THE SUPERMATRIX EIGENPROBLEM: AN INTERPRETATION OF THE PRIORITY VECTORS IN THE ANALYTIC NETWORK PROCESS

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ABSTRACT

Continuing the previous articles on interpretation of the solutions obtained in the Analytic Hierarchy Process (AHP) and in the Analytic Network Process (ANP) with outer dependence, the current work considers a general ANP problem and shows that the solution for its supermatrix, obtained by raising the supermatrix to powers is equivalent to solving the eigenproblem for this matrix. Thus, the global priority solution for an ANP model is an eigenproblem of its supermatrix, and the principal eigenvector of the supermatrix itself gives the mean priorities for the whole network of the compared items. This approach provides an easy way to describe complex ANP interconnections, and gives an explicit interpretation of the priority results convenient for practical managerial decisions.

Keywords: Analytic Network Process, Supermatrix Eigenvectors.

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Do not become the slave of your model – Vincent van Gogh

1. Introduction

This paper continues the consideration of the relationship between the Analytic Hierarchy Process (AHP) and the Analytic Network Process (ANP) started in three previous papers in this journal (Lipovetsky, 2010, 2011a, b). In an ANP problem there are two kinds of matrices to consider: the pairwise comparison matrices and the supermatrix created by combining the eigenvectors from all the pairwise comparison matrices. In the current article, we consider a general ANP problem and show that its solution can be obtained by raising its supermatrix to powers or by solving the eigenproblem for its supermatrix. The solution of an ANP problem can be obtained by solving the ejgenproblem for the supermatrix, and its principal eigenvector presents the mean priorities for the elements in the whole network. The eigenproblem approach offers a way to easily describe the complex ANP interconnections and explicitly interpret the priority results in a way that is convenient for practical managerial decisions.

2. ANP eigenproblem solution and interpretation

For purposes of this exposition, let us consider an example of a general supermatrix for a complicated network with a holarchy structure for estimating the turnaround of the US economy in 1992, described by Saaty (1996, pp.171-184). This holarchy has links from the primary factors to the subcriteria to the alternatives, and the alternatives are linked back to the primary factors. In this example, the alternatives represent time periods during which the turnaround will have occurred as measured from the time the prediction was made, and their priorities represent the likelihood that the turnaround happened during that time period. There are 15 nodes in the network structure, grouped into several blocks as listed below. A visual representation of the network containing three clusters of nodes is shown in Figure 1 and the derived priorities resulting from the pairwise comparison matrices are shown in the columns in Table 1.

Primary Factors

1) Conventional Adjustment,
2) Economic Restructuring,

Subfactors

Conventional Adjustment

3) Consumption
4) Export
5) Investment
6) Confidence
7) Fiscal Policy
8) Monetary Policy

Economic Restructuring

9) Financial Sector
10) Defense Posture
11) Global Competition,

Alternatives

12) Three Months of Adjustment
13) Six Months of Adjustment
14) One Year of Adjustment
15) Two Years and More of Adjustment

The colored windows in Figure 1 represent the clusters that contain the nodes that are the factors in the problem. Clusters may be thought of as logical collections of issues being considered in the decision problem. The arrow from one cluster to another merely indicates that some node(s) in the beginning cluster are linked to some node(s) in the terminal cluster. There are no actual links connecting clusters; links go from node to node(s). The actual details of the connections can be seen in Table 1.

The blocks of entries in Table 1 are color-coded to match the clusters in Figure 1. In Table 1 the parent node of a comparison set is listed at the top of the column and its children nodes are listed in the rows. Children nodes for a given comparison set must all be in the same cluster. For example, the Conventional Adjustment node number 1, at the top of the table, is the parent of a comparison set and its children are the nodes in rows 3 to 8: Consumption, Exports, Investment, Confidence, Fiscal Policy and Monetary Policy. The numbers in column1 are the local priority vector obtained from the pairwise comparison matrix. The question is posed like this: Which factor (child node) of Conventional Adjustment is more important in causing an economic turnaround?

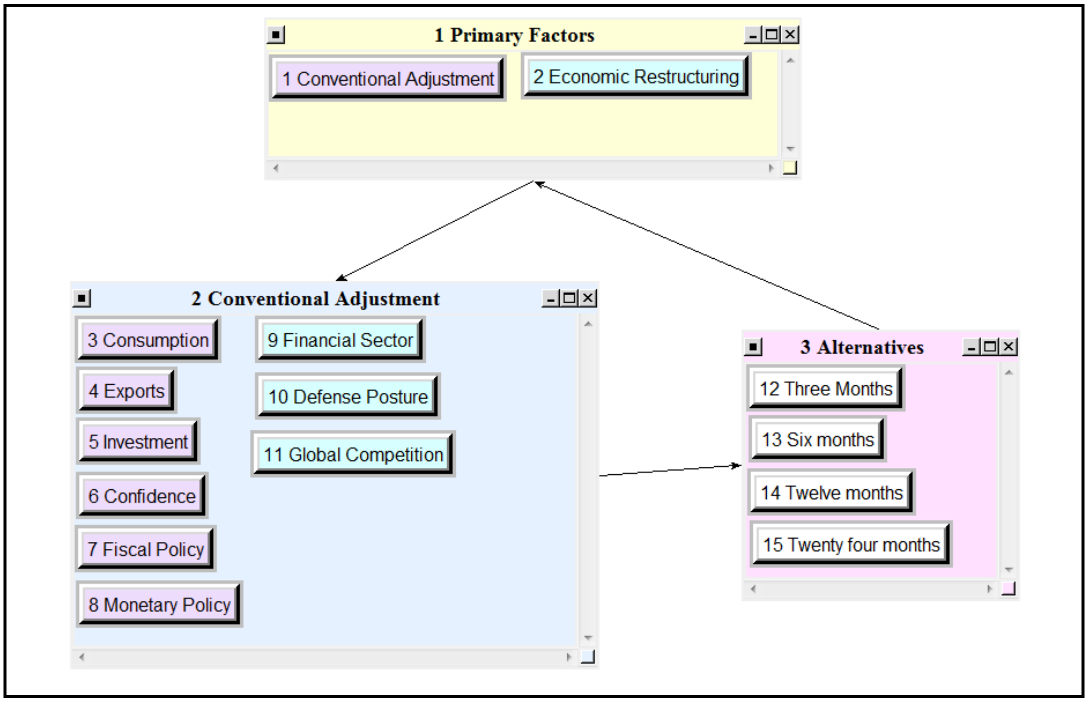

Figure 1 US Economy Holarchy of Factors (Saaty, 1996)*

The column-stochastic supermatrix containing all the local priority vectors obtained from pairwise comparison matrices in the network are in the columns in Table 1. The background color of the clusters in Figure 1 is the same color as the associated cells in Table 1. The supermatrix in Table 1 is a square matrix with a row and column for each of the nodes in Figure 1.

Raising a supermatrix *S* to powers is the simplest way to find its principal eigenvector and eigenvalue. Usually it converges after a few iterations. Sometimes, in cases of circular priorities insolvency occurs (Bar Niv and Lipovetsky, 1995) as the matrix cycles among several solutions; however, in this case it does not. The maximum eigenvalue of a

* The screenshot is of the network if from the SuperDecisions software, available free to educators and researchers from www.superdecisions.com.

column-stochastic matrix is $\lambda = 1$; its left eigenvector is the identity vector, and raising it to powers until it reaches a stable state leads to the limiting matrix shown in Table 2. Each column in the limiting matrix is the same, the normalized limiting vector *w,* obtained by raising the supermatrix to powers.

Table 1
Supermatrix of local priority vectors for "Turnaround in the US economy" (Saaty, 1996)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	0	0	0	0	.833	.833	.500	.167
2	0	0	0	0	0	0	0	0	0	0	0	.167	.167	.500	.833
3	.118	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	.029	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	.058	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	.334	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	.118	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	.343	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	.584	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	.281	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	.135	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	.043	.083	.078	.517	.099	.605	.049	.049	.089	0	0	0	0
13	0	0	.113	.083	.078	.305	.088	.262	.085	.085	.089	0	0	0	0
14	0	0	.310	.417	.305	.124	.383	.042	.236	.236	.209	0	0	0	0
15	0	0	.534	.417	.539	.054	.432	.091	.630	.630	.613	0	0	0	0

The matrix in Table 2 can be represented by the following outer product:

$$\begin{pmatrix} w_1 & w_1 & \cdots & w_1 \\ w_2 & w_2 & \cdots & w_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ w_n & w_n & \cdots & w_n \end{pmatrix} = 1 \begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{pmatrix} \begin{pmatrix} 1 & 1 \ldots. 1 \end{pmatrix} \tag{1}$$

This is the eigenvector decomposition of the rank one matrix. The vector *w* in Table 1 is constructed of an element from each row of the limiting matrix, shown in Table 2, and is the right eigenvector of the original supermatrix. The vector *w* can be obtained by solving the eigenproblem of the supermatrix:

$$S\,w = \lambda w \tag{2}$$

Table 2
Limiting supermatrix for the 15 elements in "Turnaround in the US economy" example

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
1	.161	.161	.161	.161	.161	.161	.161	.161	.161	.161	.161	.161	.161	.161	.161
2	.172	.172	.172	.172	.172	.172	.172	.172	.172	.172	.172	.172	.172	.172	.172
3	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019
4	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005	.005
5	.009	.009	.009	.009	.009	.009	.009	.009	.009	.009	.009	.009	.009	.009	.009
6	.054	.054	.054	.054	.054	.054	.054	.054	.054	.054	.054	.054	.054	.054	.054
7	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019	.019
8	.055	.055	.055	.055	.055	.055	.055	.055	.055	.055	.055	.055	.055	.055	.055
9	.101	.101	.101	.101	.101	.101	.101	.101	.101	.101	.101	.101	.101	.101	.101
10	.048	.048	.048	.048	.048	.048	.048	.048	.048	.048	.048	.048	.048	.048	.048
11	.023	.023	.023	.023	.023	.023	.023	.023	.023	.023	.023	.023	.023	.023	.023
12	.075	.075	.075	.075	.075	.075	.075	.075	.075	.075	.075	.075	.075	.075	.075
13	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051	.051
14	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067
15	.141	.141	.141	.141	.141	.141	.141	.141	.141	.141	.141	.141	.141	.141	.141

The results of the eigenproblem solution for the 15^{th} order supermatrix v in Table 1 are given in Table 3.

The principal eigenvector corresponding to the maximum real eigenvalue $\lambda = 1$ is given in column 2 of Table 3 and shown normalized to 1 in column 3. The normalized principal eigenvector in column 3 is the same as the vector obtained by raising the supermatrix to powers (as shown by any one of the columns of Table 2).

But the eigenproblem solution for the supermatrix reveals more fascinating features. The next two eigenvalues, $\lambda = -0.500 \pm 0.866i$, shown in the bottom row of Table 3, are complex conjugates, but the modulo (absolute value) of each equals one, that is $|\lambda| = \sqrt{0.5^2 + 0.866^2} = 1$. The modulus values obtained from the pairs in column 4 are given in column 5 and normalized to 1 in column 6. The values in columns 6 from the complex conjugate eigenvectors are the same as the normalized eigenvector in column 3. So we have shown that all three of these eigenvectors can be reduced to the same unique solution coinciding with the priorities in Table 2. In column 7 the values in each cluster are normalized to 1.

Table 3
Eigenvectors of the supermatrix for the "Turnaround in the US economy" example

	Principal Eigenvector of Supermatrix in Table 1		Second and Third Complex Conjugated Eigenvectors of Supermatrix			Priorities (from Table 2)
Elements	Eigen-vector	Normalized to Total =1 (Priorities)	Original Vectors	Modulo	Modulo Normalized to Total =1 (Priorities)	Priorities are normalized to 1within each Block
1	.565	.161	.594 ± .544i	.806	.161	.484
2	.603	.172	.634 ± .581i	.860	.172	.516
3	.067	.019	-.091 ± .029i	.095	.019	.057
4	.016	.005	-.022 ± .007i	.023	.005	.014
5	.033	.009	-.045 ± .014i	.047	.009	.028
6	.189	.054	-.257 ± .081i	.269	.054	.162
7	.067	.019	-.091 ± .029i	.095	.019	.057
8	.194	.055	-.264 ± .083i	.276	.055	.166
9	.352	.100	-.479 ± .151i	.502	.100	.301
10	.169	.048	-.230 ± .073i	.242	.048	.145
11	.081	.023	-.111 ± .035i	.116	.023	.070
12	.261	.074	.081 ∓ .363i	.372	.074	.223
13	.177	.051	.055 ∓ .247i	.253	.051	.152
14	.235	.067	.072 ∓ .327i	.335	.067	.201
15	.495	.141	.153 ∓ .689i	.706	.141	.424
Eigen-value	1		-.500 ± .866i			

To summarize, the supermatrix contains three non-zero blocks shown in Table 1. The variables in these blocks are the priorities obtained from pairwise comparison matrices by raising them to powers (Saaty, 1996, p.180-182). Let us take the real or imaginary part of the second or third original eigenvector and normalize them separately for the three blocks, the first having two elements, the next nine elements, and the last four elements. These normalized-by-block values are in the last column in Table 3. For the first block, for instance, the sum 0.484+0.516=1, the sum of the next nine elements in rows 3-11 is one, and the sum of the last four elements in the rows 12-15 is one as well.

But normalizing the entire vector so it sums to one, we obtain the vector presented in the second from the last column of Table 3and this vector coincides with the principal eigenvector from Table 2. Thus, all three eigenvectors (corresponding to modulo one eigenvalues) give the same solution which is the same as that obtained by raising the matrix to powers as shown in Table 2.

By defining the solution of a supermatrix in terms of its principal eigenvector we can easily and explicitly interpret the priority vector elements. The elements of an eigenvector correspond to the mean preferences of each alternative, or criterion, etc., over all the other compared items. Global vectors in ANP are the eigenvectors of the combined local eigenvectors, and they represent the mean priorities of the compared entries in the whole network. ANP can also be described as a stochastic model consisting of the priority vectors for each element of the network. A supermatrix is column-stochastic, so totals in each column equal one, and the elements of supermatrix s_{ij} are already the eigenvector preferences of each item over the others in each block of the whole network structure. It means that the transposed stochastic supermatrix itself can be considered as a matrix of transitional probabilities p_i and described in a system of Chapman-Kolmogorov equations:

$$\begin{cases} \dfrac{dp_1}{dt} = s_{12}p_2 + \ldots + s_{1n}p_n - p_1 \\ ----------- \\ \dfrac{dp_n}{dt} = s_{n1}p_1 + \ldots + s_{n,n-1}p_{n-1} - p_n \end{cases} \tag{3}$$

These equations are written in matrix form as follows:

$$\dot{p} = Sp - p \tag{4}$$

For the steady-state priorities it reduces to the eigenproblem of the supermatrix:

$$Sp = \lambda p\,, \tag{5}$$

in which the maximum eigenvalue of the column-stochastic supermatrix is $\lambda = 1$, which is associated with a unique positive eigenvector (Bar Niv and Lipovetsky, 1995; Lipovetsky and Conklin, 2003; Lipovetsky, 2005). As shown above, the solution by raising the supermatrix to powers coincides with its principal eigenvector obtained in the eigenproblem equation (2). Thus, the ANP priority vectors can be described in terms of the probability of choice among all compared items. It is important to know that dynamic equations for a time dependent supermatrix was introduced originally by Saaty (1994, Ch. 12).

3. Summary

Together with the previous articles (Lipovetsky, 2010, 2011a, b) this work has shown that AHP/ANP solutions for local and global priority vectors can be derived from the standard framework of an $n \times n$ matrix and solved in terms of its principal eigenvector. The elements of an eigenvector correspond to the mean preferences of each alternative, or criterion, over all the other compared items. Global vectors in AHP and ANP are the eigenvectors of the combined local eigenvectors, and they represent the mean priorities of the compared entries in the whole network. This interpretation of AHP and ANP priority vectors as the mean prevalence among the compared elements can be easily accepted by managers and decision makers.

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