https://doi.org/10.31436/iiumej.v24i1.2380

HIGHER DERIVATIVE BLOCK METHOD WITH GENERALISED STEP LENGTH FOR SOLVING FIRST-ORDER FUZZY INITIAL VALUE PROBLEMS

KASHIF HUSSAIN, OLUWASEUN ADEYEYE* AND NAZIHAH AHMAD

School of Quantitative Sciences, Universiti Utara Malaysia,
Kedah, Malaysia

**Corresponding author: adeyeye@uum.edu.my*

(Received: 30th March 2022; Accepted: 10th August 2022; Published on-line: 4th January 2023)

ABSTRACT: Block methods have been adopted in studies for solving first and higher order differential equations due to its impressive accuracy property. Taking a step further to improve this accuracy, researchers have considered the inclusion of higher-derivative terms in the block method, although this has been limited to the presence of one higher-derivative term in previous studies. Hence, this article aims at better accuracy by introducing two higher-derivative terms in the block method. In addition, this article presents a scheme with generalised step length such that there is flexibility on the choice of step length when developing the block method. The generalised step length scheme is adopted to develop a three-step block method for solving first-order fuzzy initial value problems. Its properties to ensure convergence and to show the region of absolute stability is investigated, and problems relating to charging and discharging of capacitor are considered. The absolute error shows the impressive accuracy of the three-step block method including obtaining the same values as the exact solution. Therefore, in addition to the new generalised algorithm presented in this article, a new three-step method for solving linear and nonlinear first order fuzzy initial value problems is presented.

ABSTRAK: Kaedah blok digunakan dalam banyak kajian untuk menyelesaikan persamaan pembezaan peringkat pertama dan peringkat tinggi kerana sifat ketepatannya yang baik. Bagi meningkatkan ketepatan ini, penyelidik telah mengambil kira dengan memasukkan terbitan peringkat tinggi dalam kaedah blok, walaupun ini terhad pada satu sebutan terbitan peringkat tinggi dalam kajian sebelum. Oleh itu, kajian ini bertujuan bagi mendapatkan ketepatan yang lebih baik dengan memperkenalkan dua sebutan terbitan peringkat tinggi dalam kaedah blok. Tambahan, kajian ini memperkenalkan skema dengan panjang-langkah kaki biasa supaya terdapat kebolehlenturan pada pilihan langkah semasa membangunkan kaedah blok. Skema ini diadaptasi bagi membangunkan kaedah blok tiga-langkah bagi menyelesai masalah nilai awal peringkat pertama secara rawak. Ciri-ciri terperinci dikaji bagi memastikan penumpuan lingkungan kestabilan mutlak, dan masalah berkaitan pengecasan dan nyahcas kapasitor juga turut diambil kira. Ralat mutlak menunjukkan ketepatan yang mengkagumkan pada kaedah blok tiga-langkah termasuk mendapatkan nilai yang sama seperti penyelesaian. Oleh itu, tambahan pada algoritma ini, kaedah tiga-langkah bagi menyelesaikan linear dan tidak linear pada masalah nilai awal peringat pertama secara rawak diperkenalkan.

KEYWORDS: *fuzzy initial value problem; generalised steplength; block method; higher derivative; charging and discharging of capacitor*

1. INTRODUCTION

The primary focus of numerical methods for solving fuzzy differential equations has been on presenting numerical methods with a higher level of accuracy. This includes providing a more accurate numerical solution for first order fuzzy initial value problems (FIVPs) of the form

$$y'(x)=f(x,y(x)),\, y(x_0)=y_0,\, x\in[x_0,X]. \tag{1}$$

Numerous researchers have developed different numerical methods [1-6] to solve problems in the form of Equation (1), however, the major problem encountered is that these existing numerical methods give a low level of accuracy in terms of absolute error due to order of the method used. Specifically, researchers considered the use of linear multistep methods implemented in predictor-corrector mode (a non-self-starting approach with low accuracy) as seen in studies [7,8]. To improve the accuracy, block methods were introduced in [9-11] and better accuracy was observed than linear multistep methods. However, there is still a need for an improvement in the solution accuracy in terms of absolute error. Hence, the motivation to develop block methods in this article with the presence of two higher derivative terms with the aim of obtaining better accuracy. In comparison to existing methods, the proposed method has the advantage of better accuracy, being self-starting, and flexibility in development and implementation of the block method.

2. PRELIMINARIES

This section recalls some basic definitions which will be adopted in this article.

Triangular Fuzzy Number [12]. Consider that $(u,v,w)\in\mathbb{R}^3$, $u\le v\le w$. Then the triangular fuzzy number, $M(x)$ is given as

$$M(x,u,v,w)=\begin{cases}0, x<u\\ \dfrac{x-u}{v-u}, u\le x\le v\\ \dfrac{w-x}{w-v}, v<x\le w\\ 0, x>w\end{cases}. \tag{2}$$

The corresponding *r*-level set of the triangular fuzzy number is denoted as

$$M_r=\left[u+r\left(v-u\right), w-r(w-v)\right],\ r\in[0,1]. \tag{3}$$

Trapezoidal Fuzzy Numbers [12]. Consider that $(u,v,w,\delta)\in\mathbb{R}^4$, $u\le v\le w\le\delta$. Then the trapezoidal fuzzy number $M(x)$ is given as

$$M(x,u,v,w,\delta)=\begin{cases}0, x<u\\ \dfrac{x-u}{v-u}, u\le x<v\\ 1, v\le x\le w\\ \dfrac{w-x}{w-v}, w<x\le\delta\\ 0, x>\delta\end{cases}. \tag{4}$$

The corresponding *r*-level set of the trapezoidal fuzzy number is denoted as

$$M_r=\left[u+r\left(v-u\right),\delta-r(\delta-w)\right],\ r\in[0,1] \tag{5}$$

Some of the basic fuzzy definitions and notions that are not included in this Section 2 are widely known. Notions of fuzzy sets, functions and their operations, fuzzy derivatives, and Zadeh's extension theory can be retrieved from literature such as [13-16].

3. METHODOLOGY

Given that the first-order FIVP of the form defined in Eq. (1) be a mapping, $f:\mathbb{R}_f\to\mathbb{R}_f$ and $y_0\in\mathbb{R}_f$, with *r*-level set $y_0\in\left(\underline{y}(0,r),\bar{y}(0,r)\right)_{\underline{r}}^{\bar{r}}$, $r\in[0,1]$. Also, denote the approximation solution as $\left(\mathrm{y}(\mathrm{x}_n,\alpha)\right)_{\underline{r}}^{\bar{r}}=\left(\underline{y}(\mathrm{x}_n,\mathrm{r}),\bar{y}(\mathrm{x}_n,\mathrm{r})\right)_{\underline{r}}^{\bar{r}}$ at points $x_n=x_0+nh$, where $0\le n\le N$ and $h=\dfrac{X-x_0}{n}$.

The generalized *k*-step block method with presence of second and third derivative in first-order form is stated below as,

$$\left(y_{n+\eta}\right)_{\underline{r}}^{\bar{r}}=\left(y_n+\sum_{d=0}^{2}\left[\sum_{v=0}^{k}\psi_{dv\eta}f_{n+v}^{(d)}\right]\right)_{\underline{r}}^{\bar{r}},\eta=1,2,3,\dots,k. \tag{6}$$

Expanding Eq. (6) gives the expression

$$\left(y_{n+\eta}\right)_{\underline{r}}^{\bar{r}}=\left(y_n+\begin{bmatrix}\psi_{00\eta}f_n+\psi_{01v}f_{n+1}+,\dots,+\psi_{0k\eta}f_{n+k}\\ +\psi_{10\eta}f'_n+\psi_{11\eta}f'_{n+1}+,\dots,+\psi_{1k\eta}f'_{n+k}\\ +\psi_{20\eta}f''_n+\psi_{21\eta}f''_{n+1}+,\dots,+\psi_{2k\eta}f''_{n+k}\end{bmatrix}\right)_{\underline{r}}^{\bar{r}}. \tag{7}$$

Consider the Taylor series expansions defined by [17]:

$$\left(\mathrm{y}(x_n+jh;\mathrm{r})\right)_{\underline{r}}^{\bar{r}}=\left(\sum_{i=0}^{n}\frac{(jh)^i}{i!}f^i(x_n;\mathrm{r})\right)_r^{\bar{r}},\ j=0,1,\dots,\mathrm{k}, \tag{8}$$

$$\left(y_{n+j}\right)_{\underline{r}}^{\bar{r}}=\left(\mathrm{y}(x_n;\mathrm{r})+jhy'(x_n;\mathrm{r})+\frac{(jh)^2}{2!}\mathrm{y}''(x_n;\mathrm{r})+\frac{(jh)^3}{3!}\mathrm{y}'''(x_n;\mathrm{r})+\dots+\frac{(jh)^n}{\mathrm{n}!}\mathrm{y}''(x_n;\mathrm{r})\right)_{\underline{r}}^{\bar{r}}. \tag{9}$$

Applying these expansions in Eqs. (8) and (9) to expand each term in Eq. (7) results in obtaining the unknown coefficients $\psi_{dv\eta}$ from $\psi_{dv\eta} = A^{-1}B$, where

$$A=\begin{pmatrix} 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & h & \dots & kh & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{h^2}{2!} & \dots & \frac{(kh)^2}{2!} & 0 & h & \dots & kh & 1 & 1 & \dots & 1 \\ 0 & \frac{h^3}{3!} & \dots & \frac{(kh)^3}{3!} & 0 & \frac{h^2}{2!} & \dots & \frac{(kh)^2}{2!} & 0 & h & \dots & kh \\ . & . & \dots & . & . & . & \dots & . & . & . & \dots & . \\ . & . & \dots & . & . & . & \dots & . & . & . & \dots & . \\ . & . & \dots & . & . & . & \dots & . & . & . & \dots & . \\ 0 & \frac{h^{3k+2}}{(3k+2)!} & \dots & \frac{(kh)^{3k+2}}{(3k+2)!} & 0 & \frac{h^{3k+1}}{(3k+1)!} & \dots & \frac{(kh)^{3k+1}}{(3k+1)!} & 0 & \frac{(h)^{3k}}{3k!} & \dots & \frac{(kh)^{3k}}{3k!} \end{pmatrix}_{\underline{r}}^{\bar{r}}, \text{ and } B=\begin{pmatrix} \eta h \\ \frac{(\eta h)^2}{2!} \\ \frac{(\eta h)^3}{3!} \\ \frac{(\eta h)^4}{4!} \\ . \\ . \\ . \\ \frac{(\eta h)^{3k+3}}{(3k+3)!} \end{pmatrix}_{\underline{r}}^{\bar{r}}.$$

The resultant values are substituted in Eq. (7) to get the desired generalized k-step block method with the presence of second and third derivatives for solving first-order FIVPs. A more detailed explanation is given in the following subsection, where the generalised step length (k-step) block method scheme with presence of second and third derivatives is adopted to develop a three-step ($k=3$) block method for first order FIVPs.

3.1 Development of Three-Step Block Method

To develop a three-step block method with second and third derivatives for first order FODEs requires substituting $k=3$ in Eq. (7) and then applying Taylor series expansions in Eqs. (8), (9). The unknown coefficients $\psi_{dv\eta}$ are obtained as follows:

$$\begin{pmatrix} \psi_{001} \\ \psi_{011} \\ \psi_{021} \\ \psi_{031} \\ \psi_{101} \\ \psi_{111} \\ \psi_{121} \\ \psi_{131} \\ \psi_{201} \\ \psi_{211} \\ \psi_{221} \\ \psi_{231} \end{pmatrix}_{\underline{r}}^{\bar{r}} = \begin{pmatrix} \frac{912523}{2395008} \\ \frac{23717}{29568} \\ \frac{-5851}{29568} \\ \frac{35339}{2395008} \\ \frac{214943}{3991680} \\ \frac{-10657}{147840} \\ \frac{10657}{147840} \\ \frac{-5941}{1330560} \\ \frac{11369}{3991680} \\ \frac{4423}{88704} \\ \frac{-7453}{443520} \\ \frac{1513}{3991680} \end{pmatrix}, \begin{pmatrix} \psi_{002} \\ \psi_{012} \\ \psi_{022} \\ \psi_{032} \\ \psi_{102} \\ \psi_{112} \\ \psi_{122} \\ \psi_{132} \\ \psi_{202} \\ \psi_{212} \\ \psi_{222} \\ \psi_{232} \end{pmatrix}_{\underline{r}}^{\bar{r}} = \begin{pmatrix} \frac{7031}{18711} \\ \frac{302}{231} \\ \frac{71}{231} \\ \frac{178}{18711} \\ \frac{544}{10395} \\ \frac{32}{1155} \\ \frac{-32}{1155} \\ \frac{92}{31185} \\ \frac{17}{6237} \\ \frac{212}{3465} \\ \frac{-19}{3465} \\ \frac{8}{31185} \end{pmatrix}, \begin{pmatrix} \psi_{003} \\ \psi_{013} \\ \psi_{023} \\ \psi_{033} \\ \psi_{103} \\ \psi_{113} \\ \psi_{123} \\ \psi_{133} \\ \psi_{203} \\ \psi_{213} \\ \psi_{223} \\ \psi_{233} \end{pmatrix}_{\underline{r}}^{\bar{r}} = \begin{pmatrix} \frac{3849}{9856} \\ \frac{10935}{9856} \\ \frac{10935}{9856} \\ \frac{3849}{9856} \\ \frac{2799}{49280} \\ \frac{-2187}{49280} \\ \frac{2187}{49280} \\ \frac{-2799}{49280} \\ \frac{153}{49280} \\ \frac{2187}{49280} \\ \frac{2187}{49280} \\ \frac{153}{49280} \end{pmatrix}.$$

Substituting the obtained coefficients in Eq. (7) for $k=3$, the three-step block scheme is derived as

$$\left(y_{n+1} = y_n + \left[\begin{array}{l} h\left(\frac{912523}{2395008} f_n + \frac{23717}{29568} f_{n+1} - \frac{5851}{29568} f_{n+2} + \frac{35339}{2395008} f_{n+3} \right) + \\ h^2\left(\frac{214943}{3991680} g_n - \frac{10657}{147840} g_{n+1} + \frac{10657}{147840} g_{n+2} - \frac{5941}{1330560} g_{n+3} \right) + \\ h^3\left(\frac{11369}{3991680} m_n + \frac{4423}{88704} m_{n+1} - \frac{7453}{443523} m_{n+2} + \frac{1513}{3991680} m_{n+3} \right) \end{array} \right] \right)_{\underline{r}}^{\bar{r}},$$

$$\left(y_{n+2} = y_n + \left[\begin{array}{l} h\left(\frac{7031}{18711} f_n + \frac{302}{231} f_{n+1} + \frac{71}{231} f_{n+2} + \frac{178}{18711} f_{n+3} \right) + \\ h^2\left(\frac{544}{10395} g_n + \frac{32}{1155} g_{n+1} - \frac{32}{1155} g_{n+2} - \frac{92}{31185} g_{n+3} \right) + \\ h^3\left(\frac{17}{6237} m_n + \frac{212}{3465} m_{n+1} - \frac{19}{3465} m_{n+2} + \frac{8}{31185} m_{n+3} \right) \end{array} \right] \right)_{\underline{r}}^{\bar{r}}, \qquad (10)$$

$$\left(y_{n+3} = y_n + \left[\begin{array}{l} h\left(\frac{3849}{9856} f_n + \frac{10935}{9856} f_{n+1} + \frac{10935}{9856} f_{n+2} + \frac{3849}{9856} f_{n+3} \right) + \\ h^2\left(\frac{2799}{49280} g_n - \frac{2187}{49280} g_{n+1} + \frac{2187}{49280} g_{n+2} - \frac{2799}{49280} g_{n+3} \right) + \\ h^3\left(\frac{153}{49280} m_n + \frac{2187}{49280} m_{n+1} + \frac{2187}{49280} m_{n+2} + \frac{153}{49280} m_{n+3} \right) \end{array} \right] \right)_{\underline{r}}^{\bar{r}}$$

The block method in Eq. (10) has corrector form

$$\left(A^0 \mathrm{Y}_{n+k}\right)_{\underline{r}}^{\bar{r}} = \left(A^1 \mathrm{Y}_{n-k}\right)_{\underline{r}}^{\bar{r}} + h\left(B^0\, \mathrm{Y}'_{n+k} + B^1\, \mathrm{Y}'_{n-k}\right)_{\underline{r}}^{\bar{r}} + h^2\left(C^0\, \mathrm{Y}''_{n+k} + C^1\, \mathrm{Y}''_{n-k}\right)_{\underline{r}}^{\bar{r}} +$$
$$h^3\left(D^0\, \mathrm{Y}''_{n+k} + D^1\, \mathrm{Y}'''_{n-k}\right)_{\underline{r}}^{\bar{r}}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\underline{r}}^{\bar{r}}, A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{\underline{r}}^{\bar{r}}, B^0 = \begin{pmatrix} \frac{23717}{29568} & \frac{-5851}{29568} & \frac{35339}{2395008} \\ \frac{302}{231} & \frac{71}{231} & \frac{178}{18711} \\ \frac{10935}{9856} & \frac{10935w}{9856} & \frac{3849}{9856} \end{pmatrix}_{\underline{r}}^{\bar{r}},$$

$$C^0 = \begin{pmatrix} \frac{-10657}{147840} & \frac{10657}{147840} & \frac{-5941}{1330560} \\ \frac{32}{1155} & \frac{-32}{1155} & \frac{-92}{31185} \\ \frac{-2187}{49280} & \frac{2187}{49280} & \frac{-2799}{49280} \end{pmatrix}_{\underline{r}}^{\bar{r}}, B^1 = \begin{pmatrix} 0 & 0 & \frac{912523}{2395008} \\ 0 & 0 & \frac{7031}{18711} \\ 0 & 0 & \frac{3849}{9856} \end{pmatrix}_{\underline{r}}^{\bar{r}}, C^1 = \begin{pmatrix} 0 & 0 & \frac{214943}{3991680} \\ 0 & 0 & \frac{544}{10395} \\ 0 & 0 & \frac{2799}{49280} \end{pmatrix}_{\underline{r}}^{\bar{r}},$$

$$D^0 = \begin{pmatrix} \frac{4423}{88704} & \frac{-7453}{443520} & \frac{1513}{3991680} \\ \frac{212}{3465} & \frac{-19}{3465} & \frac{8}{31185} \\ \frac{2187}{49280} & \frac{2187}{49280} & \frac{153}{49280} \end{pmatrix}_{\underline{r}}^{\bar{r}}, Y_{n-k} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}_{\underline{r}}^{\bar{r}}, D^1 = \begin{pmatrix} 0 & 0 & \frac{11369}{3991680} \\ 0 & 0 & \frac{17}{6237} \\ 0 & 0 & \frac{153}{49280} \end{pmatrix}_{\underline{r}}^{\bar{r}}, Y'_{n-k} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}_{\underline{r}}^{\bar{r}},$$

$$Y''_{n+k}=\begin{pmatrix} g_{n+1}\\ g_{n+2}\\ g_{n+3}\end{pmatrix}_{\underline{r}}^{\bar{r}}, Y''_{n-k}=\begin{pmatrix} g_{n-2}\\ g_{n-1}\\ g_{n}\end{pmatrix}_{\underline{r}}^{\bar{r}}, Y'''_{n+k}=\begin{pmatrix} m_{n+1}\\ m_{n+2}\\ m_{n+3}\end{pmatrix}_{\underline{r}}^{\bar{r}}, Y'''_{n-k}=\begin{pmatrix} m_{n-2}\\ m_{n-1}\\ m_{n}\end{pmatrix}_{\underline{r}}^{\bar{r}}, Y_{n+k}=\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\end{pmatrix}_{\underline{r}}^{\bar{r}}, Y'_{n+k}=\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\end{pmatrix}_{\underline{r}}^{\bar{r}}.$$

4. CONVERGENCE PROPERTIES

This section will detail the convergence properties of the developed three-step second-third derivative scheme. The following definitions are used: consistency, zero-stability, and region of absolute stability from [18]. These definitions for block methods in crisp form are adopted to the proposed method for fuzzy initial value problems to prove the convergence properties for the proposed method.

4.1 Order and Error Constant

The linear operator which is associated with Equation (6) for the three-step block method is defined as:

$$L(y(x),h)=\left(y_{n+\eta}-y_n-\sum_{d=0}^{2}\left[\sum_{v=0}^{3}\psi_{dv\eta}f_{n+v}^{(d)}\right]\right)_{\underline{r}}^{\bar{r}} \tag{13}$$

$$L(y(x),h)=\left(\mathbb{N}_0\,y(x_n)+\mathbb{N}_1hy'(x_n)+\mathbb{N}_2h^2\,y''(x_n)+,...,+\mathbb{N}_zh^zy^z(x_n)+\mathbb{N}_{z+1}h^{z+1}y^{z+1}(x_n)\right)_{\underline{r}}^{\bar{r}}.$$

The order of this method is z if $\mathbb{N}_0=\mathbb{N}_1=\mathbb{N}_2=,....,=\mathbb{N}_z=0$ and $\mathbb{N}_{z+1}$ is the error constant. By using the definition of order and error constant, the developed block method has order $z=12$ with error constant $\left[\frac{-29609}{28768836096000},\frac{23}{32108076000},\frac{9}{5637632000}\right]$. So, the developed block method is consistent.

4.2 Zero-stability

The zero-stability of the proposed method is computed from

$$p(\psi)=\left|\psi\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}\right|_{\underline{r}}^{\bar{r}}=\psi^2(\psi-1)=0.$$

The obtained roots satisfy the condition in [18]. Hence, the proposed three-step block method is zero-stable. Since the proposed method satisfies the properties of consistency and zero-stability for block methods, this implies that the method is convergent.

4.3 Region of Absolute Stability

The characteristic polynomial used to obtain the region of absolute stability of the developed block method is obtained as

$$\left(\det\left[-(w)^k+A^1+q\left[\sum_{j=0}^{k}B^jw^{k-j}\right]+q^2\left[\sum_{j=0}^{k}C^jw^{k-j}\right]+q^3\left[\sum_{j=0}^{k}D^jw^{k-j}\right]+q^4\left[\sum_{j=0}^{k}E^jw^{k-j}\right]\right]\right)_{\underline{r}}^{\bar{r}},$$
$$q=\lambda h.$$

https://doi.org/10.31436/iiumej.v24i1.2380

$$R(w)==\left(\begin{array}{l}\left[\frac{q^9}{369600}-\frac{52183q^8}{910694400}+\frac{388873q^7}{546416649}-\frac{231359q^6}{40981248}+\frac{1549771q^5}{45534720}-\frac{10104397q4}{68302080}+\frac{5805453q^3}{12142592}-\frac{47q^2}{44}-\frac{3q}{2}-1\right]w^6\\ \left[\frac{q^9}{369600}+\frac{q^8}{16800}+\frac{157q^7}{221760}+\frac{109q^6}{19008}+\frac{179q^5}{5280}+\frac{589q4}{3960}+\frac{21q^3}{44}+\frac{47q^2}{44}+\frac{3q}{2}+1\right]w^3\end{array}\right).$$

The region of absolute stability is determined by plotting the roots of the polynomial using a boundary locus approach, as shown in Fig. 1.

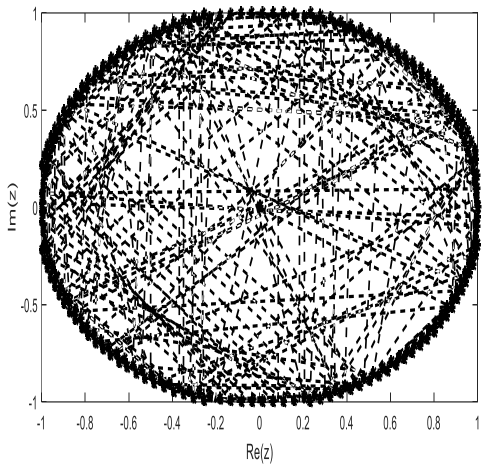

Fig. 1: Absolute stability region of three-step second-third derivative block method.

5. RESULTS AND DISCUSSION

This section details using the three-step block method to solve first-order linear and nonlinear FIVPs numerically and comparing the results to the exact solution. Tables and graphs are being used to compare exact and approximate solutions. The following notations are utilised in this section.

x-axis shows the value of approximation solution

y-axis shows the value of *r*-level set

$\underline{Y},\overline{Y}$ are the exact solution of lower and upper bound respectively

$\underline{y},\overline{y}$ are the approximation solution of lower and upper bound respectively

$|\underline{Y}-\underline{y}|$ absolute error of lower bound approximation

$|\vec{Y}-\overline{y}|$ absolute error of upper bound approximation

h is the stepsize

Example 1 [19].

Consider the following crisp capacitor model

$$\frac{d(U_c(x))}{dx}=-\frac{1}{RC}U_c(x)+\frac{1}{RC}U_G(x) \quad (14)$$

with exact solution

https://doi.org/10.31436/iiumej.v24i1.2380

$$U_c(x) = K.e^{-\int \frac{dx}{RC}} + \left[\int (\frac{U_G(x)}{RC} e^{-\int \frac{dx}{RC}}) dx \right] . e^{-\int \frac{dx}{RC}} . \tag{15}$$

For the initial condition charging of the capacitor

$$U_C(x) = U_B.[1 - e^{\frac{x}{RC}}], \tag{16}$$

while for the initial condition discharging of the capacitor

$$U_C(x) = U_{c,0}.e^{\frac{x}{RC}} . \tag{17}$$

According to [4], the crisp equation can be modelled in a fuzzy form using the definition of fuzzy theory, which is given in Section 2. The uncertain behaviour of a capacitor using the voltage, capacitance, or resistance of the circuit current is defined as triangular fuzzy numbers.

5.1 Charging of a Capacitor

The exact and approximate solutions are presented at $x = 4s$. Table 1 presents the accuracy for the lower and upper solutions of charging capacitor under DC condition with triangular fuzzy number. The corresponding graphs are shown in Fig. 2. The specifications adopted are battery voltage $= 12V$, $C = 0.25F$ (farads), $U_c(0) = 0$, and resistance with triangular fuzzy number is $R = (2 + r, 4 - r)$.

5.2 Discharging of a Capacitor

The exact and approximate solutions are presented at $x = 4s$. Table 2 presents the accuracy for the lower and upper solutions of the discharging capacitor under DC condition with triangular fuzzy number. The corresponding graphs are shown in Fig. 3. The specifications adopted as same as the charging of a capacitor scenario.

Table 1: Lower and Upper Solutions for Charging of Capacitor Problem in Example 1

r	$\underline{y}$	$\lvert\underline{Y} - \underline{y}\rvert$	$\overline{y}$	$\lvert\overline{Y} - \overline{y}\rvert$
0	11.995974448465169	0.0e+00	11.780212333335189	0.0e+00
0.2	11.991669406884018	0.0e+00	11.821937321808754	0.0e+00
0.4	11.984728394383922	0.0e+00	11.859076458515744	0.0e+00
0.6	11.974496498029186	0.0e+00	11.905917027388661	0.0e+00
0.8	11.960417930928731	0.0e+00	11.931188450176629	0.0e+00
1	11.942064600074023	0.0e+00	11.942064600074023	0.0e+00

Table 2: Lower and Upper Solutions for Discharging of Capacitor Problem in Example 1

r	$\underline{y}$	$\lvert\underline{Y} - \underline{y}\rvert$	$\overline{y}$	$\lvert\overline{Y} - \overline{y}\rvert$
0	0.00402555153483014	4.336e-18	0.21978766666481017	8.3266e-17
0.2	0.00833059311598270	7.806e-18	0.17806267819124583	8.3266e-17
0.4	0.01527160561607770	8.673e-18	0.14092354148425637	1.9428e-16
0.6	0.02550350197081452	1.737e-18	0.10850319025595753	2.2204e-16
0.8	0.03958206907126909	1.008e-17	0.08085536398902561	8.3266e-17
1	0.05793539992597728	1.048e-17	0.05793539992597728	1.5265e-16

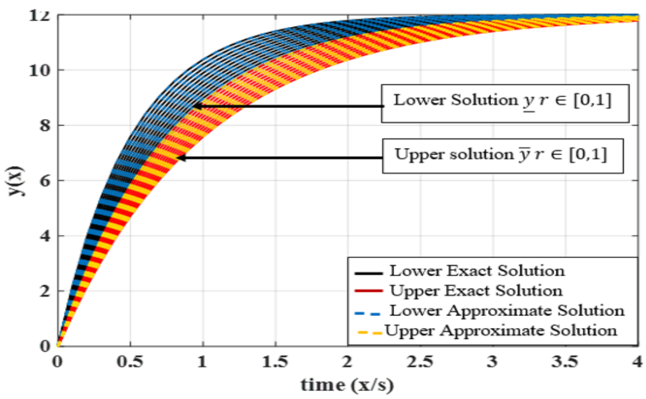

Fig. 2: Example 1 at h=0.1, $x \in [0,4], r \in [0,1]$

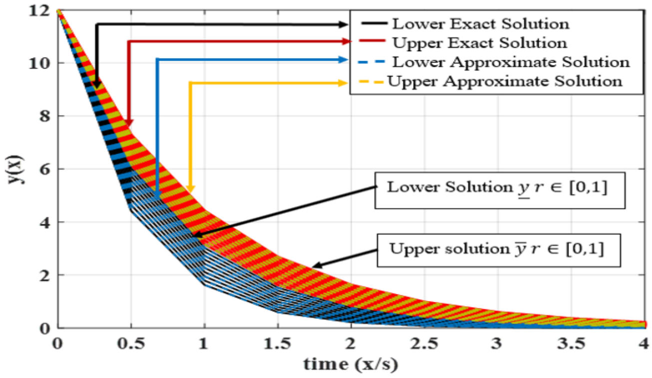

Fig. 3: Example 1 at h=0.1, $x \in [0,4], r \in [0,1]$

In subsequent examples (Examples 2 and 3), since the exact solution cannot be obtained analytically, the proposed method in this study is used to obtain the approximate solution. It is seen that the approximate solution shows a non-monotone behaviour as time increases. The approximate solution in Tables 3 and 4 shows the lower and upper solutions using triangular fuzzy numbers.

Example 2. [20].

Consider the following nonlinear FIVP

$$y'(x) = \cos(xy), y(0,r) = (\tfrac{\pi}{2} r, \pi - \tfrac{\pi}{2} r).$$

Table 3: Lower and Upper Solution for Example 2

r	$\underline{y}$	$\overline{y}$
0	0.61513329423446	2.81437834172917
0.2	0.62072922853453	1.31086566147775
0.4	0.62648141558304	0.74700523547913
0.6	0.63281743366655	0.68810151091971
0.8	0.64032241288569	0.66399579330793
1	0.6500044930335	0.65000449303352

Example 3. [21].

Consider the following nonlinear FIVP

$$y'(x) = x^2 + y^2, y(0,r) = (0.1r - 0.1, 0.1 - 0.1r).$$

Table 4: Lower and Upper Solution for Example 3

r	$\underline{y}$	$\overline{y}$
0	0.24913567333881	0.48255938900528
0.2	0.26259107127145	0.45370358499704
0.4	0.28321967366284	0.42612209786584
0.6	0.30466869011884	0.39973233756115
0.8	0.32698805629646	0.37445870012445
1	0.35023184431536	0.35023184431536

In Example 1, a crisp capacitor model was successfully solved using the proposed method with a fuzzy initial value, and the results were compared to the exact solution. The results are seen in Table 1 and 2 with charging and discharging of the capacitor. These tables, showing the comparison between exact and approximate solution, indicate that the accuracy of the solution in terms of absolute error is quite impressive. The nonlinear Examples 2 and 3, which cannot be solved exactly, are solved numerically by the proposed method. The obtained results are demonstrated in Tables 3 and 4. Although, Example 3 was solved by [21] with homotopy perturbation method, where the authors solved crisp Riccati equation with two defuzzification for FIVPs, their obtained results lie in the short time interval [0,0.5] which indicate that their proposed method is limited to the specific points with large amounts of mathematical complexity.

6. CONCLUSION

The major objective of this research is to enhance the solution accuracy in terms of absolute error for first order FIVPs. As a result, this article developed a generalised step length block method for first order fuzzy ordinary differential equations with the presence of second and third derivatives. Because the algorithm can simultaneously construct block methods of step length *k* for solving first order FIVPs, the generalised technique is considered as extremely flexible. The sample block method with second and third derivatives scheme has proven to be a viable strategy with increased accuracy for solving both linear and nonlinear FIVPs. The method was developed using a linear block approach with low computational complexity, while also satisfying all convergence conditions for the block methods. The solution of the FIVPS as seen in the tables and graphs demonstrates the applicability of the three-step implicit block method for first order FIVPs. So, this generalised approach is suitable for developing block methods for first order FIVPs.

REFERENCES

[1] Ahmady N, Allahviranloo T, Ahmady E. (2020) A modified Euler method for solving fuzzy differential equations under generalized differentiability. Computational and Applied Mathematics, 39(2):1-21. Doi: https://doi.org/10.1007/s40314-020-1112-1

[2] Kodal Sevindir H, Cetinkaya S, Tabak G. (2018) On numerical solutions of fuzzy differential equations. International Journal of Development Research, 8(9): 22971- 22979. Doi: https://www.researchgate.net/publication/329737810

[3] Ahmadian A, Salahshour S, Chan CS, Baleanu D. (2018) Numerical solutions of fuzzy differential equations by an efficient Runge–Kutta method with generalized differentiability. Fuzzy Sets and Systems, 331: 47-67. Doi: https://doi.org/10.1016/j.fss.2016.11.013

[4] Jameel AF, Saleh HH, Azmi A, Alomari AK, Anakira NR, Man NH. (2022) Efficient approximate analytical methods for nonlinear fuzzy boundary value problem. International Journal of Electrical & Computer Engineering, 12(2): 1916-1928. Doi: https://doi.org/10.11591/ijece.v12i2.pp1916-1928

[5] Maghool FH, Radhy ZH, Mehdi HA, Abass HM. (2019) Simpson's rule to solve fuzzy differential equations. International Journal of Advanced Research in Science, Engineering and Technology, 6(10): 11306-11315. Doi: https://www.researchgate.net/publication/337941473

[6] Fariborzi Araghi MA, Barzegar Kelishami H. (2021) Numerical accuracy of the predictor-corrector method to solve fuzzy differential equations based on the stochastic arithmetic. Fuzzy Information and Engineering, 12(3): 335-354. Doi: https://doi.org/10.1080/16168658.2021.1880134

[7] Shang D, Guo X. (2013) Adams predictor-corrector systems for solving fuzzy differential equations. Mathematical Problems in Engineering, 2013(1): 1-12. Doi: https://doi.org/10.1155/2013/312328

[8] Allahviranloo T, Abbasbandy S, Ahmady N, Ahmady E. (2009) Improved predictor–corrector method for solving fuzzy initial value problems. Information Sciences, 179(7): 945-55. Doi: https://doi.org/10.1016/j.ins.2008.11.030

[9] Isa S, Abdul Majid Z, Ismail F, Rabiei F. (2018) Diagonally implicit multistep block method of order four for solving fuzzy differential equations using Seikkala derivatives. Symmetry, 10(2): 42-53. Doi: https://doi.org/10.3390/sym10020042

[10] Ramli A, Abdul Majid Z. (2016) Fourth order diagonally implicit multistep block method for solving fuzzy differential equations. International Journal of Pure and Applied Mathematics, 107(3): 635-660. Doi: https://doi.org/10.12732/ijpam.v107i3.12

[11] Mohd Zawawi IS, Ibrahim ZB, Suleiman M. (2013) Diagonally implicit block backward differentiation formulas for solving fuzzy differential equations. In AIP Conference Proceedings, 1522(1), 681-687. Doi: https://doi.org/10.1063/1.4801191

[12] Barnabas B. (2013) Fuzzy sets. In Mathematics of Fuzzy Sets and Fuzzy Logic, pp. 1-12: Springer. Doi: https://doi.org/10.1007/978-3-642-35221-8_1

[13] Zimmermann HJ. (2010) Fuzzy set theory. Wiley Interdisciplinary Reviews: Computational Statistics, 2,(3): 317-320. Doi: https://doi.org/10.1002/wics.82

[14] Madan PL, Ralescu DA. (1983) Differentials of fuzzy functions. Journal of Mathematical Analysis and Applications, 91(2): 552-558. Doi: https://doi.org/10.1016/0022-247X(83)90169-5

[15] Babakordi F, Allahviranloo T. (2021) A new method for solving fuzzy Bernoulli differential equation. Journal of Mathematical Extension, 15(4): 1-20. Doi: https://doi.org/10.30495/JME.2021.1704

[16] Hashim HA, Shather A., Jameel AF, Saaban A. (2019) Numerical solution of first order nonlinear fuzzy initial value problems by six-stage fifth-order Runge-Kutta method. International Journal of Innovative Technology and Exploring Engineering, 8(5): 166-170. Doi: https://www..net/2/publication/332141468

[17] Sindu DS, Ganesan K. (2017) An approximate solution by fuzzy Taylor's method. International Journal of Pure and Applied Mathematics, 113(13): 236-44. Doi: https://acadpubl.eu/jsi/2017-113-pp/articles/13/26.pdf

[18] Adeyeye O, Omar Z. (2019) Direct solution of initial and boundary value problems of third order ODEs using maximal order fourth-derivative block method. In AIP Conference Proceedings, 2138: 030002_1-030002_6. Doi: https://doi.org/10.1063/1.5121039

[19] Hohenauer W. (2018) Hohenauer-Physical Modelling Based on First Order Odes. Doi: https://simiode.org/resources/4448
[20] Ahmad MZ, Hasan MK, De Baets B. (2013) Analytical and numerical solutions of fuzzy differential equations. Information Sciences, 236: 156-167. Doi: https://doi.org/10.1016/j.ins.2013.02.026
[21] Jameel AF, Ismail AIM. (2015) Approximate solution of first order nonlinear fuzzy initial value problem with two different fuzzifications. Journal of Uncertain Systems, 9(3): 221-229. Doi: http://www.worldacademicunion.com/journal/jus/jusVol09No3paper06.pdf