || = K_1$, we have from (2.3) and (3.3) So , by choosing

$$Ty(t) \le \lambda \int_a^b G(s, s)\, g(s)\, f(y(s))\, ds$$

$$\le \lambda \int_a^b G(s, s)\, g(s)\, (f_0 + \varepsilon)\, y(s)\, ds$$

$$\le \lambda \int_a^b G(s, s)\, g(s)\, ds\, (f_0 + \varepsilon)\, y(s)\, \|y\|$$

$$\le \|y\|$$

Consequently , $\|Ty\| \le \|y\|$. So , if we set $\Omega_1 = \{x \in B |\; ||x|| < K_1\}$

then

$$||Ty|| \le ||y|| \;, \text{ for } y \in P \cap \partial\Omega_1 . \qquad \text{..........(3.6)}$$

Next , considering f_∞ , there exist an $K_2 > 0$ such that $f(x) \ge (f_\infty - \varepsilon)\, x$,for all $x > K_2$.

Let $K_3 = \max\{2K_1 , K_2/M\}$ and let $\Omega_2 = \{ x \in B \,|\; ||x|| < K_3\}$

If $y \in P$ with $||y|| = K_3$, then $\min_{\frac{2a+1}{4} \le t \le \frac{2b+1}{4}} y(t) \ge M\|y\| = MK_3 \ge K_2$, and we have from (3.1) and (3.3) that

$$Ty(h) = \lambda \int_a^b G(h, s)\, g(s)\, f(y(s))\, ds$$

$$\ge \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h, s)\, g(s)\, f(y(s))\, ds$$

$$\ge \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h, s)\, g(s)\, (f_\infty - \varepsilon)\, y(s)\, ds$$

$$\ge \frac{\lambda}{M} \int_{(2a+1)/4}^{(2b+1)/4} G(h, s)\, g(s)\, ds\, (f_\infty - \varepsilon)\, \|y\|$$

$$\ge \|y\|$$

Thus , $\|Ty\| \geq \|y\|$. Hence ,

$\|Ty\| \geq \|y\|$, for $y \in P \cap \partial\Omega_2$ ……….(3.7)

Applying (1) of theorem 1 to (3.6) and (3.7) yields that T has a fixed point $y(t) \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$. As such , y(t) is a desired solution of 1.1 for the given λ . Further , since G (t , s) > 0 , it follows that y(t) > 0 for a < t <b . This completes the proof of the theorem .

Theorem 3 . Assume that condition (A),(B),(C) , (D) and (E) are satisfied . Then , for each λ satisfying

$$\frac{4}{\left(M\int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right) f_0} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right) f_\infty} \qquad \text{……….(3.8)}$$

there exists at least one solution of 1.1 in P .

Proof. Let λ be given as in (3.8) . Now , let ε > 0 be chosen such that

$$\frac{1}{\left(M\int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right)(f_0-\varepsilon)} < \lambda < \frac{1}{\left(\int_a^b G(s,s)g(s)\, ds\right)(f_\infty+\varepsilon)} \qquad \text{.........(3.9)}$$

Let T be the cone preserving , completely continuous operator that was defined by(3.4). Beginning with f_0 , there exists an $K_4 > 0$ such that $f(x) \geq (f_0 - \varepsilon)\, x$, for $0 < x \leq K_4$.

$y \in P$ such that $\| y \| = K_4$, we have from (3.1) and (3.9) so , for So

$$\begin{aligned} Ty(h) &= \lambda\int_a^b G(h,s)\, g(s)\, f(y(s))ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, f(y(s))ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, (f_0-\varepsilon)\, y(s)\, ds \\ &\geq M\lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\, (f_0-\varepsilon)\, \|y\| \\ &\geq \|y\| \end{aligned}$$

Thus , $\|Ty\| \geq \|y\|$. So , if we let

$\Omega_3 = \{x \in B \mid \|x\| < K_4\}$

then

$\|Ty\| \geq \|y\|$ for $y \in P \cap \partial\Omega_3$ ……. (3.10)

It remains to consider f_∞ , there exists an $K_5 > 0$ such that $f(x) \leq (f_\infty + \varepsilon)\, x$, for all $x > K_5$. There are the two cases , (a) f is bounded , and (b) f is unbounded .

For case (a) , suppose $K_6 > 0$ is such that $f(x) \leq K_6$, for all $0 < x < \infty$.

Let $K_7 = \max\{2K_4 , K_6\ \lambda\int_a^b G(s,s)\, g(s)\, f(y(s))ds\}$. Then , for $y \in P$ with $||y|| = K_7$ we have from (2.3) and (3.2)

$$Ty(t) = \lambda\int_a^b G(t,s)\, g(s)\, f(y(s))\, ds\}$$
$$\leq \lambda\, K_6 \int_a^b G(s,s)\, g(s)\, ds$$
$$\leq \|y\|$$

so that $\|Ty\| \leq \|y\|$. So if $\Omega_4 = \{x \in B|\ ||x|| < K_7\}$

then

$||Ty|| \leq ||y||$, for $y \in P \cap \partial\Omega_4$ ……….(3.11)

For case (b) , let $K_8 > \max\{2K_4 , K_5\}$ be such that $f(x) \leq f(K_8)$, for $0 < x \leq K_8$.

By choosing $y \in P$ such that $||y|| = K_8$ and we have from (2.3),(3.2) and (3.9)

$$Ty(t) = \lambda\int_a^b G(t,s)\, g(s)\, f(y(s))ds$$
$$\leq \lambda\int_a^b G(s,s)\, g(s)\, f(y(s))\, ds$$
$$\leq \lambda\int_a^b G(s,s)\, g(s)\, f(K_8)\, ds$$
$$\leq \lambda\int_a^b G(s,s)\, g(s)\, ds\, (f_\infty + \varepsilon)K_8$$

But

$$\lambda\int_a^b G(s,s)\, g(s)\, ds\, (f_\infty + \varepsilon)K_8 = \lambda\int_a^b G(s,s)\, g(s)\, ds\, (f_\infty + \varepsilon)\|y\|$$

Therefore

$$Ty(t) \leq \lambda\int_a^b G(s,s)\, g(s)\, ds\, (f_\infty + \varepsilon)\|y\|$$

and so $\|Ty\| \leq \|y\|$. For this case , if we let

$\Omega_4 = \{x \in B|\ ||x|| < K_8\}$

then

$||Ty|| \leq ||y||$, for $y \in P \cap \partial\Omega_4$ ……….(3.12)

Thus , in both cases , an applying of part (2) of theorem 1 to (3.10),(3.11) and (3.12) yields that T has a fixed point $y(t) \in P \cap (\overline{\Omega_4} \setminus \Omega_3)$. As such , y(t) is a desired solution of 1.1 for the given λ . Further , since G (t , s) > 0 , it follows that y(t) > 0 for a < t < b . This completes the proof of the theorem .

References

1. Kuiper,H. J. (1979). On Positive Solution of nonlinear elliptic eigenvalue Problems , Rend. Math. Cire. Palermo , Serie II ,Tom. XX 113-138 .
2. Erbe, L. H.; Hu, S. and Wang, H. (1994) Multiple Positive Solutions of some boundary value problems , J. Math. Anal.Appl.184: 640–748 .
3. Eloe, P.W.; Henderson, J. and Wong, P.J.Y. Positive Solutions for two – point boundary value problems , Dyn. Sys. Appl. , in press .
4. Eloe, P. W. and Henderson, J. (1995). Positive Solutions for Higher Order Differential Equations , Elec. J. Diff. Equ. 3 :1 – 8 .
5. Erbe, L. H. and Wang, H. (1994). On The Existence of Positive Solution of Ordinary Differential Equations , Proc. Amer. Math. Soc. 120: 743 – 748 .
6. Garaizar, X. (1987). Existence of Positive radial Solutions for Semilinear elliptic Problems in the annulus , J. Diff. Equ. 70: 69 – 72 .
7. Henderson, J. and Wang, H. (1997). Positive Solutions for Nonlinear Eigenvalue Problems , J. Math. Anal. APPl. 208: 252 – 259 .
8. Krasnoseelskii, M. A.(1964). Positive Solutions of Operator Equations, Noordhoff, Groning.

وجود الحلول الموجبة لمسائل القيم الحدودية

صالح محمد حسين

قسم الرياضيات، كلية التربية للعلوم الصرفة ،جامعة الانبار

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الخلاصة

هذا البحث درس وجود الحلول الموجبة للمسألة الحدودية الاتية :-

$$-y'' = \lambda\, g(t)\, f(y) \qquad a < t < b$$
$$\alpha\, y(a) - \beta\, y'(a) = 0$$
$$y(b) = 0$$

مستخدما نظرية النقطة الثابتة وتوصلت إلى أن هذه المسألة تمتلك على الأقل حلا واحدا موجبا وتم تحديد قيم المعلمة (λ) التي عندها توجد حلول موجبة للمسألة الحدودية .