

ON CALCULATION OF MULTIPLE EIGENVALUES OF THE LINEAR OPERATOR-FUNCTION BY REDUCTION PSEUDO-PERTURBATION METHODS

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ABSTRACT: This article examines the theory of bifurcations, which applies to the problem of retaining of the approximately given multiple eigenvalues and their generalized eigenvectors. This approach allows the reduction of algebraic multiplicity of eigenvalue for one and transfers the problem to the similar one but with simple eigenvalue. The method of the false perturbation is used to construct iterative processes.

ABSTRAK: Dalam artikel ini, kaedah teori bifurkasi diaplikasikan terhadap masalah untuk mengekalkan penghampiran pelbagai nilai eigen dan vektor eigen yang teritlak. Kaedah ini membenarkan pengurangan kegandaan aljabar nilai eigen kepada satu dan memindahkan permasalahan kepada yang hampir serupa tetapi dengan nilai eigen yang lebih mudah. Pengkaedahan usikan palsu digunakan untuk proses pelelaran.

KEYWORDS: *bifurcation theory methods; multiple eigenvalues; root number; reduction pseudo-perturbation method*

1. INTRODUCTION

Many of engineering tasks connected with free fluctuations are reduced to eigenvalue problems so that many of them aren't solved analytically but resort to the approached methods. The problem of retaining of the approximately given eigenvalue and its generalized eigenvectors has been considered in the articles [1-3]. The authors used method of the false perturbations which was introduced by Gavurin [4]. According to this method the operator of the false perturbation builds in a way that the known approximations of the eigenvalue and generalized eigenvectors become the exact ones but for the perturbed operator-function.

Using the method of the theory of bifurcation [5], the iteration processes are then built in order to find the exact values of the eigenvalues and generalized eigenvectors of initial operator-function. The most general operator of the false perturbations was built in [3]. In its formula the generalized eigenvectors of the direct operator-function and their adjoints were used in a symmetrical way. This operator was applied to several problems of mathematical physics [6, 7].

The method of the finite-dimensional regularization [8] allows the reduction of geometric multiple of the eigenvalue to a unit (in this case operator-function has many linearly independent eigenvectors without generalized Jordan chains' elements). Here, using the results and methods developed in [8], it will be shown that algebraic multiplicity of the eigenvalue also could be reduced to a unit. The terminology and notations of [5] are used.

2. PROBLEM SETTING

Let E_1 and E_2 - be Banach spaces, $A_0 : E_1 \supset D(A_0) \rightarrow E_2$, $A_1 : E_1 \supset D(A_1) \rightarrow E_2$ - are the densely defined closed linear operators, where $D(A_0) \subset D(A_1)$ and A_1 subordinate to A_0 (i.e. $\|A_1 x\|_{E_2} \leq \|A_0 x\|_{E_2} + \|x\|_{E_1}$ on $D(A_0)$) or $D(A_1) \subset D(A_0)$ and A_0 subordinate to A_1 (i.e. $\|A_0 x\|_{E_2} \leq \|A_1 x\|_{E_2} + \|x\|_{E_1}$ on $D(A_1)$).

We consider eigenvalue problem

$$(A_0 - tA_1)x = 0. \tag{2.1}$$

Let unknown eigenvalue λ be the Fredholm point of the linear operator-function $A_0 - tA_1$ with $N(A_0 - \lambda A_1) = \text{span}\{\varphi_1, \dots, \varphi_n\}$, $N(A_0^* - \lambda A_1^*) = \text{span}\{\psi_1, \dots, \psi_n\}$. Then corresponding A_1 - and A_1^* - Jordan chains [5] are defined by formulas:

$$(A_0 - tA_1)\varphi_i^{(s)} = A_1\varphi_i^{(s-1)}, (A_0^* - tA_1^*)\psi_i^{(s)} = A_1^*\psi_i^{(s-1)}, s = \overline{2, p_i}, i = \overline{1, n}, \tag{2.2}$$

$$K = \det\left\|\langle A_1\varphi_i^{(p_i)}, \psi_j^{(1)} \rangle\right\| \neq 0, L = \det\|L_{ij}\| \neq 0, L_{ij} = \left\|\langle A_1\varphi_i^{(p_i+1-j)}, \psi_k^{(1)} \rangle\right\|, \\ i(k) = \overline{1, n}, j(l) = \overline{2, p_i(p_k)}.$$

According to [9] elements $\varphi_i^{(j)}, \psi_k^{(l)}, j(l) = \overline{2, p_i(p_k)}, i(k) = \overline{1, n}$ of the A_1 - and A_1^* - Jordan chains, could be chosen in such a way that they satisfy following biorthogonality conditions:

$$\langle \varphi_i^{(j)}, \gamma_k^{(l)} \rangle = \delta_{ik} \delta_{jl}, \langle z_i^{(j)}, \psi_k^{(l)} \rangle = \delta_{ik} \delta_{jl}, j(l) = \overline{2, p_i(p_k)} \tag{2.3}$$

$$\gamma_k^{(l)} = A_1^* \psi_k^{(p_k+1-l)}, z_i^{(j)} = A_1 \varphi_i^{(p_i+1-j)}, i(k) = \overline{1, n}.$$

We assume that $\Lambda, \varphi_{i0}^{(s)}, \psi_{i0}^{(s)}$ are sufficiently good approximations to an unknowns eigenvalue λ and elements of Jordan chains: $\|\varphi_i^{(s)} - \varphi_{i0}^{(s)}\| \leq \varepsilon, \|\psi_i^{(s)} - \psi_{i0}^{(s)}\| \leq \varepsilon, \|\varphi_i^{(s)} - \varphi_{i0}^{(s)}\| \leq \varepsilon.$

We suppose that numbers K_0 and L_0 (which are approximations of K and L from the formula (2.2)) are close to 1.

Next lemma was proved in [3].

Lemma. Going over linear combinations, can be defined as the systems

$$\{\gamma_k^{(l)}\}_{k,l=1}^{n,p_k}, \gamma_k^{(l)} = A_1^* \psi_k^{(p_k+1-l)}, \{z_k^{(l)}\}_{k,l=1}^{n,p_k}, z_k^{(l)} = A_1 \varphi_k^{(p_k+1-l)} \tag{2.4}$$

that satisfies the following biorthogonality conditions:

$$\langle \varphi_{i0}^{(j)}, \gamma_{k0}^{(l)} \rangle = \delta_{ik} \delta_{jl}, \langle z_{i0}^{(j)}, \psi_{k0}^{(l)} \rangle = \delta_{ik} \delta_{jl}, j(l) = \overline{2, p_i(p_k)}. \tag{2.5}$$

Now we want to apply methods developed in [8] to the problem of retaining of the approximately given multiple eigenvalues and their generalized eigenvectors.

3. REDUCTION PSEUDOPERTURBATION METHODS

Let us introduce an operator-function

$$\overline{A(t)} \equiv (\overline{A_0 - tA_1}) + \sum_{k=2}^{p_1} \langle \cdot, \gamma_{10}^{(k)} \rangle z_{10}^{(p_1+1-k)} + \sum_{i=2}^n \sum_{k=1}^{p_i} \langle \cdot, \gamma_{i0}^{(k)} \rangle z_{i0}^{(p_i+1-k)} \quad (3.1)$$

which we call regularisation of the operator - function $A_0 - tA_1$.

Theorem 3.1

Unknown eigenvalue λ is the simple eigenvalue of the operator-function (3.1). Moreover, the eigenvector and the defect functional of $\overline{A(t)}$ are defined by formulas

$$\tilde{\varphi} = \varphi_1^{(p_1)} + \sum_{i=1}^n c_{i1} \varphi_i + \sum_{i=2}^n \sum_{s=2}^{p_i} c_{is} \varphi_i^{(s)} + \sum_{s=1}^{p_1-1} c_{1s} \varphi_1^{(s)}, \quad \tilde{\psi} = \psi_1 + \sum_{i=1}^n \sum_{s=2}^{p_i} d_{is} \psi_i^{(s)} + \sum_{i=2}^n d_{i1} \psi_i \quad (3.2)$$

Proof:

If the formulas (3.2) hold then

$$\begin{aligned} 0 &= \overline{A(\lambda)} \tilde{\varphi} = A(\lambda) \varphi_1^{(p_1)} + \sum_{i=2}^n \sum_{s=2}^{p_i} c_{is} A(\lambda) \varphi_i^{(s)} + \sum_{s=1}^{p_1-1} c_{1s} A(\lambda) \varphi_1^{(s)} + \\ &+ \sum_{s=2}^{p_1} \langle \varphi_1^{(p_1)}, \gamma_{10}^{(s)} \rangle z_{10}^{(p_1+1-s)} + \sum_{i=1}^n \sum_{s=2}^{p_i} c_{i1} \langle \varphi_i, \gamma_{i0}^{(s)} \rangle z_{i0}^{(p_i+1-s)} + \sum_{s=2}^{p_1-1} \sum_{k=2}^{p_1} c_{1s} \langle \varphi_1^{(s)}, \gamma_{10}^{(k)} \rangle z_{10}^{(p_1+1-k)} + \\ &+ \sum_{i=2}^n \sum_{s=2}^{p_i} \sum_{k=2}^{p_i} c_{is} \langle \varphi_i^{(s)}, \gamma_{i0}^{(k)} \rangle z_{i0}^{(p_i+1-k)} + \sum_{i=2}^n \sum_{k=1}^{p_i} \langle \varphi_1^{(p_1)}, \gamma_{i0}^{(k)} \rangle z_{i0}^{(p_1+1-k)} + \sum_{i=1}^n \sum_{s=2}^{p_i} \sum_{k=1}^{p_s} c_{i1} \langle \varphi_i^{(s)}, \gamma_{s0}^{(k)} \rangle z_{s0}^{(p_s+1-k)} + \\ &+ \sum_{i=2}^n \sum_{s=2}^{p_i-1} \sum_{k=2}^{p_i} c_{1s} \langle \varphi_1^{(s)}, \gamma_{i0}^{(k)} \rangle z_{i0}^{(p_1+1-k)} + \sum_{i=2}^n \sum_{i=2}^n \sum_{s=2}^{p_i} \sum_{k=2}^{p_i} c_{is} \langle \varphi_i^{(s)}, \gamma_{i0}^{(k)} \rangle z_{i0}^{(p_i+1-k)}, \\ 0 &= A^*(\lambda) \tilde{\psi} = \sum_{i=1}^n \sum_{s=2}^{p_i} d_{is} A_1^* \psi_i^{(s-1)} + \sum_{s=2}^{p_1} \langle z_{10}^{(p_1+1-s)}, \psi_1 \rangle \gamma_{10}^{(s)} + \sum_{i=2}^n \sum_{s=2}^{p_i} d_{i1} \langle z_{10}^{(p_1+1-s)}, \psi_1 \rangle \gamma_{i0}^{(s)} + \\ &+ \sum_{i=2}^n \sum_{s=2}^{p_i} \sum_{k=2}^{p_i} d_{is} \langle z_{10}^{(p_1+1-s)}, \psi_i^{(k)} \rangle \gamma_{i0}^{(s)} + \sum_{i=2}^n \sum_{k=1}^{p_i} \langle z_{i0}^{(p_i+1-k)}, \psi_1 \rangle \gamma_{i0}^{(k)} + \\ &+ \sum_{i=2}^n \sum_{k=1}^{p_i} \sum_{j=1}^n \sum_{s=2}^{p_j} d_{js} \langle z_{i0}^{(p_i+1-k)}, \psi_j^{(s)} \rangle \gamma_{i0}^{(k)} + \sum_{i=2}^n \sum_{j=2}^{p_i} \sum_{k=1}^{p_i} d_{j1} \langle z_{10}^{(p_1+1-k)}, \psi_i \rangle \gamma_{i0}^{(k)}. \end{aligned}$$

Applying the functional $\psi_{v0}^{(\mu)}$ to the first equality and elements $\varphi_{v0}^{(\mu)}$ to the second one, we obtain the system of the linear algebraic equations to find the unknown coefficients $c_{11}, K, c_{np_n}; d_{12}, K, d_{np_n}$:

$$\sum_{j=1}^{k(A)-1} a_{ij} x_j = e_i, \quad i = \overline{2, k(A)-1}; \quad (3.3)$$

$$\sum_{j=1}^{k(A)-1} b_{ij} y_j = e'_i, \quad i = \overline{2, k(A)-1}, \quad (3.4)$$

where $k(A) = p_1 + \Lambda + p_n$ - is a root number of the operator-function $A(t)$, $x_j = c_{1j}$, $j = \overline{1, p_1 - 1}$, $x_{p_1+j} = c_{2,j+1}$, $j = \overline{0, p_2 - 1}$, $x_{p_1+\Lambda+p_{k-1}+j} = c_{k,j+1}$, $j = \overline{0, p_k - 1}$, $k = \overline{2, n}$, $a_{i1} = \langle \varphi_1, \gamma_{10}^{(l)} \rangle$, $a_{ik} = \langle z_1^{(p_1+2-k)}, \psi_{10}^{(p_1+1-l)} \rangle + \langle \varphi_1^{(k)}, \gamma_{10}^{(l)} \rangle$, $k = \overline{2, p_1 - 1}$, $a_{l,p_1+\Lambda+p_{i-1}} = \langle \varphi_i, \gamma_{10}^{(l)} \rangle$, $a_{l,p_1+\Lambda+p_{i-1}+k} = \langle z_1^{(p_1+1-k)}, \psi_{10}^{(p_1+1-l)} \rangle + \langle \varphi_i^{(k)}, \gamma_{10}^{(l)} \rangle$, $k = \overline{2, p_1 - 1}$, $l = \overline{1, p_1 - 1}$, $i = \overline{2, n}$, $a_{p_1+\Lambda+p_{s-1}+l,1} = \langle \varphi_1, \gamma_{s0}^{(l+1)} \rangle$, $a_{p_1+\Lambda+p_{s-1}+l,k} = \langle z_1^{(p_1+2-k)}, \psi_{s0}^{(p_s+1-l)} \rangle + \langle \varphi_1^{(k)}, \gamma_{s0}^{(l)} \rangle$, $k = \overline{2, p_1 - 1}$, $a_{p_1+\Lambda+p_{s-1}+l,p_1+\Lambda+p_{i-1}} = \langle \varphi_i, \gamma_{s0}^{(l)} \rangle$, $a_{p_1+\Lambda+p_{s-1}+l,p_1+\Lambda+p_{i-1}+k} = \langle z_i^{(p_i+1-k)}, \psi_{s0}^{(p_s+1-l)} \rangle + \langle \varphi_i^{(k+1)}, \gamma_{s0}^{(l)} \rangle$, $k = \overline{1, p_i - 1}$, $i = \overline{2, n}$, $l = \overline{1, p_{s+1} - 1}$, $s = \overline{2, n}$; $y_j = d_{1,j+1}$, $j = \overline{1, p_1 - 1}$, $y_{p_1+\Lambda+p_{k-1}+j} = d_{k,j+1}$, $j = \overline{1, p_k - 1}$, $k = \overline{2, n}$, $b_{p_1+\Lambda+p_{s-1}+l,1} = \langle \varphi_{s0}^{(p_s+1-l)}, \gamma_1^{(p_1)} \rangle$, $b_{p_1+\Lambda+p_{s-1}+l,k} = \langle \varphi_{s0}^{(p_s+1-l)}, \gamma_1^{(p_1+k)} \rangle + \langle z_s^{(l)}, \psi_1^{(k)} \rangle$, $k = \overline{1, p_s - 1}$, $b_{p_1+\Lambda+p_{s-1}+l,p_1+\Lambda+p_{i-1}} = \langle z_{s0}^{(l)}, \psi_i \rangle$, $b_{p_1+\Lambda+p_{s-1}+l,p_1+\Lambda+p_{i-1}+k} = \langle \varphi_{s0}^{(p_s+1-l)}, \gamma_i^{(p_i+1-k)} \rangle + \langle z_{s0}^{(l)}, \psi_i^{(k+1)} \rangle$, $k = \overline{1, p_i - 1}$, $i = \overline{2, n}$, $l = \overline{1, p_{s+1} - 1}$, $s = \overline{2, n}$; $e_i = -\left[\langle z_1^{(2)}, \psi_{10}^{(p_1+1-i)} \rangle + \langle \varphi_1^{(p_1)}, \gamma_{10}^{(i)} \rangle \right]$, $i = \overline{1, p_1 - 1}$, $e_{p_1+i} = -\left[\langle z_1^{(2)}, \psi_{20}^{(p_2+1-i)} \rangle + \langle \varphi_1^{(p_1)}, \gamma_{20}^{(i)} \rangle \right]$, $i = \overline{1, p_2 - 1}$, $e_{p_1+\Lambda+p_{s-1}+i} = -\left[\langle z_1^{(2)}, \psi_{20}^{(p_2+1-i)} \rangle + \langle \varphi_1^{(p_1)}, \gamma_{s0}^{(i)} \rangle \right]$, $i = \overline{1, p_s - 1}$, $s = \overline{2, n}$; $e'_i = \langle z_{10}^{(i)}, \psi_1 \rangle$, $i = \overline{1, p_1 - 1}$, $e'_{p_1+\Lambda+p_{s-1}+i} = -\langle z_{s0}^{(i)}, \psi_1 \rangle$, $i = \overline{0, p_s - 1}$, $s = \overline{2, n}$.

It is not difficult to show that $|a_{ii}| = 1 + O(\varepsilon)$, $|a_{i,i+1}| = 1 + O(\varepsilon)$, $|a_{ij}| = O(\varepsilon)$, $j \neq i, i + 1$, $i, j = \overline{1, k(A) - 1}$, $|b_{ii}| = 1 + O(\varepsilon)$, $|b_{i+1,i}| = 1 + O(\varepsilon)$, $i = \overline{1, p_1 - 2}$, $|b_{p_1 p_1}| = 1 + O(\varepsilon)$, $|b_{i,i+1}| = 1 + O(\varepsilon)$,

$|b_{ii}| = O(\varepsilon)$ for $j \neq i, i + 1$, and for all other cases. Consequently, $\det \|a_{is}\| = 1 + O(\varepsilon)$, $\det \|b_{is}\| = 1 + O(\varepsilon)$, i.e. systems (3.3) and (3.4) have unique solutions.

Because of $|e_{p_1-1}| = \left| \langle \varphi_1^{(p_1)}, \gamma_{10}^{(p_1-1)} \rangle + \langle z_1^{(2)}, \psi_{10}^{(2)} \rangle \right| = 1 + O(\varepsilon)$, and $|e_i| = O(\varepsilon)$, for all $i \neq p_1 - 1$, we obtain $c_i = O(\varepsilon)$, $i \neq p_1 - 1$ and $c_{p_1-1} = -1 + O(\varepsilon)$. Therefore, $\tilde{\varphi}_0 = \varphi_{10}^{(p_1)} - \varphi_{10}^{(p_1-1)}$ and $\tilde{\psi}_0 = \psi_{10}$ should be taken as initial approximations for φ and ψ . The solution of the equation $\langle \overline{A(t)} \tilde{\varphi}_0, \tilde{\psi}_0 \rangle = 0$, is taken as the initial approximation for eigenvalue λ , i.e. $\lambda_0 = \frac{\langle A_0 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle + 1}{\langle A_1 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle}$.

Choosing $\tilde{\gamma}_0 = \frac{1}{k_0} A_1^* \psi_{10}$ and $\tilde{z}_0 = \frac{1}{k_0} A_1 \tilde{\varphi}_0$, we find that $\langle \tilde{\varphi}_0, \tilde{\gamma}_0 \rangle = 1$, $\langle \tilde{z}_0, \tilde{\psi}_0 \rangle = 1$, because $k_0 = \langle A_1 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle = \langle A_1 \varphi_{10}^{(p_1)}, \tilde{\psi}_0 \rangle - \langle A_1 \varphi_{10}^{(p_1-1)}, \tilde{\psi}_0 \rangle = \langle A_1 \varphi_{10}^{(p_1)}, \tilde{\psi}_0 \rangle \neq 0$.

We define pseudoperperturbed operator using formulas

$$D_0 x = \langle x, \tilde{\gamma}_0 \rangle \overline{A(\lambda_0)} \tilde{\varphi}_0 + \langle x, A^*(\lambda_0) \tilde{\psi}_0 \rangle \tilde{z}_0, \quad D_0^* x = \langle \overline{A(\lambda_0)} \tilde{\varphi}_0, y \rangle \tilde{\gamma} + \langle \tilde{z}_0, y \rangle \overline{A^*(\lambda_0)} \tilde{\psi}_0.$$

Then, $D_0 \tilde{\varphi}_0 = \overline{A(\lambda_0)} \tilde{\varphi}_0$, $D_0^* \tilde{\psi}_0 = \overline{A^*(\lambda_0)} \tilde{\psi}_0$, i.e. $N(\overline{A(\lambda_0)} - D_0) = \{\tilde{\varphi}_0\}$, $N(\overline{A^*(\lambda_0)} - D_0^*) = \{\tilde{\psi}_0\}$.

Using the Schmidt's regularization [5] the equation $\overline{A(t)}x = 0$ could be reduced to the system:

$$\begin{cases} x = \xi [I + \bar{\Gamma}_0 (D_0 + \overline{A(t)} - \overline{A(\lambda_0)})]^{-1} \tilde{\varphi}_0, \\ \xi = \langle x, \tilde{\gamma}_0 \rangle. \end{cases} \quad (3.5)$$

where, $\bar{\Gamma}_0 = [\overline{A(\lambda_0)} - D_0 + \langle \cdot, \tilde{\gamma}_0 \rangle \tilde{z}_0]^{-1}$.

If we substitute x in the second equation (3.5) with the value found from the first, we construct the equation:

$$F(t) \equiv 1 - \langle [I + \bar{\Gamma}_0 (D_0 + \overline{A(t)} - \overline{A(\lambda_0)})]^{-1} \tilde{\varphi}_0, \tilde{\gamma}_0 \rangle = 0, \quad (3.6)$$

which is called "bifurcation equation". The exact eigenvalue λ is a simple root of bifurcation equation.

Let $S(\lambda_0; \rho)$ - be the ball of radius ρ centred at λ_0 .

Theorem 3.2

If initial approximations is the sufficiently good, then there is the ball $S(\lambda_0; r)$, where $r = \frac{1 - \sqrt{1 - 4h}}{2h} \eta \leq \rho$, $h = |F'(\lambda_0)|^{-1} L |F(\lambda_0)| \leq \frac{1}{4}$, $\eta = |F'(\lambda_0)|^{-1} |F(\lambda_0)|$, in which equation (3.6) has unique solution $t = \lambda$. The iterations defined by the Newton's modified method, given below,

$$\lambda_{m+1} = \lambda_m - [F'(\lambda_0)]^{-1} F(\lambda_m), \quad m = 0, 1, 2, \dots, \quad (3.7)$$

will converge toward this solution.

Proof:

Firstly we verify the conditions of the Theorem 3.2 [10, p.446]:

$$1) F(\lambda_0) = 1 - \langle [I + \bar{\Gamma}_0 D_0]^{-1} \tilde{\varphi}_0, \tilde{\gamma}_0 \rangle = 1 - \langle \tilde{\varphi}_0, \tilde{\gamma}_0 \rangle + \langle D_0 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle - \Lambda = O(\|D_0\|);$$

$$|F(\lambda_0)| \leq C_1 (\|D_0\|);$$

$$2) F'(\lambda_0) = \langle [I + \bar{\Gamma}_0 D_0]^{-1} \bar{\Gamma}_0 A_1 [I + \bar{\Gamma}_0 D_0]^{-1} \tilde{\varphi}_0, \tilde{\gamma}_0 \rangle = \langle A_1 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle -$$

$$- 2 \langle D_0 \bar{\Gamma}_0 D_0 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle + \Lambda = k_0 - 2 \langle D_0 \bar{\Gamma}_0 D_0 \tilde{\varphi}_0, \tilde{\psi}_0 \rangle + \Lambda,$$

$$|F'(\lambda_0)| \geq |k_0| - C_2 (\|D_0\|),$$

$$M_0 = |F'(\lambda_0)|^{-1} \leq [|k_0| - C_2 (\|D_0\|)]^{-1};$$

$$3) |F'(t_1) - F'(t_2)| = |F''(t_1 + \theta(t_1 - t_2))(t_1 - t_2)| \leq L_1 \cdot |t_1 - t_2|, \text{ where } L = \sup_{t \in G} |F''(t)|.$$

If initial approximations are sufficiently good, we can choose \mathcal{E} such that $M_0LC_1 \leq \frac{1}{4}$. Then, according to Theorem 3.2 [10, p.446]-where, there is unique solution of the equation (3.6) in the ball $S(\lambda_0; r)$, where $r = \frac{1 - \sqrt{1 - 4h}}{2h} \eta \leq \rho$. The iterations calculated by formula (3.7) converge toward this solution. It should be noted that on every step of iterative process it is necessary to solve only one equation:

$$\left[A(\lambda_m) + \langle \cdot, \tilde{\gamma}_0 \rangle \tilde{z}_0 \right] x = \tilde{z}_0.$$

Theorem 3.3

The elements of GJS $\{\varphi_i^{(j)}\}_{i,j=1,1}^{n,p_i}$, $\{\psi_i^{(j)}\}_{i,j=1,1}^{n,p_i}$ are the solutions of the following recurrent systems:

$$\begin{aligned} \left[A_0 - \lambda A_1 + \sum_{s=1}^n \langle \cdot, \gamma_{s0} \rangle z_{s0} \right] x &= z_{i0}, & \left[A_0^* - \lambda A_1^* + \sum_{s=1}^n \langle z_{s0}, \cdot \rangle \gamma_{s0} \right] y &= \gamma_{i0}, \\ \left[A_0 - \lambda A_1 + \sum_{s=1}^n \langle \cdot, \gamma_{s0} \rangle z_{s0} \right] x_{j,i} &= A_1 x_{j-1,i} + z_{i0}, & x_{1i} &= \varphi_i, & x_{ji} &= \varphi_i^{(j)}, \\ \left[A_0^* - \lambda A_1^* + \sum_{s=1}^n \langle z_{s0}, \cdot \rangle \gamma_{s0} \right] y_{ji} &= A_1^* y_{j-1,i} + \gamma_{i0}, & y_{1i} &= \psi_i, & y_{ji} &= \psi_i^{(j)}, & j = \overline{2, p_i}, & i = \overline{1, n}. \end{aligned}$$

Proof.

We want to apply method of false perturbations to initial equations (2.1) and (2.2). In order to do this we are using pseudo perturbed operator D_0 suggested by Loginov [8]:

$$\begin{aligned} D_0 x &= \sum_{i=1}^n \sum_{j=1}^{p_i} \langle x, \gamma_{i0}^{(j)} \rangle \left[\sigma_{i0}^{(j)} - \sum_{k=1}^n \sum_{s=1}^{p_k} \langle \sigma_{i0}^{(j)}, \psi_{k0}^{(s)} \rangle z_{k0}^{(s)} \right] + \sum_{i=1}^n \sum_{j=1}^{p_i} \langle x, \tau_{i0}^{(j)} \rangle z_{i0}^{(j)}, \\ D_0^* x &= \sum_{i=1}^n \sum_{j=1}^{p_i} \left\langle \sigma_{i0}^{(j)} - \sum_{k=1}^n \sum_{s=1}^{p_k} \langle \sigma_{i0}^{(j)}, \psi_{k0}^{(s)} \rangle z_{k0}^{(s)}, y \right\rangle \gamma_{i0}^{(j)} + \sum_{i=1}^n \sum_{j=1}^{p_i} \langle z_{i0}^{(j)}, y \rangle \tau_{i0}^{(j)}, \end{aligned}$$

which, according to theorem 1 [3], has the following properties: $D_0 \varphi_{i0}^{(j)} = \sigma_{i0}^{(j)}$, $D_0^* \psi_{i0}^{(j)} = \tau_{i0}^{(j)}$. Then $N(A_0 - \lambda A_1 - D_0) = \{\varphi_{i0}\}_{i=1}^n$, $N(A_0^* - \lambda A_1^* - D_0^*) = \{\psi_{i0}\}_{i=1}^n$.

The equation $(A_0 - \lambda A_1)x = 0$ is reduced to the system

$$\begin{cases} x = \sum_{s=1}^n \xi_s [I + \Gamma_0 D_0]^{-1} \varphi_{s0}, \\ \xi_s = \langle x, \gamma_{s0} \rangle, & s = \overline{1, n}, \end{cases}$$

Where $\Gamma_0 = \left[A_0 - \lambda A_1 - D_0 + \sum_{s=1}^n \langle \cdot, \gamma_{s0} \rangle z_{s0} \right]^{-1}$ -is the bounded linear operator which exists according to the E. Schmidt's lemma [5]. Then

$$\xi_k = \sum_{s=1}^n \xi_s \langle [I + \Gamma_0 D_0]^{-1} \varphi_{s0}, \gamma_{k0} \rangle, \quad k = 1, 2, \dots, n.$$

Because the initial equation has exactly n linearly independent solutions, the last system also has n linearly independent solutions. It means that the rank of the matrix of the coefficients of this system is equal to zero and therefore it is possible to assume

$$\varphi_i = [I + \Gamma_0 D_0]^{-1} \varphi_{i0}.$$

Similarly (2.2) could be reduced to the system

$$\begin{cases} x_s = \sum_{k=1}^n \xi_{sk} [I + \Gamma_0 D_0]^{-1} \varphi_{k0} + [I + \Gamma_0 D_0]^{-1} \Gamma_0 A_1 x_{s-1}, \\ \xi_{sk} = \langle x_s, \gamma_{k0} \rangle, \quad k = \overline{1, n}. \end{cases}$$

Substituting x_s in the second part of the system with the value found from the first one we can simplify the system:

$$\xi_{sk} = \sum_{l=1}^n \xi_{sl} \langle [I + \Gamma_0 D_0]^{-1} \varphi_{l0}, \gamma_{k0} \rangle + \langle [I + \Gamma_0 D_0]^{-1} \Gamma_0 A_1 x_{s-1}, \gamma_{k0} \rangle.$$

Because it was mentioned earlier that the rank of the matrix of this system equals 0 for each s , we can assume that

$$x_{sk} = [I + \Gamma_0 D_0]^{-1} \varphi_{k0} + [I + \Gamma_0 D_0]^{-1} \Gamma_0 A_1 x_{s-1,k}, \quad k = \overline{1, n}, \quad s > 1.$$

Similarly, applying the same arguments to equations $(A_0^* - \lambda A_1^*)y = 0$ and $(A_0^* - \lambda A_1^*)y_s = A_1^* y_{s-1}$ we come to recurrent equations:

$$\psi_k = [I + \Gamma_0^* D_0^*]^{-1} \psi_{k0},$$

$$y_{sk} = [I + \Gamma_0^* D_0^*]^{-1} \psi_{k0} + [I + \Gamma_0^* D_0^*]^{-1} \Gamma_0^* A_1^* y_{s-1,k}, \quad k = \overline{1, n}, \quad s > 1.$$

Example:

In order to illustrate these results, we consider the eigenvalue problem

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u'(x_0) = u'(1)$$

in the space $C^2[(0, x_0) \cup (x_0, 1)] \cap C^1[0; 1]$. Detailed study of this problem could be found in [11]. If $x_0 = \frac{2s - 2m + 1}{2s + 2m + 1}$ and $0 < m < s + \frac{1}{2}$, the exact value of eigenvalue is

$$\mu_m = \mu_s = \mu_0 \quad (\text{here } \mu_m = \frac{2m\pi}{1 + x_0}, \mu_s = \frac{(2s + 1)\pi}{1 + x_0}).$$

This is the eigenvalue of multiplicity 2 and the corresponding eigenvector and Jordan chain elements are

$$\varphi^{(1)} = \sin \mu_0 x, \quad \varphi^{(2)} = -\frac{1}{2\mu_0} x \cos \mu_0 x,$$

$$\psi^{(1)} = \begin{cases} 0, & 0 \leq x \leq x_0, \\ \frac{8\mu_0}{1-x_0^2} \cos \mu_0 x, & x_0 \leq x \leq 1, \end{cases} \quad \psi^{(2)} = \begin{cases} -\frac{4}{1+x_0} \sin \mu_0 x, & 0 \leq x \leq x_0, \\ -\frac{4(1-x)}{1-x_0^2} \sin \mu_0 x, & x_0 \leq x \leq 1. \end{cases}$$

As initial approximations of functions $\varphi^{(1)}$, $\tilde{\psi}^{(1)}$ and elements of Jordan's $\varphi^{(2)}$, $\tilde{\psi}^{(2)}$, have been taken the initial segments of their Taylor series with 25 terms. All calculation experiments were carried out using program Maple 11.

Under $m=s=1$, $x_0 = \frac{1}{5}$, $\lambda = \frac{25\pi^2}{4} = 61,6850275$:

$\lambda_0 = 61,8321723$; $\lambda_1 = 61,6728554$; $\lambda_2 = 61,6850467$; $\lambda_3 = 61,6850275$;

$\lambda_4 = 61,6850275$.

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