REFLECTIONS ON SCALES FROM MEASUREMENTS, NOT MEASUREMENTS FROM SCALES

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ABSTRACT

This essay will discuss the different kinds of scales that are frequently used, and what can and cannot be done with them. We will also explore what absolute numbers are and how to work with them to deal with quantitative and qualitative data. From my point of view, this is the starting point for everything else in the AHP/ANP; this knowledge is crucial to understanding why and how AHP was built. Indeed, it gives the key for knowing how to deal with the data that we have to face in real-life problems.

Keywords: Scales; measurement; AHP

1. Introduction

First of all, I would like to interpret what Dr. Tom Saaty meant with the title of his paper, "*Scales From Measurement, Not Measurement From Scales.*" (Saaty, 2004) With this title, Tom pointed out the importance of being careful how scales are used when measuring. There are many different kinds of scales and they present different properties or qualities that one must be aware of.

For instance, let us explore the following sentence (Holder, 1990):

> If A is weakly more important than B (3, on Saaty's scale), and B is weakly more important than C (again 3, on Saaty's scale), it implies that A is absolutely more important that C (9, in Saaty's scale)…Indeed, the above logic does violence to the normal usage of the English language. (p. 1074)

As the reader can see, Holder's question is; how do two weak relations (3) make an extreme one (9)?

Of course, if we do not realize that Saaty's fundamental scale is an absolute-ratio scale instead of an ordinal scale (or just a semantic and non-continuous scale in Holder's words), and assuming we don't have a good understanding of what a ratio scale, an absolute ratio scale and an ordinal scale are, then this confusion of "how is it possible that two weak pair-comparisons make an extreme" can easily arise. If we think that Saaty's fundamental scale is an ordinal scale, then we are in trouble and this sentence is significant. On the other hand, if we know that Saaty´s scale is an absolute ratio scale, and we know what that means, then we can easily understand why two "weak" comparisons (moderate in Saaty's scale) can (and must) make an extreme comparison. I think this is one of the things Tom was thinking about when he wrote this paper.

So, in the end, if we build scales from measurement, we will get scales that have properties that allow them to be used rationally. For instance, a scale where 3 times 3 is 9, not 3 or 4 or 5 as Holder would expect (he considered Saaty's scale too short, or too easy to get the extreme or last value of the scale). This issue can be explained using apples too (as Tom liked to do). If we have a melon that weighs as much as 3 apples, and a watermelon that weighs as much as 3 melons, then we have a watermelon that weighs as much as 9 apples. By the way, using a very wide scale (an exponential scale for instance) may produce a big loss of precision in the pair comparison process; this happens because the human being is able to compare only homogeneous objects. (Homogeneous refers to objects that belong to the same order of magnitude, that is, in a ratio of 1 to 9).

One question has always intrigued me about this issue of scales and numbers. Why in school (or University), in mathematics classes, does nobody teach about this fundamental issue? Do we know what kind of numbers we are using? Where do they come from? What is it that you can or cannot do with them? In brief, what are those "absolute" things that we call numbers, and are continuously working with in such familiar ways?

Saaty said (2004),

> Human beings have been genetically endowed with the talent to compare things, and it does not seem that that talent will leave us soon. We need to make comparisons to survive, since there are no absolutes. The AHP is the way to make comparisons scientifically. With this method we deduce a scale of priorities by comparing objects in terms of their relative dominance with respect to some attribute. What is established here, even if it seems surprising, first comes the metric to perform the measurements then (and only then), comes a unit and a convenient zero to put in the scale. (p. 1)

Therefore, first we need to build the metric, and then we may build the scale (related to this metric) to be able to use it to make measurements. So, first comes the measurement from which we can later build the scale, and then (and only then), comes the scale of measurement (not inversely).

Again, in Saaty's words (2004):

> Traditionally, measurements are associated with an existing physical scale, from which we can assign to any object a number. Measurement is a transformation from objects to numbers in a one-to-one correspondence. However, this is not always the only way, particularly when the property that is being measured is an intangible and there are no scales that we can use to make the measurement. In this case, the objects to be measured can be used to establish the different values of the scale that they would receive as a group, not individually. (p. 2)

Saaty concluded by saying (2004),

> The relative scales are much more general, since a measurement performed in a physical scale with a unit can be reduced to a relative measurement, but not inversely. When passing from a physical to relative scale we simple discard the information related to the concept of its dimensional unit and its zero, and this process does not affect the resulting relative values or priorities. It is relative measurement that allows us to measure intangibles. If they were tangibles, a scale would probably already exist and it would not be necessary to create a special one to measure them. Sometimes relative measurement would "*seem*" to work against personal system values and understanding, because they are not based on an arbitrary unit to make measurements as happens with physical scales. However, they might be used to give a completely different interpretation about what the measurement means, through a scheme or formula based on understanding in a decision. Finally, in the decision making process, the criteria need to be measured in relative terms with respect to superior objectives to obtain their weights, and use them

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> to weight the priorities of the alternatives. Therefore, the construction of scales is an inescapable issue. (p. 2)

This essay will show the difference between having a scale before making a measurement and building a scale as part of the measurement process. Tom presented a paper about this at the 17th International MCDM conference held at Whistler, British Columbia, Canada and the contents of that paper is the focus of this article.

2. Measurements

The measurement, or the result of the act of measuring, has been defined in many different ways:

> Measuring is the process of discovering an attribute's dimensions, extension, quantity, degree or capacity in an object under observation and then representing these elements in a data language based on qualitative or quantitative terms," or "Measure is the process of assigning to every element from a list of **n** elements or individuals one and just one from **k** categories." or "Measure is to know or give a value through a comparison with a known pattern, that is, to apply a metric," or "A measurement is the result of an objective process the purpose of which is to determine a characteristic number for a specific physical situation," or "Measurement is any set of rules for assigning numbers to attributes of objects. (Saaty 2004, p.2)

In spite of this variety of definitions, there is in all of them the notion (perhaps limited and narrow) of what a measure is, for example, measure is a function which assigns a number to an object. In general, measurements are values that belong to a certain kind of scale that characterizes all that measures. Frequently, the measurements are represented by numbers, but sometimes they are represented by words. Underlying any measure is the idea of "greater than" or "less than", which in combination with some ordering of the objects gives the possibility of selecting one of the measured objects according to some merit. For instance, "the bigger the number, the better the object will be considered", or vice versa.

When an object is measured using a physical instrument, the assigned number is a multiple or a fraction of the arbitrarily chosen unit. Hence, that measure is arbitrary itself and has to be interpreted in terms of its meaning or contribution to the goal in mind. The interpretation of a measurement using a physical scale depends on our physical senses (or our knowledge or experience with that scale) to be interpreted and then correctly used. For a blind person, a centimeter has no more sense than a color,

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unless he or she uses their fingers or some other part of the body to internalize it. This shows clearly the magnitude of the dependence of measurement on our body and not just on our rational minds, as our mechanism for interpreting the world.

3. Scales

There is a clear distinction between reading the numbers of a scale, which is what we usually mean when we say we are measuring something, and the mathematical properties behind the scale. There is confusion in the way we use the concept of scale. On the one hand, we have a clear mathematical definition of a scale, invariant under a transformation, (by the way, this invariant concept which is the core of mathematical definition of scale, is not present in the current definition of scale in Google or in any "scales for dummies"), or what we are doing when we measure using a scale (assigning numbers from the scale to the objects). On the other hand, we have the lay use of the word scale, by which we mean one of the many existing scales that human beings have built that are frequently related to some dispositive or physical instrument. They are more properly called physical or objective scales.

Mathematical Definition of a Scale: Mathematically, a scale is a triplet composed of a set of numbers, a set of objects, and a transformation from the numbers to the objects. The scale also has a more abstract interpretation that only refers to the nature of the numbers and not to the objects, or how the numbers are assigned to the objects. The kind of transformation or the forms to create the numbers admissible for a particular measurement define what is called a scale of measurement for a measurement operation. We have different kinds of transformation and scales described as follows:

Nominal Scale: Invariant under a one-to-one correspondence; for example, when a name, or a telephone number is assigned to an object, there is one and only one name and telephone number assigned to the objects in the set.

Ordinal Scale: Invariant under monotone transformation, where the numbers order the objects, but the magnitude of those numbers are useful only for defining whether the order is increasing or decreasing; for example, when assigning numbers 1 and 2 to two people to indicate that one is taller than the other, without including more information about their real height. The minor number can be assigned to the taller person or vice versa.

Interval Scale: Invariant under a positive linear transformation. For instance, the linear transformation F = (9/5) + 32, which transform readings of temperature from Celsius to Fahrenheit. Notice that it is not possible to add two measures x_1 and x_2 in an interval scale, because then $y_1 + y_2 = (ax_1+b) + (ax_2 + b) = a(x_1+x_2) + 2b$, which take the form of $(ax_3 + 2b)$ which is not in the form of $(ax + b)$ anymore. However, we can take the average of both readings, because after dividing by 2 we are back to the original form. This is why 10 degrees of temperature plus 15 degrees of temperature does not produce 25 degrees of temperature (at most, its sum can make 15 degrees).

Proportional Scale: Invariant under homogenous transformation, $y = ax$, $a>0$. An example is a transformation from pounds to kilograms, with the transformation K = 2.2P. The proportion of the weights of two objects is the same and does not depend on whether the measurement was in pounds or kilos. The zero has no correspondence with any measurement of a real object, it is applied only to objects that do not present

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the property also, it is not possible to divide by zero, and get back a result we can interpret. We also may note that we can add two measures of the same scale $a(x_1+x_2) = a(x_3)$ which have the same form of a. We can also multiply and divide different readings from the same proportional scales. When dividing two measures of the same proportional scale, the ratio of any two measures within it belongs to a proportional absolute scale. For example, 6kg /3kg = 2. The number “2” belongs to a proportional absolute scale, showing that the object weighing 6kg is double the weight of the object weighing 3kg. The number 2 is an absolute number because it cannot be transformed to any different number. The idea of n “absolute number” gives us the entry to present the following scale.

Absolute Scale: Invariant under the identity transformation $y = x$, (with $a = 1$, coming from the ratio: $ax_1/ax_2 = 1x_3$). Examples of this scale are the numbers used to count people in a room, and the natural and real numbers (that is, those used to resolve equations). These absolute numbers are defined in terms of correspondence and equivalence classes of one-to-one correspondence (the Peano postulates), not in terms of some unit of measure starting from an origin at zero.

There are other well-known scales too, like the logarithmic or semi-log scales. The complexity of the scale defines what kind of arithmetic operations can be carried out over the set of numbers in a valid way.

None of the absolute scales have a need for a zero or a unit. In spite of the fact that it is said that a proportional scale has an absolute zero, this is only a supposition that makes it easy to work with the scale. Nowhere in the mathematical definition of a scale is it stated that it should have a unit and an origin with a zero value.

In general, people use the term objective scale when referring to scales that use some type of dispositive measurement (that is, it resolves a situation once and for all), then the measurement made on it is said to be an objective measurement. However, it is significant that not any measure made with a dispositive measurement is an objective measurement. For instance, the Likert scale is used when the intention is to measure subjective intensity-of-preference judgments using an objective scale (it is an absolute scale), measuring all the objects one by one. Such a scale lacks the definition of a unit and a point of reference for the precise representation of the state of the mind of the person giving the judgments; it is measuring his real understanding about something. But even in the best case, the resulting scale would only be an ordinal scale.

To allow the correct manipulation of the measurements and capture the underlying order is a complex problem; cardinal mental measurements are required, not ordinal. This is the origin of a second kind of scale. Besides the physical scales, relative scales emerge. Relative scales also possess the properties of a mathematical scale, particularly the properties of the absolute mathematical scale. Relative scales do not exist objectively out of the time-line and are not valid for any object of the set; they only serve for a given interval of time and for a given set of objects.

There are two ways to obtain a relative scale. The first is to normalize the elements from an objective scale by dividing each by their sums. The resulting scale may be described as an absolute relative scale. The other way to obtain a relative absolute scale is very different. The origin of this relative scale is not from an objective scale, though its readings belong to a mathematical scale and, further, are an absolute mathematical scale with relative values. The relative measures are obtained from the

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real world too by comparing a pair of objects that share some property and assuming that the smaller member defines the unit value for the property. Every pair of objects of some particular set have to be related to each other with a number from the fundamental 1-9 scale of the AHP that indicates how many times larger the dominant member is than the smaller member. Then the measures obtained have to be related with real numbers, in such a way as to build or create a relative scale of absolute numbers.

So, there are these two ways to make measurements; one through an instrument and making the correspondence in a direct way. And the other through judgments that are then processed to make a correspondence from the members of the set to numbers in an indirect way. The next figure represents the two kinds of scale construction:

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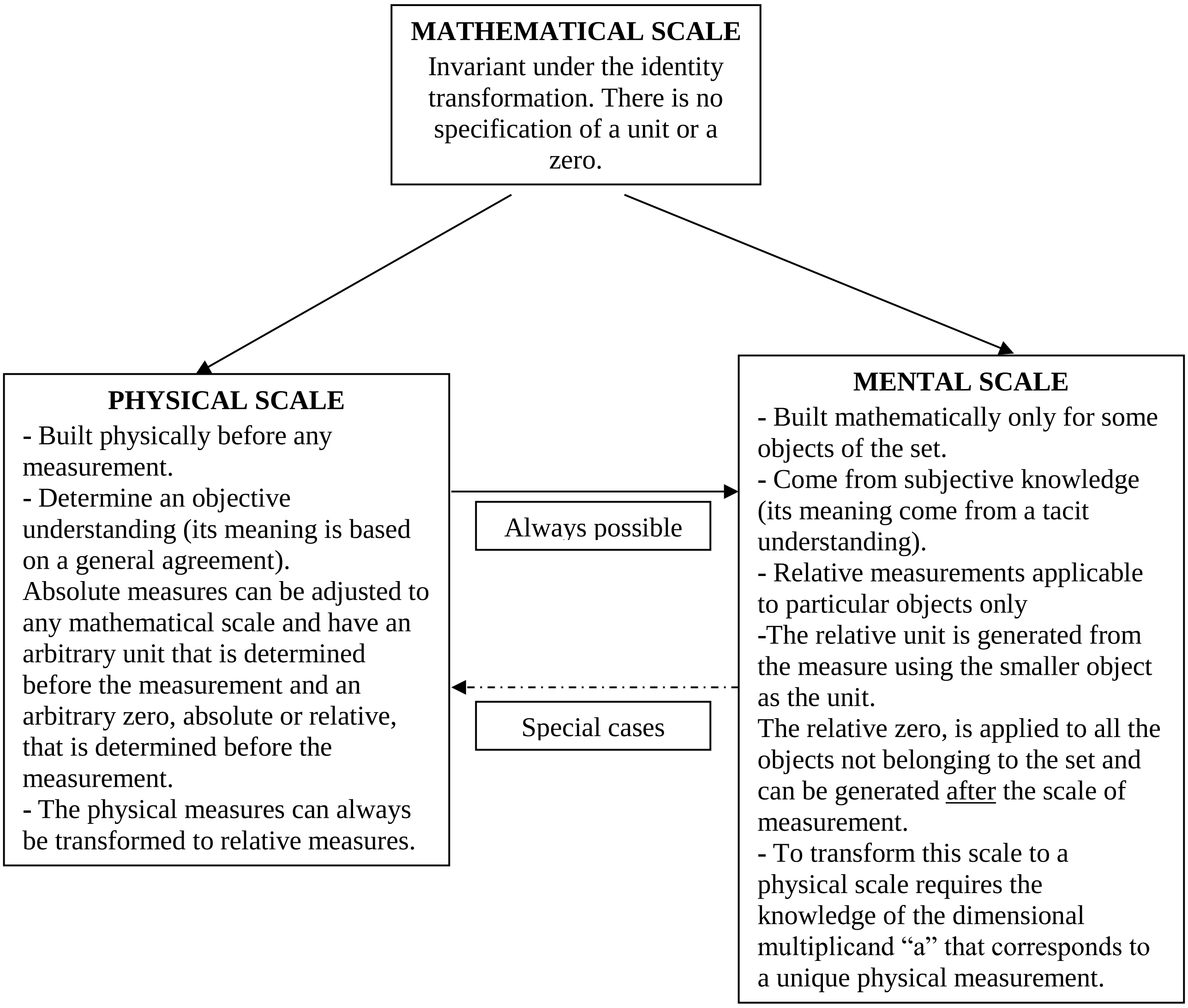

Figure 1. Classification of scales (Saaty 2004, p.6)

Relative Deduced Scales:
From my perspective, the most important part of Tom’s paper is relative deduced scales. Here is where you may find the power of the paired comparison process and its application in AHP/ANP in relative and absolute measurement mode. Let´s start from the beginning. There are two ways to obtain relative deduced scales. The first and most well-known is the typical scale deduced by the combination of two or more basic or primitive relative scales. Examples of these deduced scales are velocity that comes from the relation of distance over time, pressure as force per square inch, the quantity of alcohol in the blood, the density of some liquids, etc.

There is an alternative way to define relative deduced scales. In spite of being widely known in the literature, this approach causes suffering to the critic coming from the old school, who is addicted to the way of measurement used in engineering or affine areas, and reluctant to see the differences between how people think and measure and how a machine of low understanding makes measurements through a mechanism.

Let me explain through an example related to the height of a person. When comparing two persons, both considered to be very tall, one of them will have a relative value less than the other. The relative value the person would receive in this comparison is one of many possible values; it depends on the height of the other person involved in the comparison. The relative value of the tallest person is not unique, as it would be if we made a direct measurement.

As many people know, the AHP uses a fundamental scale of absolute numbers for making comparisons between pairs of objects. Using the scale, one assigns a number to answer the question of how many times one object dominates the other with respect to a certain property. The fundamental scale has the following intensities: 1 equal (neither object dominates the other under the property), 3 moderate, 5 strong, 7 very strong, 9 extreme domination (objects should be grouped for pairwise comparison so that only one order of magnitude is required). The fundamental scale is for judgments, in the absence of measurements from an existing scale, in which case the values themselves can be used (by taking their ratio) to make the comparison. These numbers are entered in a reciprocal positive matrix.

Of course, all these numbers are absolute numbers, whether from judgment, or by taking the ratio of existing measures from some pre-defined scale. Moreover, when an absolute scale already exists, the meaning of a measurement using it, its intrinsic value, may or may not reflect our value system (which, by the way, is the basis of, and can be captured by the paired comparison process).

Let me show an example about the difference between direct data and one possible representation of our value system; the way prices in stores are measured in ratio scales (they have a zero and a unit). Anyone can say how many times more expensive one object is than the other by dividing one price by the other. Nevertheless, this process does not necessarily represent our perception. Imagine that you are buying a cell phone of middle price, about US$500. If you find a phone of similar characteristics for $1,500 you probably will not say it is 3 times (moderate) more expensive, but extremely more expensive and you will assign a 9 (extreme) to that comparison instead of 3. But if you find another cell-phone for US$600, you will say that is a little more expensive and you will probably assign a value between 1 and 2 when comparing with the initial phone. This example shows that our perception or system value is the one that has to be used to interpret numbers, even when it is about

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money, to reflect our perceptions. Hard data (like the price), in general does not have to be translated directly by a normalization process to relative numbers, unless we explicitly accept that the price or market value exactly reflects our value system. Thus, even hard data should first be interpreted through our preferences using the fundamental scale of AHP, and then the relative numbers deduced that more faithfully reflect our value scale for the measured property.

The next example considers a case of temperature reading. A temperature scale is a monotone function with unique values that are linearly related to the temperature being measured. There are the absolute scales of temperature Kelvin (K), and Rankine (Ra) where K = 1.8Ra, with an absolute zero that represents the total ceasing of molecular activity. Both define a proportional scale. There are 3 more temperature scales: the familiar Celsius scale (C), the Fahrenheit scale (F) and Reamur (R), where F = 1.8C+32, and R = 0.8C. All are interval scales. Since C = K - 273.15, then both scales have the same unit. If we work with Kelvin values, then it is possible to produce ratios as we did in the price example. But, if we use the Celsius or Fahrenheit scale, one cannot simply divide the values since they belong to an interval scale. If ratios are required, we first have to form differences of temperature and divide it by some other difference of temperature. This is because $(ax_1+b) / (ax_2+b)$ has no numerical interpretation, as happens also with $(ax_1+b) + (ax_2+b) = a(x_1+ x_2) + 2b$, which does not have the form of the original scale $(ax+b)$. However, $(ax_1+b) - (ax_2+b) = a(x_1- x_2) = ax_3$, does have the form of a proportional scale and can be used to make ratios. Imagine a typical temperature reading from a cold day of February is 5 degrees. As the temperature rises, it is legitimate to ask how much preferable the difference between 15 and 5 degrees is than the difference between 10 and 5 degrees. The numerical proportion of 2 (10/5) is mathematically correct. But here, the exactness of mathematics does not reflect our real sensation about the warmer situation. One physically feels the difference to be "strongly warmer" so using the number 5 from Saaty's scale represents our judgment better than the "slightly warmer" number 2.

The output of a set of paired comparisons is a priority vector. Saaty would say it is comprised of relative absolute numbers. Its properties correspond to a mathematical scale where it is possible to perform arithmetic operations but it has no unit. The relative scales deduced in this manner always define an absolute proportional scale, since this is the only scale where we can talk about the relationship of one value to another through division. The priority vector is calculated with mathematical rigor. This is done using the concept of order of dominance (from order topology). Thus, the value of one paired comparison itself is not relevant, but the set of comparisons as a whole, the relative absolute priority vector, is relevant. Its derivation from the set of paired comparisons is based on graph and system theory. This concept of dominance is also strongly related to the consistency/inconsistency measure, and how new information can be linked with old information in a dynamic system without corrupting the data.

Once it is available for one set of objects, the deduced measure might be used as the value scale for those objects and for other objects that are going to be pairwise compared with the first set. Sometimes the measure relating the old objects and the new ones is kept unaltered, and in some other cases the measure changes as new objects come into the original set. The scale is revised and expanded any time it is necessary in order to include the new values that come from recently performed paired comparisons. Unlike physical scales, the values of the relative scale (the

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priority vector) do not exist before the objects. They appear at the same time the paired comparisons of the objects are finished and transformed into a priority vector using the AHP process of finding the eigenvector associated to the maximum eigenvalue of the reciprocal pairwise comparison matrix. Once the priority vector is finished, any object outside of the set of objects has the value zero. If it is ever considered as a candidate to be measured, then it has to be moved into the set of objects and all the previous measures of the priority vector have to be revised. In other words, the deduction of relative measurement is an iterative process, and it depends on the particular objects under consideration for the measurement.

The Relative Mode of Measurement:
Relative measurements are useful for providing priorities, for instance, for issues like beauty or quality of life. These values give information about the relative intensity of beauty or quality of life, not its absolute value taken from some origin or zero value (what is the zero value for beauty?). This is because for intangible properties we don’t have a scale of measurement with a zero and unit before knowing what objects are to be measured. By definition, relative priorities do not relate the measured objects to an absolute pattern (which most of the times doesn’t even exist), but only in relation to the other objects of the set. This is precisely the goal of deducing or building a relative metric. The relative measures obtained were deduced within a specific context and they only have sense inside that context, in the same way that decisions have sense within a given context.

The Absolute Mode of Measurement:
There is a second way to obtain a priority vector; this is by using the absolute mode of the AHP. In this mode, the alternatives are considered one at the time, though the criteria and sub-criteria are prioritized as usual with paired comparisons. Each terminal criterion has its own scale of intensities or gradation levels, these levels have to be pairwise compared to establish their priorities; that is, we deduce the scale of measurement for each specific criterion. Then a level of intensity is assigned for each alternative under every terminal criterion. Each intensity priority scale is weighted by the global terminal criterion weight and summed for all the terminal criteria. Thus, a total priority value for each alternative is determined independently of the rest of the alternatives. This is the normative scheme of the AHP since it requires expert knowledge to define and prioritize the intensities. The final synthesis may or may not differ from the relative measurement mode, this will depend on the degree of relation among the alternatives, the criteria and eventually the scales and the way we represent those relations.

The paired comparisons and their respective final priorities (deduced without the use of some tangible measurement tool), give rise to a relative value scale, just like percentages (deduced from tangible measurements), or probabilities (deduced from the frequency of occurrence of some event). Relative value scales do not depend on some unique arbitrary unit applied repetitively over the whole set or in some part of it in a linear way (indeed, these deduced scales rarely are linear). The ideal unit of the scale is produced by dividing each value of the scale by its greatest value (maximum or infinite norm). Potentially, the measurement of any element could serve as a unit of measure, dividing by its value, thus the unit does not need to be unique and is not used to measure anything else, as happens in the case of measurements with physical scales. However, it is possible to compare any new element with the ideal and assign the proper value, greater or lower than that unit.

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In summary, the usual way to make absolute measurements is that we first have to create the scale of measurement, often involving a pattern or arbitrary unit with respect to which all the rest of the measures are related linearly, and then assign a value of the scale to each of the elements that are being measured. On the other hand, with the relative way of measuring presented here, that allows measuring both tangibles and intangibles, we first specify the objects that will be measured, and then we pairwise compare the objects according to some property they have in common, through assigning values from the fundamental scale, and only then do we deduce a common scale of relative values.

These are two opposite ways to make a measurement. A relative scale is a triplet of objects that correspond; first to a fundamental scale, second a transformation that assigns numbers from the fundamental scale to a pair of objects, and third a transformation that assigns real numbers to these numbers which come from the fundamental scale. It means an indirect transformation from the initial numbers to the real numbers. These ideas are the very basis on which the AHP has been developed.

4. Epilogue

I was so lucky to have learned these things from Tom in a simple and clear way. In my 25 years spreading, teaching (and applying) AHP/ANP in many different Universities, in undergraduate and graduate courses, in Chile and also in other countries, I have never seen anybody taking care to explain about this fundamental issue, not even in the curricula of scientific careers like engineering, mathematics, or physics. This "hole" in the curricula has always left me intrigued. As a matter of fact, I have a real story about this issue. Once while teaching AHP, for an MBA course in the Federico Santa Maria University in Chile, one student came to me at the end of the lesson to ask a question. I was sitting at my desk, and she leaned over to me and asked me (a little bit perplexed), if I was sure about what I had just said in class. Since (in her words), "*This is my third graduate program and I have never heard anything like that, and this could change my way of looking at and understanding real life problems.*" From my perspective, I do believe that the problem is the knowledge behind the numbers. By the way, I remember having told her, *"As far as I know, this is true and moreover is something that we must never forget."*

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