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C-CLASS FUNCTIONS ON SHORTER PROOFS OF SOME EVEN-TUPLED COINCIDENCE THEOREMS IN ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to prove some even tupled coincidence theorems for mappings with one variable in ordered complete metric spaces by using the concept of C-class functions. Our results generalize and improve several results in the literature.

1. Introduction

Ran and Reurings [30] extended the Banach contraction principle on ordered metric spaces for continuous monotone mappings with some applications to matrix equations. Thereafter Nieto and López [25] modified Ran and Reurings' fixed point theorem for an increasing mapping not necessarily continuous by assuming an another hypothesis on the ordered metric space and proved some fixed point theorems besides giving some applications to ordinary differential equations. In the same development, Nieto and López [26] analogously proved a fixed point theorem for a decreasing mapping on ordered metric space. In recent years, Nieto and López's [25] fixed point theorems were extended and refined by many authors ([1, 2, 7], [11]-[13], [18, 19, 24, 27]).

The idea of a coupled fixed point was introduced by Guo and Lakshmikantham [10] which was well followed by Bhaskar and Lakshmikantham [5] where the authors introduced the notion of mixed monotone property and proved some coupled fixed point theorems for weakly linear contractions enjoying mixed monotone property in ordered complete metric spaces. In [23], Lakshmikantham and Ćirić generalized these results for nonlinear contraction mappings by introducing the notion of coupled coincidence point and mixed g-monotone property.

Recently, Berzig and Samet [6] extended and generalized some fixed point results to higher dimensions. However, they used permutations of variables and distinguished between the first and last variables. Further, Roldán *et al.* [31] proved some existence and uniqueness theorems for nonlinear mappings of any number of arguments, not necessarily permuted or ordered. For more details see ([20, 31, 32, 33]).

Recently, Imdad *et al.* [16] extended the idea of mixed g-monotone property to the mapping $F : X^n \to X$ (where n is even natural number) and proved an even-tupled coincidence point theorem for nonlinear contraction mappings satisfying mixed g-monotone property. Basically their results are true for only even n but not for odd ones (for details see [14]-[17]). Very recently, Samet *et al*. [36] have shown that the coupled (analogously n-tupled) fixed results can be more easily obtained by using well known fixed point theorems on ordered metric spaces (see also [9, 28, 29]).

The concept of C-class functions was introduced by Ansari [3] which actually covers a large class of contractive conditions. In this paper, we generalize the results of Sharma *et al.* [37] by using the concept of C-class functions.

2. Preliminaries

With a view to make, our presentation self-contained, we collect some basic definitions and needed results which will be used frequently in the text later.

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DEFINITION 2.1. Let X be a non-empty set. A relation '$\preceq$' on X is said to be a partial order if the following properties are satisfied:
(i) reflexive: $x \preceq x$ for all $x \in X$,
(ii) anti-symmetric: $x \preceq y$ and $y \preceq x$ implies $x = y$,
(iii) transitive: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in X$.

A non-empty set X together with a partial order '$\preceq$' is said to be an ordered set and we denote it by $(X, \preceq)$.

DEFINITION 2.2. Let $(X, \preceq)$ be an ordered set. Any two elements x and y are said to be comparable elements in X if either $x \preceq y$ or $y \preceq x$.

DEFINITION 2.3. ([27]) A triplet $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set. Moreover, if d is a complete metric on X, then we say that $(X, d, \preceq)$ is an ordered complete metric space.

Recently, Kutbi *et al.* [22] introduced the concept of regular map.

DEFINITION 2.4. ([22]) An ordered metric space $(X, d, \preceq)$ is said to be nondecreasing regular (resp. nonincreasing regular) if it satisfies the following property:
if $\{x_m\}$ is a nondecreasing (resp. nonincreasing) sequence and $x_m \to x$, then $x_m \preceq x$ (resp. $x \preceq x_m$) $\forall m \in \mathbb{N} \cup \{0\}$.

DEFINITION 2.5. ([22]) An ordered metric space $(X, d, \preceq)$ is said to be regular if it is both nondecreasing regular and nonincreasing regular.

DEFINITION 2.6. Let $(X, d, \preceq)$ be an ordered metric space and $g : X \to X$ be a mapping. Then X is said to be nondecreasing g-regular (resp. nonincreasing g-regular) if it satisfies the following property:
if $\{x_m\}$ is a nondecreasing (resp. nonincreasing) sequence and $x_m \to x$, then $gx_m \preceq gx$ (resp. $gx \preceq gx_m$) $\forall m \in \mathbb{N} \cup \{0\}$.

DEFINITION 2.7. An ordered metric space $(X, d, \preceq)$ is said to be g-regular if it is both nondecreasing g-regular and nonincreasing g-regular.

Notice that, on setting $g = I$ (identity mapping on X), Definitions 2.6 and 2.7 reduce to Definitions 2.4 and 2.5 respectively.

Throughout the paper, n stands for a general even natural number. Let us denote by X^n the product space $X \times X \times \ldots \times X$ of n identical copies of X.

DEFINITION 2.8. ([16]) Let $(X, \preceq)$ be an ordered set and $F : X^n \to X$ and $g : X \to X$ two mappings. Then F is said to have the mixed g-monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even position arguments, that is, for $x^1, x^2, x^3, ..., x^n \in X$, if
for all $x_1^1, x_2^1 \in X$, $gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n)$
for all $x_1^2, x_2^2 \in X$, $gx_1^2 \preceq gx_2^2 \Rightarrow F(x^1, x_2^2, x^3, ..., x^n) \preceq F(x^1, x_1^2, x^3, ..., x^n)$
for all $x_1^3, x_2^3 \in X$, $gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n)$

$$\vdots$$

for all $x_1^n, x_2^n \in X$, $gx_1^n \preceq gx_2^n \Rightarrow F(x^1, x^2, x^3, ..., x_2^n) \preceq F(x^1, x^2, x^3, ..., x_1^n)$.

For $g = I$ (identity mapping), Definition 2.8 reduces to mixed monotone property (for details see [16]).

DEFINITION 2.9. ([34]) An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = x^1 \\ F(x^2, x^3, ..., x^n, x^1) = x^2 \\ F(x^3, ..., x^n, x^1, x^2) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = x^n. \end{cases}$$

DEFINITION 2.10. ([16]) An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled coincidence point of mappings $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = g(x^1) \\ F(x^2, x^3, ..., x^n, x^1) = g(x^2) \\ F(x^3, ..., x^n, x^1, x^2) = g(x^3) \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = g(x^n). \end{cases}$$

REMARK 2.1. For $n = 2$, Definitions 2.9 and 2.10 yield the definitions of coupled fixed point and coupled coincidence point respectively while on the other hand, for $n = 4$ these definitions yield the definitions of quadrupled fixed point and quadrupled coincidence point respectively.

DEFINITION 2.11. An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled common fixed point of mappings $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = g(x^1) = x^1 \\ F(x^2, x^3, ..., x^n, x^1) = g(x^2) = x^2 \\ F(x^3, ..., x^n, x^1, x^2) = g(x^3) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = g(x^n) = x^n. \end{cases}$$

DEFINITION 2.12. ([14]) Let X be a non-empty set. Then the mappings $F : X^n \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{cases} \lim_{m\to\infty} d(g(F(x_m^1, x_m^2, ..., x_m^n)), F(gx_m^1, gx_m^2, ..., gx_m^n)) = 0 \\ \lim_{m\to\infty} d(g(F(x_m^2, ..., x_m^n, x_m^1)), F(gx_m^2, ..., gx_m^n, gx_m^1)) = 0 \\ \vdots \\ \lim_{m\to\infty} d(g(F(x_m^n, x_m^1, ..., x_m^{n-1})), F(gx_m^n, gx_m^1, ..., gx_m^{n-1})) = 0, \end{cases}$$

where $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ are sequences in X such that

$$\begin{cases} \lim_{m\to\infty} F(x_m^1, x_m^2, ..., x_m^n) = \lim_{m\to\infty} g(x_m^1) = x^1 \\ \lim_{m\to\infty} F(x_m^2, ..., x_m^n, x_m^1) = \lim_{m\to\infty} g(x_m^2) = x^2 \\ \vdots \\ \lim_{m\to\infty} F(x_m^n, x_m^1, ..., x_m^{n-1}) = \lim_{m\to\infty} g(x_m^n) = x^n, \end{cases}$$

for some $x^1, x^2, ..., x^n \in X$ are satisfied.

The following families of control functions are indicated in Choudhury *et al.* [8].

(1) $\Im := \{\zeta : [0, \infty) \to [0, \infty) : \zeta$ is continuous and $\zeta(t) = 0$ if and only if $t = 0\}$
(2) $\Omega := \{\varphi : [0, \infty) \to [0, \infty) : \varphi$ is continuous and monotone nondecreasing and $\varphi(t) = 0$ if and only if $t = 0\}$
(3) $\Im_u := \{\zeta : [0, \infty) \to [0, \infty) : \zeta$ is continuous and $\zeta(t) > 0$, t > 0 and $\zeta(0) \geq 0\}$.

Notice that members of Ω are called altering distance functions (cf. [21]).

Ansari [3] introduced the concept of C-class functions which covers a large class of contractive conditions (see Example 2.1 (1),(2),(9),(15)).

DEFINITION 2.13. ([3]) A continuous function $\mathcal{F} : [0,\infty)^2 \to \mathbb{R}$ is called a C-function if $\mathcal{F}$ is continuous and satisfies the following:
(1) $\mathcal{F}(s,t) \leq s$;
(2) $\mathcal{F}(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.

An extra condition on $\mathcal{F}$ is that $\mathcal{F}(0,0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ denotes the class of all C-functions.

EXAMPLE 2.1. ([3]) Define $\mathcal{F} : [0,\infty)^2 \to \mathbb{R}$ by
(1) $\mathcal{F}(s,t) = s - t$, $\mathcal{F}(s,t) = s \Rightarrow t = 0$;
(2) $\mathcal{F}(s,t) = ms$, $0<m<1$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$;
(3) $\mathcal{F}(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $\mathcal{F}(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(5) $\mathcal{F}(s,t) = \ln(1 + a^s)/2$, $a > e$, $\mathcal{F}(s,1) = s \Rightarrow s = 0$;
(6) $\mathcal{F}(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $\mathcal{F}(s,t) = s \Rightarrow t = 0$;
(7) $\mathcal{F}(s,t) = s \log_{t+a} a$, $a > 1$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(8) $\mathcal{F}(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $\mathcal{F}(s,t) = s \Rightarrow t = 0$;
(9) $\mathcal{F}(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, and is continuous, $\mathcal{F}(s,t) = s \Rightarrow s = 0$;
(10) $\mathcal{F}(s,t) = s - \frac{t}{k+t}$, $\mathcal{F}(s,t) = s \Rightarrow t = 0$;
(11) $\mathcal{F}(s,t) = s - \varphi(s)$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(12) $\mathcal{F}(s,t) = sh(s,t)$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) < 1$ for all $t,s > 0$;
(13) $\mathcal{F}(s,t) = s - (\frac{2+t}{1+t})t$, $\mathcal{F}(s,t) = s \Rightarrow t = 0$;
(14) $\mathcal{F}(s,t) = \sqrt[n]{\ln(1 + s^n)}$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$;
(15) $\mathcal{F}(s,t) = \phi(s), \mathcal{F}(s,t) = s \Rightarrow s = 0$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
(16) $\mathcal{F}(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $\mathcal{F}(s,t) = s \Rightarrow s = 0$;
(17) $\mathcal{F}(s,t) = \vartheta(s)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function, $\mathcal{F}(s,t) = s \Rightarrow s = 0$;
(18) $\mathcal{F}(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}\, dx$, where Γ is the Euler Gamma function;

for all $s,t \in [0,\infty)$. Then $\mathcal{F}$ is an element of $\mathcal{C}$.

3. MAIN RESULTS

(A) Let $(X, \preceq)$ be an ordered set. Define the following partial order $\sqsubseteq$ on the product space X^n, for $U = (x^1, x^2, \ldots, x^n),\ V = (y^1, y^2, \ldots, y^n) \in X^n$

$$U \sqsubseteq V \Leftrightarrow x^1 \preceq y^1,\ y^2 \preceq x^2,\ x^3 \preceq y^3, \ldots, y^n \preceq x^n.$$

(B) Let (X, d) be a metric space. Define the following metric $\tilde{D}$ on the product space X^n, for $U = (x^1, x^2, \ldots, x^n),\ V = (y^1, y^2, \ldots, y^n) \in X^n$,

$$\tilde{D}(U, V) = \max_{1 \leq i \leq n} d(x^i, y^i).$$

The proof of the following lemmas are immediately. We note the same idea here, but in the case of coupled and tripled fixed point theorems, we have been first used in ([4], [28], [35]).

LEMMA 3.1. *Let $(X, d, \preceq)$ be an ordered complete metric space. Then $(X^n, \tilde{D}, \sqsubseteq)$ is an ordered complete metric space.*

LEMMA 3.2. *Let $(X, d, \preceq)$ be an ordered metric space and $F : X^n \to X$ and $g : X \to X$ be two mappings. Define mappings $T_F : X^n \to X^n$ and $T_g : X^n \to X^n$ by*

$$T_F(x^1, x^2, \ldots, x^n) = (F(x^1, x^2, \ldots, x^n), F(x^2, \ldots, x^n, x^1), \ldots, F(x^n, x^1, \ldots, x^{n-1}))$$

and $T_g(x^1, x^2, \ldots, x^n) = (gx^1, gx^2, \ldots, gx^n)$. Then the following hold:
(1) If F has the mixed g-monotone property, then T_F is monotone T_g-nondecreasing with respect to $\sqsubseteq$.
(2) If F and g are compatible, then T_F and T_g are compatible.
(3) If g is continuous, then T_g is continuous.
(4) If F is continuous, then T_F is continuous.
(5) If $(X, d, \preceq)$ is g-regular, then $(X^n, \tilde{D}, \sqsubseteq)$ is nondecreasing g-regular.
(6) A point $(x^1, x^2, \ldots, x^n) \in X^n$ is an n-tupled coincidence point of F and g iff $(x^1, x^2, \ldots, x^n)$ is a coincidence point of T_F and T_g.

The following lemma is crucial for our main result.

LEMMA 3.3. *Let $(X, d, \preceq)$ be an ordered complete metric space and f and g be two self-mappings on X. Suppose that the following conditions are satisfied:*
(i) $f(X) \subseteq g(X)$,
(ii) f is monotone g-nondecreasing,
(iii) f and g are compatible,
(iv) g is continuous,
(v) either f is continuous or X is nondecreasing g-regular,
(vi) there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$,
(vii) there exist $\varphi \in \Omega$ and $\zeta \in \Im_u$ and $\mathcal{F} \in \mathcal{C}$ such that for all $x, y \in X$,

$$\varphi(d(f(x), f(y))) \leq \mathcal{F}(\varphi(d(g(x), g(y))), \zeta(d(g(x), g(y)))), \text{ with } g(x) \preceq g(y). \tag{3.1}$$

Then f and g have a coincidence point.

PROOF. In view of assumption (vi), if $g(x_0) = f(x_0)$, then x_0 is a coincidence point of f and g and hence proof is finished. On the other hand if $g(x_0) \neq f(x_0)$, then we have $g(x_0) \prec f(x_0)$. So according to assumption (i), that is, $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $g(x_1) = f(x_0)$. Again from $f(X) \subseteq g(X)$, we can choose $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process, we define a sequence $\{x_m\} \subset X$ of joint iterates such that

$$g(x_{m+1}) = f(x_m) \ \ \forall m \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Now, we assert that $\{g(x_m)\}$ is a non-decreasing sequence, that is

$$g(x_m) \preceq g(x_{m+1}) \ \ \forall m \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

We prove this fact by mathematical induction. On using (3.2) for $m = 0$ and assumption (vi), we have

$$g(x_0) \preceq f(x_0) = g(x_1).$$

Thus, (3.3) holds for $m = 0$. Suppose that (3.3) holds for $m = r > 0$, that is,

$$g(x_r) \preceq g(x_{r+1}). \tag{3.4}$$

Then we have to show that (3.3) holds for $m = r + 1$. To accomplish this we use (3.2), (3.4) and assumption (ii) so that

$$g(x_{r+1}) = f(x_r) \preceq f(x_{r+1}) = g(x_{r+2}).$$

Thus, by induction, (3.3) holds for all $m \in \mathbb{N} \cup \{0\}$.
If $g(x_m) = g(x_{m+1})$ for some $m \in \mathbb{N}$, then by using (3.2), we have $g(x_m) = f(x_m)$, that is, x_m is a coincidence point of f and g and hence proof is finished. On the other hand if $g(x_m) \neq g(x_{m+1})$ for each $m \in \mathbb{N} \cup \{0\}$, we can define a sequence

$$\delta_m := d(g(x_m), g(x_{m+1})), \ \ m \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

On using (3.2), (3.3), (3.5) and assumption (vii), we obtain

$$\begin{aligned}\varphi(\delta_{m+1}) = \varphi(d(g(x_{m+1}), g(x_{m+2}))) &= \varphi(d(f(x_m), f(x_{m+1}))) \\ &\leq \mathcal{F}(\varphi(d(g(x_m), g(x_{m+1}))), \zeta(d(g(x_m), g(x_{m+1})))) \\ &= \mathcal{F}(\varphi(\delta_m), \zeta(\delta_m)) \leq \varphi(\delta_m). \end{aligned} \tag{3.6}$$

On using the property of φ, we have $\varphi(\delta_{m+1}) \leq \varphi(\delta_m)$, which implies that $\delta_{m+1} \leq \delta_m$. Therefore $\{\delta_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $\delta \geq 0$ such that $\delta_m \to \delta$ as $m \to \infty$. Taking limit as $m \to \infty$ in (3.6) and using the continuities of φ and ζ, we have $\varphi(\delta) \leq \mathcal{F}(\varphi(\delta), \zeta(\delta))$, so $\varphi(\delta) = 0$, or $\;, \zeta(\delta) = 0$, therefore $\delta = 0$, which is a contradiction . Therefore

$$\lim_{m\to\infty} \delta_m = \lim_{m\to\infty} d(g(x_m), g(x_{m+1})) = 0. \tag{3.7}$$

Now, we show that $\{g(x_m)\}$ is a Cauchy sequence. On contrary suppose that $\{g(x_m)\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{t(k)\}$ such that for all positive integers k, $t(k) > m(k) > k$, such that

$$\eta_k = d(g(x_{m(k)}), g(x_{t(k)})) \geq \epsilon, \text{ and } d(g(x_{m(k)}), g(x_{t(k)-1})) < \epsilon.$$

Now,

$$\begin{aligned}\epsilon \leq \eta_k &= d(g(x_{m(k)}), g(x_{t(k)})) \\ &\leq d(g(x_{m(k)}), g(x_{t(k)-1})) + d(g(x_{t(k)-1}), g(x_{t(k)})) \\ &< \epsilon + \delta_{t(k)-1}\end{aligned}$$

that is,

$$\epsilon \leq \eta_k < \epsilon + \delta_{t(k)-1}.$$

Letting $k \to \infty$ in above inequality and using (3.7), we get

$$\lim_{k\to\infty} \eta_k = \epsilon. \tag{3.8}$$

Again,

$$\begin{aligned}\eta_{k+1} &= d(g(x_{m(k)+1}), g(x_{t(k)+1})) \\ &\leq d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(x_{m(k)}), g(x_{t(k)})) + d(g(x_{t(k)}), g(x_{t(k)+1})) \\ &< \delta_{m(k)+1} + \eta_k + \delta_{t(k)+1}\end{aligned}$$

$$\Rightarrow \eta_{k+1} < \delta_{m(k)+1} + \eta_k + \delta_{t(k)+1}.$$

Letting $k \to \infty$ in above inequality and using (3.7) and (3.8), we get

$$\lim_{k\to\infty} \eta_{k+1} = \epsilon. \tag{3.9}$$

Since $t(k) > m(k)$, hence by (3.3), we get $g(x_{m(k)}) \leq g(x_{t(k)})$. Therefore, owing to (3.1) and assumption (vii), we get

$$\begin{aligned}\varphi(\eta_{k+1}) &= \varphi(d(g(x_{m(k)+1}), g(x_{t(k)+1}))) = \varphi(d(f(x_{m(k)}), f(x_{t(k)}))) \\ &\leq \mathcal{F}(\varphi(d(g(x_{m(k)}), g(x_{t(k)}))), \zeta(d(g(x_{m(k)}), g(x_{t(k)})))) \\ &= \mathcal{F}(\varphi(\eta_k), \zeta(\eta_k))\end{aligned}$$

that is,

$$\varphi(\eta_{k+1}) \leq \mathcal{F}(\varphi(\eta_k), \zeta(\eta_k)).$$

Letting $k \to \infty$ in above inequality and using (3.8), (3.9) and continuities of φ and ζ, we get

$$\varphi(\epsilon) \leq \mathcal{F}(\varphi(\epsilon), \zeta(\epsilon))$$

so $\varphi(\epsilon) = 0$, or $\;, \zeta(\epsilon) = 0$ thus $\epsilon = 0$ which is a contradiction . Therefore the sequence $\{g(x_m)\}$ is Cauchy. From the completeness of X, there exists $x \in X$ such that

$$\lim_{m\to\infty} f(x_m) = \lim_{m\to\infty} g(x_m) = x. \tag{3.10}$$

Since F and g are compatible, we have from (3.10),

$$\lim_{m\to\infty} d(f(gx_m), g(fx_m)) = 0. \tag{3.11}$$

Now, we use assumption (v). Firstly, we assume that f is continuous. Then for all $m \in \mathbb{N} \cup \{0\}$, we have

$$d(g(x), f(gx_m)) \leq d(g(x), g(fx_m)) + d(g(fx_m), f(gx_m)).$$

Taking $k \to \infty$ in above inequality and using (3.10), (3.11) and continuities of f and g, we get $d(g(x), f(x)) = 0$, that is, $g(x) = f(x)$. Hence the element $x \in X$ is a coincidence point of f and g. Next, we suppose that X is nondecreasing g-regular. From (3.3) and (3.10), we get

$$g(gx_m) \preceq g(x). \tag{3.12}$$

Since f and g are compatible and g is continuous by (3.10) and (3.11), we have

$$\lim_{m\to\infty} g(gx_m) = g(x) = \lim_{m\to\infty} g(fx_m) = \lim_{m\to\infty} f(gx_m). \tag{3.13}$$

Now, using triangle inequality, we have

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), g(gx_{m+1})) + d(g(gx_{m+1}), g(x)) \\ &= d(f(x), g(fx_m)) + d(g(gx_{m+1}), g(x)). \end{aligned}$$

Taking $k \to \infty$ in above inequality and using (3.13), we have

$$\begin{aligned} d(f(x), g(x)) &\leq \lim_{m\to\infty} d(f(x), g(fx_m)) + \lim_{m\to\infty} d(g(gx_{m+1}), g(x)) \\ &= \lim_{m\to\infty} d(f(x), f(gx_m)). \end{aligned}$$

Since φ is continuous and monotone nondecreasing, from the above inequality we have

$$\begin{aligned} \varphi(d(f(x), g(x))) &\leq \varphi(\lim_{m\to\infty} d(f(x), f(gx_m))) \\ &= \lim_{m\to\infty} \varphi(d(f(x), f(gx_m))). \end{aligned}$$

By (3.12) and assumption (vii), we get

$$\begin{aligned} \varphi(d(f(x), g(x))) &\leq \lim_{m\to\infty} \varphi(d(f(x), f(gx_m))) \\ &\leq \lim_{m\to\infty} \mathcal{F}(\varphi d(g(x), g(gx_m))), \zeta(d(g(x), g(gx_m)))) \\ &= \mathcal{F}(\lim_{m\to\infty} \varphi(d(g(x), g(gx_m))), \lim_{m\to\infty} \zeta(d(g(x), g(gx_m)))) \\ &= \mathcal{F}(\varphi(d(f(x), g(x))), \zeta(d(f(x), g(x))). \end{aligned}$$

so $\varphi(d(f(x), g(x))) = 0$, or , $\zeta(d(f(x), g(x)) = 0$,which implies that $d(f(x), g(x)) = 0$, that is, $g(x) = f(x)$. Hence $x \in X$ is a coincidence point of f and g.

Lemma 3.4. *In addition to the hypotheses of Lemma 3.3, suppose that for real $x, y \in X$ there exists, $z \in X$ such that $f(z)$ is comparable to $f(x)$ and $f(y)$. Then f and g have a unique common fixed point.*

Proof. The set of coincidence points of f and g is non-empty due to Lemma 3.3. Assume now, x and y are two coincidence points of f and g, that is,

$$f(x) = g(x) \text{ and } f(y) = g(y).$$

Now we will show that $g(x) = g(y)$. By assumption, there exists $z \in X$ such that $f(z)$ is comparable to $f(x)$ and $f(y)$. Put $z_0^1 = z$ and choose $z_1 \in X$ such that $g(z_1) = f(z_0)$. Further define sequence $\{g(z_m)\}$ such that $g(z_{m+1}) = f(z_m)$. Further set $x_0 = x$ and $y_0 = y$. In the same way, define the sequences $\{g(x_m)\}$ and $\{g(y_m)\}$. Then it is easy to show that

$$g(x_{m+1}) = f(x_m) \text{ and } g(y_{m+1}) = f(y_m).$$

Since $f(x) = g(x_1) = g(x)$ and $f(z) = g(z_1)$ are comparable, we have

$$g(x) \preceq g(z_1).$$

It is easy to show that $g(x)$ and $g(z_m)$ are comparable, that is, for all $m \in \mathbb{N}$,

$$g(x) \preceq g(z_m).$$

Thus from (3.1) we have

$$\begin{aligned}\varphi(d(g(x),g(z_{m+1}))) &= \varphi(d(f(x),f(z_m)))\\ &\le \mathcal{F}(\varphi(d(g(x),g(z_m))),\zeta(d(g(x),g(z_m)))).\end{aligned}$$

Let $R_m = d(g(x), g(z_{m+1}))$. Then

$$\varphi(R_m) \le \mathcal{F}(\varphi(R_{m-1}), \zeta(R_{m-1})). \tag{3.14}$$

Using the property of φ, we have $\varphi(R_m) \le \varphi(R_{m-1})$, which implies that $R_m \le R_{m-1}$ (by the property of φ). Therefore $\{R_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $r \ge 0$ such that $R_m \to r$ as $m \to \infty$. Taking the limit as $m \to \infty$ in (3.14) and using the continuities of φ and ζ, we have $\varphi(r) \le \mathcal{F}(\varphi(r), \zeta(r))$,so $\varphi(r) = 0, , \zeta(r) = 0$ thus $r = 0$ which is a contradiction . Therefore $R_m \to 0$ as $m \to \infty$, that is,

$$\lim_{m\to\infty} d(g(x), g(z_{m+1})) = 0.$$

Similarly we can prove that

$$\lim_{m\to\infty} d(g(y), g(z_{m+1})) = 0.$$

Therefore by triangle inequality

$$d(g(x), g(y)) \le d(g(x), g(z_{m+1})) + d(g(z_{m+1}), g(y)) \to 0 \text{ as } m \to \infty.$$

Hence

$$g(x) = g(y). \tag{3.15}$$

Since $g(x) = f(x)$ and f and g are compatible, we have $gg(x) = f(gx)$. Write $g(x) = a$, then we have

$$g(a) = f(a). \tag{3.16}$$

Thus a is the coincidence point of f and g. Then owing to (3.15) with $y = a$, it follows that $g(x) = g(a)$, that is,

$$g(a) = a. \tag{3.17}$$

Using (3.16) and (3.17), we have $a = g(a) = f(a)$. Thus a is the common fixed point of f and g. To prove the uniqueness, assume that b is another common fixed point of f and g. Then by (3.15), we have

$$b = g(b) = g(a) = a.$$

This completes the proof of Lemma.

THEOREM 3.1. *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$ and $g : X \to X$ be two mappings. Suppose that the following conditions are satisfied:*
(i) $F(X^n) \subseteq g(X)$,
(ii) F and g are compatible,
(iii) F has the mixed g-monotone property,
(iv) g is continuous,
(v) either F is continuous or X is g-regular,
(vi) there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq gx_0^2 \\ gx_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq gx_0^n, \end{cases} \tag{3.18}$$

(vii) there exist $\varphi \in \Omega$ and $\zeta \in \Im_u$ and $\mathcal{F}$ a C-function such that

$$\varphi(d(FU, FV)) \le \mathcal{F}(\varphi(\max d(gx^i, gy^i)), \zeta(\max d(gx^i, gy^i)))$$

for all $U = (x^1, x^2, ..., x^n)$, $V = (y^1, y^2, ..., y^n) \in X^n$ with $gy^1 \preceq gx^1, gx^2 \preceq gy^2, gy^3 \preceq gx^3, \ldots, gx^n \preceq gy^n$.

Then F and g have an n-tupled coincidence point.

PROOF. Consider the product space $Y = X^n$ equipped with the metric $\tilde{D}$ (given by (B)) and the partial order $\sqsubseteq$ (given by (A)). Then by Lemma 3.1, $(Y, \tilde{D}, \sqsubseteq)$ is an ordered complete metric space. Also F and g induce mappings $T_F : Y \to Y$ and $T_g : Y \to Y$ (defined in Lemma 3.2). Clearly,

- (i) implies that $T_F(Y) \subseteq T_g(Y)$,
- (ii) implies that T_F is monotone T_g-nondecreasing (by item (1) of Lemma 3.2),
- (iii) implies that T_F and T_g are compatible (by item (2) of Lemma 3.2),
- (iv) implies that T_g is continuous (by item (3) of Lemma 3.2),
- (v) implies that either T_F is continuous (by item (4) of Lemma 3.2) or $(Y, \tilde{D}, \sqsubseteq)$ is nondecreasing g-regular (by item (5) of Lemma 3.2),
- (vi) is equivalent to the condition: there exists $U_0 = (x_0^1, x_0^2, \ldots, x_0^n) \in Y$ such that $T_g(U_0) \sqsubseteq T_F(U_0)$.

Now, in view of (vii), for given $U, V \in Y$ such that $T_g(U) \sqsubseteq T_g(V)$ implies that

$$(gx^1, gx^2, \ldots, gx^n) \sqsubseteq (gy^1, gy^2, \ldots, gy^n).$$

It follows that for odd i,

$$(gx^i, gx^{i+1}, \ldots, gx^n, gx^1, \ldots, gx^{i-1}) \sqsubseteq (gy^i, gy^{i+1}, \ldots, gy^n, gy^1, \ldots, gy^{i-1}), \tag{3.19}$$

and for even i,

$$(gy^i, gy^{i+1}, \ldots, gy^n, gy^1, \ldots, gy^{i-1}) \sqsubseteq (gx^i, gx^{i+1}, \ldots, gx^n, gx^1, \ldots, gx^{i-1}). \tag{3.20}$$

If i is odd, then by using (3.19) and (vii), we get

$$\begin{aligned}
&d(F(x^i, x^{i+1}, \ldots, x^n, x^1, x^2, \ldots, x^{i-1}), F(y^i, y^{i+1}, \ldots, y^n, y^1, y^2, \ldots, y^{i-1}))\\
&\leq \varphi(\max\{d(gx^i, gy^i), d(gx^{i+1}, gy^{i+1}), \ldots, d(gx^n, gy^n), d(gx^1, gy^1),\\
&\mathcal{F}(d(gx^2, gy^2), \ldots, d(gx^{i-1}, gy^{i-1})\}), \zeta(\max\{d(gx^i, gy^i), d(gx^{i+1}, gy^{i+1}), \ldots,\\
&d(gx^n, gy^n), d(gx^1, gy^1), d(gx^2, gy^2), \ldots, d(gx^{i-1}, gy^{i-1})\}))\\
&= \mathcal{F}(\varphi(\max_{1\leq i\leq n} d(gx^i, gy^i)), \zeta(\max_{1\leq i\leq n} d(gx^i, gy^i))).
\end{aligned}$$

If i is even, then by using (3.20) and (vii), we get

$$\begin{aligned}
&d(F(x^i, x^{i+1}, \ldots, x^n, x^1, x^2, \ldots, x^{i-1}), F(y^i, y^{i+1}, \ldots, y^n, y^1, y^2, \ldots, y^{i-1}))\\
&= d(F(y^i, y^{i+1}, \ldots, y^n, y^1, y^2, \ldots, y^{i-1}), F(x^i, x^{i+1}, \ldots, x^n, x^1, x^2, \ldots, x^{i-1}))\\
&\leq \varphi(\max\{d(gy^i, gx^i), d(gy^{i+1}, gx^{i+1}), \ldots, d(gy^n, gx^n), d(gy^1, gx^1),\\
&d(gy^2, gx^2), \ldots, d(gy^{i-1}, gx^{i-1})\}) - \zeta(\max\{d(gy^i, gx^i), d(gy^{i+1}, gx^{i+1}), \ldots,\\
&d(gy^n, gx^n), d(gy^1, gx^1), d(gy^2, gx^2), \ldots, d(gy^{i-1}, gx^{i-1})\})\\
&= \varphi(\max_{1\leq i\leq n} d(gx^i, gy^i)) - \zeta(\max_{1\leq i\leq n} d(gx^i, gy^i)).
\end{aligned}$$

Hence, in both the cases, for each i $(1 \leq i \leq n)$, we have

$$\begin{aligned}
&d(F(x^i, x^{i+1}, \ldots, x^n, x^1, x^2, \ldots, x^{i-1}), F(y^i, y^{i+1}, \ldots, y^n, y^1, y^2, \ldots, y^{i-1}))\\
&\leq \varphi(\max_{1\leq i\leq n} d(gx^i, gy^i)) - \zeta(\max_{1\leq i\leq n} d(gx^i, gy^i)).
\end{aligned} \tag{3.21}$$

Hence by using (3.21), we have

$$
\begin{aligned}
&\tilde{D}(T_F(U), T_F(V)) \\
&= \max_{1\le i\le n} d(F(x^i, x^{i+1}, \dots, x^n, x^1, x^2, \dots, x^{i-1}), F(y^i, y^{i+1}, \dots, y^n, y^1, y^2, \dots, y^{i-1})) \\
&\le \mathcal{F}(\max_{1\le i\le n}[\varphi(\max_{1\le i\le n} d(gx^i, gy^i)), \zeta(\max_{1\le i\le n} d(gx^i, gy^i))]) \\
&= \mathcal{F}(\varphi(\max_{1\le i\le n} d(gx^i, gy^i)), \zeta(\max_{1\le i\le n} d(gx^i, gy^i))) \\
&= \mathcal{F}(\varphi(\tilde{D}(T_g(U), T_g(V)), \zeta(\tilde{D}(T_g(U), T_g(V))).
\end{aligned}
$$

Thus all conditions of Lemma 3.3 are satisfied for ordered complete metric space $(Y, \tilde{D}, \sqsubseteq)$ and mappings $T_F : Y \to Y$ and $T_g : Y \to Y$. Therefore T_F and T_g have a coincidence point in $Y = X^n$. According to item (6) of Lemma 3.2, the mappings F and g have an n-tupled coincidence point.

Corollary 3.1. ([37]) *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$ and $g : X \to X$ be two mappings. Suppose that the following conditions are satisfied:*
(i) $F(X^n) \subseteq g(X)$,
(ii) F and g are compatible,
(iii) F has the mixed g-monotone property,
(iv) g is continuous,
(v) either F is continuous or X is g-regular,
(vi) there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that (3.18) holds,
(vii) there exist $\varphi \in \Omega$ and $\zeta \in \Im$ such that

$$\varphi(d(FU, FV)) \le \varphi(\max_{1\le i\le n} d(gx^i, gy^i)) - \zeta(\max_{1\le i\le n} d(gx^i, gy^i)),$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $gy^1 \preceq gx^1, gx^2 \preceq gy^2, gy^3 \preceq gx^3, \dots, gx^n \preceq gy^n$.

Then F and g have an n-tupled coincidence point.

Proof. It is sufficient to take $\mathcal{F}(s,t) = s - t$ in Theorem 3.1.

Corollary 3.2. *Corollary 3.1 remains true if condition (vii) is replaced by the following:*
(vii)' there exist $\varphi \in \Omega$ such that

$$\varphi(d(FU, FV)) \le k\varphi(\max d(gx^i, gy^i)),\ 0 < k < 1,$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $gy^1 \preceq gx^1, gx^2 \preceq gy^2, gy^3 \preceq gx^3, \dots, gx^n \preceq gy^n$.

Proof. It is sufficient to take $\mathcal{F}(s,t) = ks,\ 0 < k < 1$ in Theorem 3.1.

Corollary 3.3. *Corollary 3.1 remains true if condition (vii) is replaced by the following:*
(vii)" there exist $\varphi \in \Omega$ and $\beta : [0, \infty) \to [0, 1)$ which is semi-continuous such that

$$\varphi(d(FU, FV)) \le \varphi(\max d(gx^i, gy^i))\beta(\varphi(\max d(gx^i, gy^i)))$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $gy^1 \preceq gx^1, gx^2 \preceq gy^2, gy^3 \preceq gx^3, \dots, gx^n \preceq gy^n$.

Proof. It is sufficient to take $\mathcal{F}(s,t) = s\beta(s)$ (where $\beta : [0, \infty) \to [0, 1)$ and semi-continuous) in Theorem 3.1.

Corollary 3.4. *Corollary 3.1 remains true if condition (vii) is replaced by the following:*
(vii)"' there exist $\varphi \in \Omega$ and $\phi : [0, \infty) \to [0, \infty)$ which is an upper semi-continuous function such that

$\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$ such that

$$\varphi(d(FU, FV)) \leq \phi(\varphi(\max d(gx^i, gy^i)))$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $gy^1 \preceq gx^1, gx^2 \preceq gy^2, gy^3 \preceq gx^3, \ldots, gx^n \preceq gy^n$.

Proof. It is sufficient to take $\mathcal{F}(s, t) = \phi(s)$ (where $\phi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$) in Theorem 3.1.

Corollary 3.5. *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$ be a mapping. Suppose that the following conditions are satisfied:*
(i) F has the mixed monotone property,
(ii) either F is continuous or X is regular,
(iii) there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq x_0^2 \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq x_0^n, \end{cases}$$

(iv) there exist $\varphi \in \Omega$ and $\zeta \in \Im_u$ and $\mathcal{F}$ a C-function such that

$$\varphi(d(FU, FV)) \leq \mathcal{F}(\varphi(\max d(x^i, y^i)), \zeta(\max d(x^i, y^i)))$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \ldots, y^n \preceq x^n$.

Then F has an n-tupled fixed point.

Proof. It is sufficient to take $g = I$ (identity mapping) in Theorem 3.1.

Corollary 3.6. *Corollary 3.5 remains true if condition (iv) is replaced by the following:*
(iv)' there exists $\zeta \in \Im_u$ such that

$$d(FU, FV) \leq \mathcal{F}(\max_{1 \leq i \leq n} d(x^i, y^i), \zeta(\max_{1 \leq i \leq n} d(x^i, y^i))),$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \ldots, y^n \preceq x^n$.

Proof. It is sufficient to take φ and g to be identity mappings in Theorem 3.1.

Corollary 3.7. *Corollary 3.1 remains true if condition (iv) is replaced by the following:*
(iv)" there exists $k \in (0, 1)$ such that

$$d(FU, FV) \leq k \max_{1 \leq i \leq n} d(x^i, y^i),$$

for all $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \ldots, y^n \preceq x^n$.

Proof. It is sufficient to take φ and g to be identity mappings and $\zeta(t) = (1 - k)t,\ k \in (0, 1)$ in Theorem 3.1.

Now we shall prove the uniqueness of n-tupled fixed point.

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that for real $(x^1, x^2, ..., x^n)$ and $(y^1, y^2, ..., y^n) \in X^n$ there exists, $(z^1, z^2, ..., z^n) \in X^n$ such that $(F(z^1, z^2, ..., z^n), F(z^2, ..., z^n, z^1), ..., F(z^n, z^1, ..., z^{n-1}))$ is comparable to $(F(x^1, x^2, ..., x^n), F(x^2, ..., x^n, x^1), ..., F(x^n, x^1, ..., x^{n-1}))$ and $(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1}))$. Then F and g have a unique n-tupled common fixed point.*

PROOF. Set $U = (x^1, x^2, \ldots, x^n)$, $V = (y^1, y^2, \ldots, y^n)$ and $W = (z^1, z^2, \ldots, z^n)$. Then we have

$$T_F(W) \sqsubseteq T_F(U) \text{ or } T_F(U) \sqsubseteq T_F(W)$$

and

$$T_F(W) \sqsubseteq T_F(V) \text{ or } T_F(V) \sqsubseteq T_F(W).$$

Hence by using Lemma 3.4, T_F and T_g have a unique n-tupled common fixed point.

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