A PERSONAL VIEW OF THE DEVELOPMENT OF THE AHP

Significant milestones during the evolution of the AHP

Submitted by
Rozann Saaty
IJAHP Managing Editor
Creative Decisions Foundation
rsaaty@ijahp.org

This essay is to tell you what I saw as the AHP evolved. I had an inside view as Tom and I were married in 1964 before he started developing the AHP. It was an interesting forty-five years from 1972 until 2017 when Tom passed away. Tom became interested in how to measure intangibles when he worked for the Arms Control and Disarmament Agency in the U.S. State Department in the 1960s. This agency conducted the arms control negotiations between the Soviets and the Americans, and Tom said he learned there that most decisions come down to intangibles and emotions, and that there was no way to quantify them. Which armaments? How many of theirs equaled how many of ours? These questions were decided by lawyers who came up with the terms, not scientists or mathematicians, or even military people. Tom headed a research team that included three Nobel Prize winners, and even they were not of much help in the negotiations. It took them too long to model a situation and find an answer, and by the time they were ready the negotiations would have moved on.

Tom left the Arms Control Agency in 1969 because the climate was changing in Washington; a Republican President, Richard Nixon, was elected to office after Democratic presidents Jack Kennedy and Lyndon B. Johnson had held the office for eight years. As the Democrats were out of power in the 1970s, the arms talks were sidelined, so Tom moved to academia at the Wharton School of the University of Pennsylvania in the fall of 1969.

It was with a sense of awe and amazement that just last month I came across a copy of a report, typewritten and bound with a soft tan cover, that Tom had written in the early 1970s while consulting for LMI, the Logistics Management Institute, a think tank for the Navy. The report was titled *Identification of War Reserve Stock, Task 72-04 of Contract No. SD-271*. I was amazed on two counts; first, that I had found the birth document of the AHP, and second that he still had the report from 45 years ago. The object of the job was to prioritize all the Navy war materiel ranging from toilet seats to tanks, and LMI was hired as the consultant.

Tom laid out some of the core concepts of the AHP for the first time in *Appendix 3: Mathematical Method for Quantifying Essentiality Judgments*. These concepts included the pairwise comparison matrix formed of ratios from which priority vectors are obtained and simple hierarchic composition. He does not use the word hierarchy, though he illustrates a weight and add process of prioritized properties and prioritized alternatives for each property that is essentially synthesis for a two-level hierarchy. He gives credit to

previous works: the theorem of Perron and Frobenius which he used (from Gantmacher, *The Theory of Matrices*, Vol. II, p. 53-54) and graph and matrix theoretic concepts (from C. Berge, *The Theory of Graphs*, Wiley, 1962, p. 135-138 and D. Gale and L. S. Shapley, College Admissions and the Stability of Marriage, *Am. Math. Monthly*, *69*, 1962, p. 9-14). In the report where he is giving credit to others he says that the extension for use to assign weights to properties is due to him.

To me, the evolution from the AHP to the ANP to complex ANP models to Neural Networks is an interesting story somewhat analogous to how life evolved, starting with a simple cell (the pairwise comparison matrix) and evolving into more complex cells (AHP hierarchies), then to collections of complex cells to form simple organisms (simple ANP networks), to complex organisms like animals and plants (complex ANP models of many simple networks) and finally to societies of networked organisms formed of individuals with communication links (NNP – neural network process).

The reciprocal pairwise comparison matrix was the basic building block of the theory. For example, if $a_{ij} = 5$ the ratio 5/1 is entered in the (i,j) position in the matrix, and 1/5 is entered in the reciprocal (j,i) position. The entries are ratios of absolute numbers, either formed from two measurements of a property using a traditional ratio scale or by using judgments from the fundamental 1-9 scale of the AHP where, 1 – Equal, 3 – Moderate, 5– Strong, 7 – Very Strong, 9 – Extreme. A judgment from the scale is a ratio indicating how many times the dominant element is larger than the dominated one. In each comparison based on judgments the dominated element is the unit in the denominator. But note that in every cell, the unit in the ratio is different, but the totality of the individual ratios is synthesized into an overall relative priority vector. The only role zero plays is when two elements cannot be compared it is entered into the matrix. There is no starting point, no zero, for relative scales.

People have often used the scale as if it is an ordinal scale; for example, when one chooses a number from 1 to 5, where 5 is the best, bigger numbers are only known to be better, but not by how much. Numbers from an ordinal scale are not the same as the absolute numbers expressed as ratios from the fundamental scale of the AHP. The invention of the fundamental scale of absolute numbers associated with words was a major leap into the unknown by Tom. It was the device that allowed him to bring intangibles into the picture along with measurables and tangibles. Getting people to use this idea, and being able to understand it required patience, persistence and persuasion.

A relative scale is a different way of measuring. When making pairwise comparisons adding or subtracting elements gives completely different priorities. With a traditional ratio scale, such as a "ruler", when something is measured it stays measured, the measurement does not change regardless of what items are added or removed. Tom's first book on the AHP, *The Analytic Hierarchy Process*, from McGraw Hill International, appeared in 1980.

This new scale of relative measurement led to the Rank Reversal wars of the 1980s and 1990s, and some people are still revisiting that issue today. Valerie Belton and T. Gear wrote a critical article about rank reversal in the AHP titled, '*On a Short-coming of Saaty's Method of Analytic Hierarchies'*, Omega, *11(3)*, 228-230, 1983. This became the

seminal reference in almost every article critical of rank reversal, and it is still cited in articles written today. However, an interesting thing happened at an MCDM meeting in Canada, chaired by Bill Wedley, in 2004. At this meeting, I delivered a paper in a session that was critical of the rank reversal critics of AHP. Valerie Belton was in the audience and she stood up and said she was so sorry about her paper and that she had not understood, but there was no way she could get it back or undo it as it had taken on a life of its own.

The AHP with its single kind of matrix, the pairwise comparison matrix, and a hierarchical model with its top-down weight and add synthesis process to obtain the priorities for the alternatives was followed by the ANP (Analytic Network Process) with Tom's first book about it, *The Analytic Network Process: Decision Making with Dependence and Feedback*, RWS Publications, 1996. This book dealt with single network models, the structure changed from being a hierarchy of elements arranged in levels to a network of clusters of elements with links between the elements. A new data structure was introduced, the supermatrix, a square matrix with all the elements in the network as row and column elements arranged by cluster. The priority vectors are vertical priority vectors stacked in a column derived by pairwise comparing the elements in a cluster that are connected from the column element.

"Why call it a supermatrix?" people asked. Tom defended the name, saying that it was a matrix of matrices in a sense, not his attempt to be grandiose. The supermatrix is the second kind of matrix in the theory, being made up of priority vectors derived from pairwise comparison matrices, quite different from the pairwise comparison matrices composed of judgments. But the eigenvector is again used to derive an overall vector of priorities of all the elements in the network, including the alternatives. His colleague at the Wharton School, James Bennett–who Tom liked and admired him for his brilliance–was very struck by the supermatrix and complimented Tom mightily on having come up with it.

Let me mention that Tom was superbly equipped to generalize the hierarchical structures of the AHP to networks. He wrote a book with Robert Busacker before the AHP titled, *Finite Graphs and Networks* by McGraw Hill in 1965. This book was heavy with the mathematics for networks and at the same time as he worked out the mathematics for the AHP and ANP, shoring up his intuition with theoretical proofs, he was also delving deeper into modeling, writing these four books: *Mathematical Models of Arms Control and Disarmament*, (translated to Russian), John Wiley and Sons, 1968; *The Four-Color Problem; Assaults and Conquest*, with Paul C. Kainen, McGraw-Hill International, in 1977; *Thinking with Models* with Joyce Alexander, Pergamon Press, 1981, and *Conflict Resolution: The Analytic Hierarchy Process* with Joyce Alexander, Praeger, 1989.

Tom was honing another skill along the way. Starting in 1964 he collected jokes, putting out a new joke book every two years or so to a total of 23 ending with three volumes published in July 2017. Of course, he mostly collected the jokes, he did not invent many of them, but he came up with the titles, practicing capturing the essence of a joke with short succinct phrases. I think this was a very important skill to have as he broke ground in this new field of relative measurement and wrote countless papers about it. He had to

invent some of the language. The ability to write concisely and with drama about a dense mathematical subject was a great asset, and I think it came partly from his work on jokes. One might argue that the name Analytic Hierarchy Process for the process he invented is no evidence of this skill, but no one has come up with a better name.

Continuing the development of the ANP, a major step Tom had to overcome was how to get the supermatrix to converge to a solution when it is raised to powers. The supermatrix converges only if each column of the supermatrix sums to 1.0 (i.e., is stochastic). A column may have several priority vectors stacked on top of each other and as each priority vector sums to 1.0, the entire column sums to something more than one. He had to find a way to get every column to sum to one, therefore he invented the cluster matrix. By pairwise comparing the impact of the clusters linked from a given cluster for their impact on the "parent" cluster, with the usual pairwise comparison matrix of judgments, one gets a priority matrix for the clusters that becomes a column in the cluster matrix. Arranging the cluster comparison priority vectors in a square matrix and multiplying the elements in it times the respective components in the supermatrix accomplished this aim and "Voila!", the weighted supermatrix was born. It has the desired property that all its columns sum to 1.0 and it converges when raised to powers to the limit supermatrix which has priorities for all the elements in the network.

It turned out that not only does the supermatrix converge after multiplying its elements by the cluster matrix, but humans do have the ability to pairwise compare clusters, judging which one has more impact on the parent cluster. I don't know how Tom thought of this, as he never talked about the insight that led to it. This one is mystical, to me at least.

The next evolutionary step was the complex ANP model of separate networks Benefits, Opportunities, Costs and Risks (BOCR) which knits together different individual networks using algebraic equations.

Following that was Tom's work on the brain, the NNP – neural network process. He believed that the transmission of signals in the brain depended on having some way to synthesize the electrical and chemical inputs to and outputs from a neuron. In essence, the brain is a highly complex network with different kinds of stimuli including visual, auditory, sensory, memories and so on. These stimuli must be synthesized to produce the signals throughout the brain's network of cells and neurons, and perhaps it is the state of all this at any one moment of time that is what we call consciousness. Tom had bookcases full of works on the brain and nervous system, and he had read them all. Many of the books have scraps of paper sticking out of them to mark where he found something interesting.

To conclude this essay, let me summarize what I think were Tom's contributions and significant advances:

1. Developing the reciprocal pairwise comparison matrix of elements where the entries in the cells are ratios of tangibles or human judgments and finding its solution of priorities with the eigenvector (and years went into justifying that the eigenvector is the correct way for inconsistent matrices). The solution is an absolute relative ratio scale vector of priorities.

2. Inventing the fundamental scale of the AHP comprised of absolute numbers that brought human judgment into the pairwise comparison matrix; specifying that this scale should be used for homogeneous groups of elements.
3. Using the eigenvalue from the eigenvector solution to create an index for inconsistency in a pairwise comparison matrix of judgments.
4. Developing the compatibility index for determining closeness of two priority vectors.
5. Creating a hierarchic structure for decision models and a synthesis method to obtain the overall priorities of the alternatives at the bottom of the hierarchy – hierarchies are not the same as trees beyond three levels, a little understood fact. Most current AHP software packages use a tree structure. The Superdecisions software for ANP (www.superdecisions.com) developed by Tom and me and is for modeling networks which are a generalization of hierarchies, so it can be used to model true hierarchies.
6. Axiomatizing the AHP There are 4 or 5 simple axioms that are fundamental to describing AHP models, and somehow, with his training in classical mathematics, he was able to boil down all the ideas to these few axioms.
7. Generalizing from hierarchies to single ANP networks with hierarchic levels being replaced by clusters of elements and links between the elements. ANP networks in general do not have goals. It is a relative world where priorities are determined by the interactions among the elements.
8. Developing the Conflict Resolution approach from the AHP, a process that he and his colleague Dean Jerry Zoffer used in several trial negotiations between teams from Israel and Palestine to prioritize things they might trade and arrange them into equitable swaps. Tom and Jerry considered this to be practice for when the time ever came that the leaders of the two countries truly wanted peace; real negotiations needed to have a process that had been tested for doing tradeoffs. His insight was that there is sometimes a retributive aspect to resolving conflicts. Not only do the parties want to receive something of value, they want to inflict costs on the giving party, so every swap is analyzed from the perspective of each party for gain to them and cost to their opponent.
9. Inventing the supermatrix that contains the priority vectors from all the pairwise comparison matrices, and the cluster matrix formed by pairwise comparing clusters for their impact on a parent cluster.
10. Using the supermatrix to find priorities for all the elements in the network by raising the weighted supermatrix to powers until it converges. Finding the solution this way is the same as finding the eigenvector of the supermatrix. Surprising fact: the eigenvector is also the solution for the pairwise comparison matrix.
11. Developing the complex ANP BOCR model where formulas are used in the top level to combine priority vectors from lower level networks.
12. Developing the Hypermatrix for the brain. Tom wrote two books about the brain: *The Brain* in 2000 and *The Neural Network Process* in 2014.

It would take many more pages than are allowed in an essay to discuss all the creativity Tom had and all the intricacies of mathematics he knew and used. But I would like to close by mentioning that he published three important papers in 2016 and 2017 when he was in his nineties, being an inspiring role model for the rest of us.

1. Saaty, T. (2016). Seven is the magic number in nature, *Proceedings of the American Philosophical Society*, *160(4)*, 335-360.

2. Saaty, T. (2017). Neurons the decision makers, Part I: The firing function of a single neuron, *Neural Networks*, *86*, 102–114.

3. Saaty, T. (2017). Part 2—The firings of many neurons and their density; the neural network its connections and field of firings, *Neural Networks*, *86*, 115–122.

Tom liked to be called the "Father of AHP" rather than the creator of the method. In this sense, he was right since AHP, ANP and now NNP have taken on a life of their own and will transcend, through their worldwide use, the unavoidable finite human life span of Tom. It has been said of the Greeks that they took no rest themselves and gave none to others. This could also be said of Tom. May his soul rest in peace.