# AN ANALYTICAL TECHNIQUE TO OBTAIN HIGHER-ORDER APPROXIMATE PERIODS FOR THE NONLINEAR OSCILLATOR

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***ABSTRACT:*** In this paper, an analytical technique has been proposed to obtain higher-order approximate periods for the nonlinear oscillator with the square of the angular frequency depending quadratically on the velocity which is based on the harmonic balance method (HBM). Analytical investigation of the appeared set of nonlinear algebraic equations is usually cumbersome, which is addressed by the proposed technique in a novel way. In this paper, this limitation is eradicated and provides desired results without much numerical complexity. Additionally, a new suitable truncation formula has been introduced in which the approximate periods measure much better results than existing periods. The proposed technique is applied to the benchmark nonlinear oscillatory problem where the square of the angular frequency depends quadratically on the velocity to illustrate its novelty, reliability, and wider applicability. It is remarkably improtant to note that, using the proposed technique, a third-order approximate period gives an excellent agreement as compared with the exact ones.

***ABSTRAK:*** Kertas cadangan ini membincangkan tentang teknik analitikal bagi menghasilkan tempoh anggaran lebih tinggi pada sistem bukan linear dengan kuasa dua frekunsi angular halaju kuadratik, iaitu bergantung pada kaedah imbangan harmoni (HBM). Penyelidikan analitikal pada set persamaan algebra bukan linear selalunya adalah rumit, iaitu dibincangkan secara teknik cadangan baru. Kertas ini, menghapuskan kekurangan ini dan menghasilkan keputusan tanpa banyak kerumitan numerik. Selain itu, formula baru yang ringkas dan sesuai ini diperkenalkan dengan tempoh anggaran ukuran keputusan lebih bagus daripada tempoh sedia ada. Teknik yang dicadangkan ini digunakan sebagai penanda aras pada masalah osilator bukan linear dengan frekunsi angular halaju kuadratik bagi memperlihatkan kebaharuan, kebergantungan dan keluasan pada kegunaannya. Pengunaan kaedah ini adalah sangat luar biasa.

## 1. INTRODUCTION

Nonlinear oscillations have important aspects for areas including physical sciences, mechanical structures, engineering and other disciplines [1,2] which appear mathematically in the form of nonlinear differential equations (NDEs). The solution procedure of obtaining approximate solutions of linear differential equations is comparatively easy and well established. In contrast, the solution procedure of obtaining approximate solutions of NDEs remains less available to this day. It is often more difficult to get an analytic approximation than a numerical one. A few nonlinear systems can be solved explicitly, and the numerical methods especially the most well-known Runge-Kutta fourth order method are frequently used to calculated approximate solutions. However, in stiff differential equations and chaotic differential equations, the numerical schemes do not always give accurate results, thus presenting a big challenge to numerical analysis. In this situation, many researchers have shown an intensifying interest in the field of analytical approximate techniques. The most widely used analytical technique for solving nonlinear equations associated with oscillatory systems is the perturbation method [3-6], which is the most versatile tool available in nonlinear analysis of engineering problems, and it is constantly being developed and applied to ever more complex problems. However, the standard perturbation methods have many limitations, and they do not yield for strongly nonlinear oscillators.

As a result, to overcome this shortcoming, in recent years, a large variety of modified perturbation techniques are commonly used in nonlinear systems, especially for strongly nonlinear oscillators. There modified methods include optimal homotopy asymptotic method [7], homotopy perturbation method [8-10], modified homotopy perturbation method [11], modified He's homotopy perturbation method [12-14], modified Lindsted-Poincare method [15], He's modified Lindsted-Poincare method [16], and modified multiple time scale method [17].

In the recent past, some other approximation techniques have been investigated. These techniques include the He's max-min approach [18], elliptic balance [19], algebraic [20], the differential transform approach [21], He's frequency-amplitude formulation [22], iteration [23], the variational approach [24], energy balance [25-29] methods, and the rational harmonic balance method [30]. All have been paid much attention in order to determine periodic solutions of strongly nonlinear oscillatory problems. In fact, to the best of our knowledge, in the energy balance method and some other methods, there is no clear idea to obtain higher-order approximate solutions. Moreover, only first-order approximation has been considered, which does not provides sufficient accuracy.

In this situation, an analytical technique has been proposed based on the harmonic balance method [31-38] to obtain approximate periods to the nonlinear oscillator with the square of the angular frequency depending quadratically on the velocity. The higher-order approximate period (mainly third-order approximation) has been obtained. The proposed technique not only provides accurate results but is also more convenient and efficient for solving more complex nonlinear oscillatory problems. Moreover, using a suitable truncation formula gives approximate periods that are very near to the next higher-order approximation and avoids a lot of calculation. This is the main advantage of the proposed technique presented in this article.

The rest of this paper is organized as follows: In section 2, we give the outline of the solution approaches of the harmonic balance method. In section 3, we implement the harmonic balance method to the nonlinear oscillator with the square of the angular frequency depending quadratically on the velocity. The results and a detailed discussion have been explained extensively in section 4. Concluding remarks are given in section 5.

## 2.   SOLUTION APPROACHES

Consider a general second-order nonlinear differential equation and initial conditions as follows:

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad [x(0) = a_0, \ \dot{x}(0) = 0] \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x})$, $\omega_0 \geq 0$ and $\varepsilon$ is a constant.

A periodic solution of Eq. (1) can be assumed as

$$x = a_0 (\rho \cos(\omega t) + u \cos(3\omega t) + v \cos(5\omega t) + w \cos(7\omega t) + z \cos(9\omega t) \cdots) \tag{2}$$

where $a_0$, $\rho$ and $\omega$ are constants. If $\rho = 1 - u - v - \cdots$, then the solution to Eq. (2) easily satisfies the initial conditions given in Eq. (1). Substituting Eq. (2) into Eq. (1) and expanding $f(x, \dot{x})$ in a Fourier series, reduces it to an algebraic identity

$$a_0 [\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)\cos(3\omega t) + \cdots \qquad \cdots$$
$$+ F_3(a_0, u, \cdots \qquad \cdots \tag{3}$$

By comparing the coefficients of equal harmonic terms of Eq. (3), the following nonlinear algebraic equations are obtained.

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \tag{4}$$

Using the first equation, $\omega^2$ is eliminated from all the remaining equations of Eq. (4). Thus, Eq. (4) takes the following form

$$\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9u F_1), \quad 24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25v F_1), \cdots \tag{5}$$

Substituting $\rho = 1 - u - v - \cdots$, and then simplifying, second-, third- equations of Eq. (5) takes the following form.

$$u = G_1(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \cdots \tag{6}$$

where $G_1, G_2, \cdots$ exclude, respectively, the linear terms of $u, v, \cdots$.

Whatever the values of $\omega_0$, $\varepsilon$ and $a_0$, there exists a parameter $\lambda_0(\omega_0, \varepsilon, a_0) \ll 1$, such that $u, v, \cdots$ are expandable in the following power series in terms of $\lambda_0$ as

$$u = U_1 \lambda_0 + U_2 \lambda_0^2 + \cdots, \quad v = V_1 \lambda_0 + V_2 \lambda_0^2 + \cdots, \quad \cdots \tag{7}$$

where $U_1, U_2, \cdots, V_1, V_2, \cdots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from Eq. (7) into the first equation of Eq. (5), the approximate angular $\omega$ is determined. This completes the determination for the approximate periods obtained by the relation $T = \dfrac{2\pi}{\omega}$.

## 3.  NUMERICAL EXAMPLE

Consider a nonlinear oscillator which was studied in [6,15,31-32] as:

$$\ddot{x} + (1 + \dot{x}^2)x = 0 \tag{8}$$

In Eq. (2), a second-order approximate solution of Eq. (8) is

$$x = a_0(\rho\cos(\omega t) + u\cos(3\omega t)) \tag{9}$$

Substituting Eq. (9) along with $\rho = 1 - u$ into Eq. (8), then simplifying and equating the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$ to zero, the following residuals are obtained.

$$
\begin{aligned}
&1 - \omega^2 + a_0^2\omega^2/4 - u + \omega^2 u + a_0^2\omega^2 u/2 + 11a_0^2\omega^2 u^2/4 - 7a_0^2\omega^2 u^3/2 = 0, \\
&-a_0^2\omega^2/4 + u - 9\omega^2 u + 5a_0^2\omega^2 u/4 - 7a_0^2\omega^2 u^2/4 + 3a_0^2\omega^2 u^3 = 0.
\end{aligned} \tag{10}
$$

From the first equation of Eq. (10), it becomes

$$\omega^2 = (1-u)/(1 - a_0^2/4 - u - a_0^2 u/2 - 11a_0^2 u^2/4 + 7a_0^2 u^3/2). \tag{11}$$

Substituting Eq. (11) into the second equation of Eq. (10), the nonlinear algebraic equation of $u$ is obtained as:

$$u = \lambda_0(-1 + 5u - 14u^2 + 32u^2/a_0^2 + 8u^3 + 2u^4), \quad \lambda_0 = a_0^2/32. \tag{12}$$

Therefore, the power series solution of Eq. (12) in terms of $\lambda_0$ is

$$u = -\lambda_0 - 5\lambda_0^2 + (-39 + 32/a_0^2)\lambda_0^3 + \cdots \tag{13}$$

Substituting the value of $u$ from Eq. (13) into Eq. (11) and then simplifying the second-order approximate angular frequency results in:

$$\omega = \left(1 + \frac{a_0^2}{8} + \frac{3a_0^4}{256} + \frac{3a_0^6}{4096} + \frac{a_0^8}{8192} + \cdots\right), \tag{14}$$

and using the relation $T = \dfrac{2\pi}{\omega}$, the second-order approximate period of Eq. (8) is:

$$T_2 = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{a_0^6}{4096} - \frac{7a_0^8}{65536} + \cdots\right) \tag{15}$$

Considerable calculation is saved and improved results are obtained if we use the truncation principle in Eq. (10). The higher-order terms of $u$ greater than second-order have no effect on the value of the unknowns $u$ and $\omega$. So, we may ignore greater than second-order terms of $u$; but half of the second-order terms are considered. This is called the truncation principle. After using the truncation principle, Eq. (10) can be transformed into

$$
\begin{aligned}
&1 - \omega^2 + a_0^2\omega^2/4 - u + \omega^2 u + a_0^2\omega^2 u/2 + 11a_0^2\omega^2 u^2/8 = 0, \\
&-a_0^2\omega^2/4 + u - 9\omega^2 u + 5a_0^2\omega^2 u/4 - 7a_0^2\omega^2 u^2/8 = 0.
\end{aligned} \tag{16}
$$

From the first equation of Eq. (16), it can be easily written as

$$\omega^2 = (1-u)/(1 - a_0^2/4 - u - a_0^2 u/2 - 11 a_0^2 u^2/8).$$ (17)

Substituting $\omega^2$ into the second equation of Eq. (16), the nonlinear algebraic equation of $u$ is reduce to:

$$u = \lambda_0(-1 + 5u - 21u^2/2 + 32u^2/a_0^2 - 2u^3)$$ (18)

where $\lambda_0$ is given in Eq. (12).

The power series solution of Eq. (18) in terms of $\lambda_0$ is:

$$u = -\lambda_0 - 5\lambda_0^2 + (-71/2 + 32/a_0^2)\lambda_0^3 + \cdots$$ (19)

Substituting the value of $u$ from Eq. (19) into Eq. (17) the second-order approximate angular frequency becomes:

$$\omega = \left(1 + \frac{a_0^2}{8} + \frac{3a_0^4}{256} + \frac{a_0^6}{16384} - \frac{35a_0^8}{131072} - \frac{2065a_0^{10}}{16777216} + \cdots\right)$$ (20)

and the approximate period of oscillation in using the truncation principle is:

$$T_2^{truc} = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5.6a_0^6}{6144} + \frac{15a_0^8}{131072} + \cdots\right)$$ (21)

By the same mathematical manipulation as stated above, the higher-order approximations have been obtained using the proposed technique. In this paper, a third-order approximation is

$$x(t) = a_0 \cos(\omega t) + a_0 u(\cos(3\omega t) - \cos(\omega t)) + a_0 v(\cos(5\omega t) - \cos(\omega t))$$ (22)

Substituting Eq. (22) into Eq. (8), then simplifying and equating the coefficients of $\cos(\omega t)$, $\cos(3\omega t)$ and $\cos(5\omega t)$ equal to zero, the related equations are

$$
\begin{aligned}
&1 - \omega^2 + a_0^2\omega^2/4 - u + \omega^2 u + a_0^2\omega^2 u/2 + 11 a_0^2\omega^2 u^2/4 - 7a_0^2\omega^2 u^3/2 \\
&- v + \omega^2 v + \cdots = 0, \\
&- a_0^2\omega^2/4 + u - 9\omega^2 u + 5a_0^2\omega^2 u/4 - 7a_0^2\omega^2 u^2/4 + 3a_0^2\omega^2 u^3 + 3a_0^2\omega^2 v \\
&- a_0^2\omega^2 uv/2 + \cdots = 0, \\
&- 7a_0^2\omega^2 u/4 + 11 a_0^2\omega^2 u^2/4 - a_0^2\omega^2 u^3 + v - 25\omega^2 v + a_0^2\omega^2 v/2 + 5a_0^2\omega^2 uv/2 \\
&+ \cdots = 0
\end{aligned}
$$ (23)

From the first equation of Eq. (23), it can be written as:

$$\omega^2 = (1-u-v)/(1 - a_0^2/4 - u - a_0^2 u/2 - 11 a_0^2 u^2/4 - v + 3a_0^2 v/4 + \cdots).$$ (24)

With the help of Eq. (24), $\omega^2$ is eliminated from the second and third equations of Eq. (23) and then simplifying, the nonlinear algebraic equations of $u$ and $v$ are obtained as:

$$u = \lambda_0(-1 + 5u - 14u^2 + 32u^2/a_0^2 + 8u^3 + 2u^4 + 13v - 16uv - 31u^2v + \cdots), \qquad (25)$$

$$v = \mu_0(-7u + 18u^2 - 15u^3 + 4u^4 + v + 13uv - 23u^2v - 3v^2 - 39uv^2 - \cdots), \qquad (26)$$

where $\mu_0 = a_0^2/96$ and $\lambda_0$ is given in Eq. (12). The algebraic relation between $\lambda_0$ and $\mu_0$ is:

$$\mu_0 = \lambda_0/3 \qquad (27)$$

Therefore, Eq. (26) takes the form:

$$v = \lambda_0(-7u + 18u^2 - 15u^3 + 4u^4 + v + 13uv - 23u^2v - 3v^2 - 39uv^2 - \cdots)/3. \qquad (28)$$

The power series solution of Eq. (25) and Eq. (28) in terms of $\lambda_0$ are:

$$u = -\lambda_0 - 5\lambda_0^2 + (-26/3 + 32/a_0^2)\lambda_0^3 + \cdots, \qquad (29)$$

$$v = 7\lambda_0^2/2 + 166\lambda_0^3/9 + (2194/27 - 448/3a_0^2)\lambda_0^4 + \cdots. \qquad (30)$$

Substituting the values of $u$ and $v$ from Eqs. (29)-(30) into Eq. (24), the third-order approximate angular frequency is:

$$\omega = \left(1 + \frac{a_0^2}{8} + \frac{3a_0^4}{256} + \frac{a_0^6}{6144} - \frac{305a_0^8}{2359296} - \frac{161a_0^{10}}{3538944} + \cdots\right). \qquad (31)$$

Thus, the third-order approximate period of Eq. (8) is:

$$T_3 = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5a_0^6}{6144} + \frac{0.55a_0^8}{262144} + \cdots\right). \qquad (32)$$

The third-order approximation of Eq. (22) measures a more correct result when a suitable truncation principle is used. Using the truncation principle, Eq. (23), takes the following form:

$$1 - \omega^2 + a_0^2\omega^2/4 - u + \omega^2 u + a_0^2\omega^2 u/2 + 11a_0^2\omega^2 u^2/4 - 7a_0^2\omega^2 u^3/4$$
$$- v + \omega^2 v - 3a_0^2\omega^2 v/4 + 15a_0^2\omega^2 uv/4 = 0,$$
$$- a_0^2\omega^2/4 + u - 9\omega^2 u + 5a_0^2\omega^2 u/4 - 7a_0^2\omega^2 u^2/4 + 3a_0^2\omega^2 u^3/2$$
$$+ 3a_0^2\omega^2 v - a_0^2\omega^2 uv/4 = 0, \qquad (33)$$
$$- 7a_0^2\omega^2 u/4 + 11a_0^2\omega^2 u^2/4 - a_0^2\omega^2 u^3/2 + v - 25\omega^2 v + a_0^2\omega^2 v/2$$
$$+ 5a_0^2\omega^2 uv/4 = 0 .$$

The first equation of Eq. (33), can be written as:

$$\omega^2 = (1 - u - v)/(1 - a_0^2/4 - u - a_0^2 u/2 - 11a_0^2 u^2/4 - v + 3a_0^2 v/4 + \cdots). \qquad (34)$$

By using the same mathematical manipulation discussed above, the nonlinear algebraic equations of $u$ and $v$ are:

$$u = \lambda_0(-1 + 5u - 14u^2 + 32u^2/a_0^2 + 2u^3 + u^4 + 13v - 15uv - 7u^2v + \cdots), \tag{35}$$

$$v = \mu_0(-7u + 18u^2 - 13u^3 + 2u^4 + v + 8uv - 27u^2v + v^2 - 20uv^2 + \cdots), \tag{36}$$

where $\lambda_0$ and $\mu_0$ are given in Eqs. (25)-(26).

The power series solution of Eqs. (35)-(36) in terms of $\lambda_0$ are:

$$u = -\lambda_0 - 5\lambda_0^2 + (-26/3 + 32/a_0^2)\lambda_0^3 + (804/9 + 1216/3a_0^2)\lambda_0^4 + \cdots, \tag{37}$$

$$v = 7\lambda_0^2/2 + 166\lambda_0^3/9 + (2281/27 - 448/3a_0^2)\lambda_0^4 + \cdots. \tag{38}$$

Substituting the values of $u$ and $v$ from Eqs. (37)-(38) into the Eq. (34), the third-order approximate angular frequency by using truncation principle is:

$$\omega = \left(1 + \frac{a_0^2}{8} + \frac{3a_0^4}{256} + \frac{a_0^6}{6144} - \frac{53a_0^8}{2359296} - \frac{263a_0^{10}}{56623104} + \cdots\right), \tag{39}$$

$$T_3^{truc} = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5a_0^6}{6144} - \frac{27.4a_0^8}{262144} + \cdots\right). \tag{40}$$

## 4.  RESULTS AND DISCUSSION

The accuracy of the approximate periods has been illustrated by comparing with the exact period $T_{ex}$ that is stated in [6]. For this nonlinear problem, the exact period is

$$T_{ex} = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5a_0^6}{6144} - \frac{7a_0^8}{262144} - \frac{133a_0^{10}}{10485760} + \cdots\right). \tag{41}$$

The second- and third-order approximate periods obtained by applying truncation and without truncation principle to the nonlinear oscillator are defined in Eq. (8).

$$T_2 = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{a_0^6}{4096} - \frac{7a_0^8}{65536} + \cdots\right), \tag{42}$$

$$T_3 = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5a_0^6}{6144} + \frac{0.55a_0^8}{262144} + \cdots\right). \tag{43}$$

In the case of truncation principle:

$$T_2^{truc} = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5.6a_0^6}{6144} + \frac{15a_0^8}{131072} + \cdots\right), \tag{44}$$

$$T_3^{truc} = 2\pi\left(1 - \frac{a_0^2}{8} + \frac{a_0^4}{256} + \frac{5a_0^6}{6144} - \frac{27.4a_0^8}{262144} + \cdots\right). \tag{45}$$

Comparing all the approximate periods, the accuracy of the proposed analytical technique using the truncation principle is better than without the truncation principle. It is highly remarkable that, using the truncation principle, the third-order approximate period gives almost the same result as the exact period. High accuracy period and very simple solution procedure reveal the novelty and reliability of the proposed harmonic balance method. The advantages of the proposed technique include its simplicity and computational efficiency, and the ability to objectively find better agreement in third-order approximate period by applying the truncation principle.

## 5. CONCLUSION

In this paper, an analytical technique has been proposed based on the harmonic balance method to determine approximate periods to the nonlinear oscillator with the square of the angular frequency depending quadratically on the velocity. The solution procedure of the proposed technique is very simple and straightforward. In the presented problem, the approximate periods obtained using the proposed technique shows much better agreement with the corresponding exact period. It is noted that, the third-order approximate period obtained using the truncation principle is almost identical compared with the numerically obtained exact period. High accuracy of the approximate periods obtained from the problem reveals the versatility of the proposed technique in solving strongly nonlinear classes of problems. It can be concluded that the proposed technique is a better and efficient alternative to the existing Lindstedt–Poinaré′ perturbation-based method and classical harmonic balance method for approximating solutions for strongly nonlinear classes of problems.

## ACKNOWLEDGEMENT

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