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Axiomatic Theory of Complex Fuzzy Logic and Complex Fuzzy Classes

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Abstract: Complex fuzzy sets, classes, and logic have an important role in applications, such as prediction of periodic events and advanced control systems, where several fuzzy variables interact with each other in a multifaceted way that cannot be represented effectively via simple fuzzy operations such as union, intersection, complement, negation, conjunction and disjunction. The initial formulation of these terms stems from the definition of complex fuzzy grade of membership. The problem, however, with these definitions are twofold: 1) the complex fuzzy membership is limited to polar representation with only one fuzzy component. 2) The definition is based on grade of membership and is lacking the rigor of axiomatic formulation.
A new interpretation of complex fuzzy membership enables polar and Cartesian representation of the membership function where the two function components carry uncertain information. Moreover, the new interpretation is used to define complex fuzzy classes and develop an axiomatic based theory of complex propositional fuzzy logic. Additionally, the generalization of the theory to multidimensional fuzzy grades of membership has been demonstrated.
In this paper we propose an axiomatic framework for first order predicate complex fuzzy logic and use this framework for axiomatic definition of complex fuzzy classes. We use these rigorous definitions to exemplify inference in complex economic systems. The new framework overcomes the main limitations of current theory and provides several advantages. First, the derivation of the new theory is based on axiomatic approach and does not assume the existence of complex fuzzy sets or complex fuzzy classes. Second, the new form significantly improves the expressive power and inference capability of complex fuzzy logic and class theory. The paper surveys the current state of complex fuzzy sets, complex fuzzy classes, and complex fuzzy logic; and provides an axiomatic basis for first order predicate complex fuzzy logic and complex class theory.
Keywords: Fuzzy Logic, Fuzzy Class Theory, Complex Fuzzy Logic, Complex fuzzy Class theory.

1 Introduction

In 1965, L. A. Zadeh has established the theory of fuzzy sets where the degree of membership of an item in a set can get any value in the interval [0, 1] rather than the two values $\{\notin, \in\}$ [1,2].

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In addition he introduced the notion of fuzzy logic [1]- [4]. Which is a multilevel extension of classical logic where propositions can get truth-values in the interval [0, 1], and are not limited to one of the two values {True, False} (or {0, 1}) [3]. The four decades that followed his pioneering work has shown a multitude of research work and applications related to signal processing [5], knowledge representation [6], control theory [4], reasoning [7]- [9], and data mining [10]. In 1975 Zadeh introduced the concept of linguistic variable and the induced concept of type-2 (type-n) fuzzy sets [11, 12]. In recent years, type-1 and type-2 along with interval type1/type-2 fuzzy logic and fuzzy systems have been applied in many areas including signal processing [12], fuzzy clustering [13], data mining [10], and software testing [14].

Complex fuzzy sets, classes, and logic have an important role in applications, such as prediction of periodic events and advanced control systems, where several fuzzy variables interact with each other in a complex way that cannot be represented effectively via simple fuzzy operations such as union, intersection, complement, negation, conjunction and disjunction. The initial formulation of these terms stems from the definition of complex fuzzy grade of membership.

Ramot et al. propose an extension of fuzzy set theory and fuzzy logic where the range of degrees of membership and the range of truth-values is the complex unit circle [15, 16]. These definitions, however, have several constraints. First, the derivation relies on a complex fuzzy membership which is limited to polar representation and carries fuzzy information only in the magnitude component of the function. Second, they do not provide an axiomatic approach for their theory. Finally, the derivation of complex fuzzy logic assumes complex fuzzy relations thereby presumes the existence of complex fuzzy sets. Dick has extended the work by Ramot et al., [17], yet his approach is also limited to polar representation with single fuzzy component.

Tamir et al., provide further generalization of the concept of complex fuzzy membership function and use a Cartesian complex fuzzy membership function where both the real part and the imaginary part can be fuzzy functions. Alternatively, polar representation where both the magnitude and the phase values of the complex membership function convey fuzzy information, can be utilized [18]. Furthermore, they provide a new interpretation of complex fuzzy grades of membership as a representation of a complex fuzzy class along with complex fuzzy class operations. While this formulation has several advantages over previous formalisms, it is still based on the definition of grade of membership which is a limiting factor in the ability to provide a rigorous, axiomatic based, theory. In ref. [19], Tamir et al., develop an axiomatic based propositional complex fuzzy logic theory which is independent of complex fuzzy sets, classes and relations. In addition, they demonstrate the potential use of this formalism for inference in complex systems.

In this paper we expand the definition of complex fuzzy logic to include axiomatic based definition of first order predicate complex fuzzy logic and use the new formalism to establish an axiomatic framework for complex fuzzy classes. The new theory is compatible with classical logic, as well as with first order predicate fuzzy logic [20]- [24]. Furthermore, the new theory supports Cartesian as well as polar representation of complex logical fuzzy propositions with two components of ambiguous information. Hence, this form significantly improves the expressive power and inference capability of complex fuzzy logic.

The paper reviews the current state of complex fuzzy sets theory, provides a brief overview of complex fuzzy classes, and complex fuzzy logic; and introduces a new and generalized complex fuzzy propositional logic theory. The new formalism can be used in advanced complex fuzzy logic systems and provides ways for extension into multidimensional fuzzy propositional and first order logic. Furthermore, it can be used for inference with type 2 (or higher) fuzzy sets [11, 12].

The rest of the paper is organized in the following way: section 2 introduces current research based on the concepts of complex fuzzy sets [15]- [18]. Section 3 provides the axiomatic definition of first order complex fuzzy predicate logic [19]. Based on the formulation in section 3, section

4 contains an axiomatic theory of complex fuzzy classes while section 5 includes conclusions and directions for further research.

2 Current and Related Research on Complex Fuzzy Sets

Ramot et al., observe that the expressive power of fuzzy set theory and fuzzy logic and the utility of derived applications can be significantly improved via the introduction of complex fuzzy sets [15,16]. Their observation is mainly motivated by fuzzy processes that contain periodical behavior such as the cycles in economic markets. In order to capture these phenomena in reasoning, they introduce a complex grade of membership and derive the definition of complex fuzzy sets. Later, they introduce complex fuzzy logic via relations on complex fuzzy sets. Their formalism, however, is limited due to the fact that they restrict the membership function to representation using polar coordinate system where only the magnitude carries fuzzy information. Motivated by similar considerations, Tamir et al., extends the formalism proposed by Ramot et al., and introduce complex class theory where both component of a complex fuzzy grade of membership carries fuzzy information [18]. This enables reasoning about processes with multi-dimensional components where each component is carrying fuzzy information and the interaction between the components cannot be decomposed and represented via primitive, one dimensional, fuzzy set theory and fuzzy logic operations such as union, intersection, conjunction, and disjunction. The current formulation of complex fuzzy sets, complex fuzzy classes, and propositional fuzzy logic is based on the definition of grade of membership. This section briefly reviews the first two formalisms [15]- [18]. The next section introduces axiomatic fuzzy logic and complex fuzzy logic theory and includes a discussion on propositional complex fuzzy logic [19].

2.1 Membership-Grade based Definition of Complex Fuzzy Sets

A complex fuzzy set S on a universe of discourse U is defined by a complex-valued grade of membership function $\mu_S(x)$; (Ramot et al. [15,16]):

$$\mu_S(x) = r_S(x)e^{j\omega_S(x)} \tag{1}$$

Where $j = \sqrt{-1}$. This definition utilizes polar representation of complex numbers along with conventional fuzzy set definition; where $r_S(x)$, the amplitude part of the grade of membership, is a fuzzy function defined in the interval [0, 1]. And $\omega_S(x)$ is a real number standing for the phase part of the grade of membership. Ramot et al. propose the following operations on complex fuzzy sets:

1) A directional fuzzy complement which induces the following membership function

$$c(\mu_S(x)) = c(r_S(x))e^{j(\omega_s(x)+\theta)} \tag{2}$$

Where C(f(x))=1-f(x).

2) The union function, ∪, of two complex fuzzy sets A and B induces a membership function $\mu_{A\cup B}(x)$ defined to be:

$$\mu_{A\cup B}(x) = [r_A(x) \oplus r_B(x)] \cdot e^{j\omega_{A\cup B}(x)} \tag{3}$$

where $\oplus$ represents a *t-conorm* function [24]; and $\omega_{A\cup B}(x)$ is a real function.

3) The intersection function ∩, of two complex fuzzy sets A and B induces a membership function $\mu_{A\cap B}(x)$ defined to be:

$$\mu_{A\cap B}(x) = [r_A(x) \oplus r_B(x)] \cdot e^{j\omega_{A\cap B}(x)} \tag{4}$$

where $\oplus$ represents a *t-conorm* function [24]; and $\omega_{A\cap B}(x)$ is a real function.

Dick expands the research on complex fuzzy sets [17], [25]- [30]. In addition, he observes the relations between complex fuzzy sets and complex neural networks where the excitation, the outputs, and weights can obtain complex values [31]- [35]. Buckley introduces the definition of complex fuzzy numbers [36,37]. Complex fuzzy numbers have been utilized in several numerical applications [38,39]. Nevertheless, the concept of complex fuzzy numbers is different from the concept of complex fuzzy sets, classes, and logic [15]- [19], it is also different from the concepts presented in the current paper. Buckley is concerned with generalizing number theory, while references [15]- [19] as well as the current paper are concerned with the generalization of fuzzy set theory and fuzzy logic.

2.2 Membership-Grade based Definition of Complex Fuzzy Classes

In the following sections, a class is denoted by an upper case Greek letter, a crisp set is denoted by an upper case Latin letter, and a member of a set is denoted by a lower case Latin letter.

The Cartesian and polar representation of a complex grade of membership, respectively are given by:

$$\mu(V,z) = \mu_r(V) + j\mu_i(z), \mu(V,x) = r(V)e^{j\sigma\phi(z)} \tag{5}$$

where $\mu_r(V), \mu_i(z), r(V)$ and $\phi(z)$, the real and imaginary components as well as the amplitude and phase components of the complex fuzzy grade of membership, are real value functions over the interval [0,1]. The scaling factor σ is in the interval $(0, 2\pi]$. It is used to control the behavior of the phase within the unit circle according to the specific application.

We adopt the definition of fuzzy classes given by Běhounek et al., and define a Complex Fuzzy class Γ to be a fuzzy class of order 2 [44,45]. Formally, let U be a universe of discourse and let 2^U be the power set of U. Let f_1 be a function from 2^U to [0, 1] and let f_2 be a function that maps elements of U to the interval [0, 1]. For $V \in 2^U$ and z $\in$ U define $\mu_\Gamma(V,z)$ to be:

$$\mu_\Gamma(V,z) = \mu_r(V) + j\mu_i(z) = f_1(V) + jf_2(z) \tag{6}$$

Then, $\mu_\Gamma(V,z)$ defines a fuzzy class of order two, where for every $V \in 2^U$, and for every z $\in$ U, $\mu_\Gamma(V,z)$; is the degree of membership of z in V and the degree of membership of V in Γ. Hence, a complex fuzzy class Γ can be represented as the set of ordered triples:

$$\Gamma = \{V, z, \mu_\Gamma(V,z) | V \in 2^U, z \in U\} \tag{7}$$

Depending on the form of $\mu_\Gamma()$ (Cartesian or polar), $\mu_r()$, $\mu_r()$, r(), and $\phi()$ denote the degree of membership of z in V and / or the degree of membership of V in Γ. The role of the real and imaginary parts of the membership function can be interchanged, the same applies to the components of the polar representation. Furthermore, the usual definition of coordinate transformation:

$$\mu_\Gamma(V,z) \Leftrightarrow r(V)e^{j\phi(x)}; \mu_\Gamma(V,z) \Leftrightarrow r(V) + j\phi(x) \tag{8}$$

maintains the semantics of complex fuzzy classes. Thus, without loss of generality we assume that: $\mu_r(V)$ and r(V) denote the degree of membership of V in Γ and that $\mu_i(z)$ and $\phi(z)$ denote the degree of membership of membership of z in V for the Cartesian and the polar representations respectively. One can consider the grade of membership based definition of a fuzzy class of order 1 as a mapping into a one dimensional space and the grade based definition of fuzzy class of order 2 (see [40]) as a mapping into a two dimensional space. Hence, it is possible to consider a degree of membership of order n as a mapping into an n-dimensional space.

Operations on Complex Fuzzy Classes.

Consider the two complex fuzzy classes: 1. $\Gamma = \{V, z, \mu_\Gamma(V, z) | V \in 2^U, z \in U\}$. 2. $\Psi = \{T, z, \mu_\Psi(T, z) | T \in 2^U, z \in U\}$. Where V and T are fuzzy classes. Assume that Γ and Ψ are defined over a universe of discourse U and let 2^U denote the power set of U. Further assume that the degree of membership of an object $z \in V$, and an object $y \in T$ is given by: $\mu_\Gamma(V, z) = \mu_{\Gamma r}(V) + j\mu_{\Gamma i}(z); \mu_\Psi(T, y) = \mu_{\Psi r}(T) + j\mu_{\Psi i}(y)$ respectively, where $\mu_{\Gamma r}(\alpha), \mu_{\Psi r}(\alpha), \mu_{\Gamma i}(\alpha)$ and $\mu_{\Psi i}(\alpha)$ stand for the real and imaginary parts of $\mu_\Gamma(V, x)$ and $\mu_\Psi(T, y)$. Finally, let $W \in 2^U$, let $\oplus$ denote a t-conorm operation, and let $\oplus$ denote a t-norm operation.

1. The complex fuzzy class complement is defined to be:

$$c(\mu_\Gamma(V, z)) = c(\mu_r(V)) + jc(\mu_i(z)) \tag{9}$$

2. The complex fuzzy class union is defined to be:

$$\mu_{\Gamma\cup\Psi}(W, z) = (\mu_{\Gamma r}(V) \oplus \mu_{\Psi r}(T) + j(\mu_{\Gamma i}(z) \oplus \mu_{\Psi i}(z))) \tag{10}$$

3. The complex fuzzy class intersection operation is given by:

$$\mu_{\Gamma\cap\Psi}(W, z) = (\mu_{\Gamma r}(V) \oplus \mu_{\Psi r}(T) + j(\mu_{\Gamma i}(z) \oplus \mu_{\Psi i}(z))) \tag{11}$$

Reference [18] includes several examples of complex fuzzy classes and their applications.

3 Axiomatic Complex Fuzzy Logic

Two main considerations are motivating this section. The first relates to the observations stated in section 2. Namely, there are multidimensional fuzzy processes which cannot be decomposed into simple single dimensional logical operations. Second, many researchers has observed that the grade based membership approach is limited in its capability to deliver a concise and precise formalism for fuzzy logic [20]- [24], [40]. Instead, current research in fuzzy logic, fuzzy class theory, fuzzy mathematics, and its applications is based on axiomatic theory. In ref. [19] we have presented an axiomatic framework for propositional complex fuzzy logic (CFL). In this section we extend the definition to include first order predicates CFL. The presented framework is based on the basic fuzzy propositional and predicate logic (BL) along with the fuzzy Łukasiewicz (Ł) and fuzzy product (Π) logical systems [21, 22, 40]. We refer to the propositional logic system as ŁΠ and to the first order predicate fuzzy logic system as ŁΠ∀. This section introduces the ŁΠ and ŁΠ∀ fuzzy logic system as well as the ŁΠ and ŁΠ∀ CFL. We use the following notation: a variable is denoted by a lower case Latin letter and a predicate/proposition is denoted by an upper case Latin letter.

3.1 Propositional Fuzzy Logic

Several axiom based logical systems have been investigated [20]- [24], [40]. Běhounek, et al., ([40]) use the ŁΠ/ ŁΠ∀ as the basis for the definition of fuzzy class theory (FCT). In this section we closely follow ŁΠ the system used by Běhounek, et al. For clarity, we reintroduce some of the important notions, notations, and concepts from that paper.

A fuzzy proposition P can get any truth-value in the real interval [0,1], where '0' denotes "False," and '1' denotes "True." Furthermore, the relation $\leq$, over the interval [0,1] implies a monotonically increasing ordering on the truth-values associated with the proposition. A fuzzy interpretation of a proposition P is an assignment of a fuzzy truth-value to P. Let P and Q

denote fuzzy propositions and let i(P) denotes the fuzzy interpretation of P. Table 1, includes the basic connectives of ŁΠ. Table 2 includes connectives that can be derived from the basic connectives. The constant 0 is assumed; and the constant 1 can be derived from 0 and the basic connectives.

Operation	Interpretation
Ł-Implication	$i(P \to_L Q) = \min(1, 1 - i(P) + i(Q))$
Π-Implication	$i(P \to_\Pi Q) = \min(1, i(P)/i(Q))$
Π-Conjunction	$i(P \otimes Q) = i(P) \cdot i(Q)$

Table 1: Basic ŁΠ Connectives

Operation	Interpretation
Ł-Negation	$i(\neg P) = 1 - i(P)$
Π-Delta	$\Delta(i(P)) = 1$ if $i(P)=1$; else $\Delta(i(P))=0$
Equivalence	$i(P \leftrightarrow Q) = i(P \to_L Q) \otimes i(Q \to_L P)$
$P \ominus Q$	$i(P \ominus Q) = \max(0, i(P) - i(Q))$

Table 2: Derived ŁΠ Connectives

Běhounek, et al., use the basic and derived connectives along the truth-constants and the following set of axioms [40]:

1) The Łukasiewicz set of axioms [21].
2) The product set of axioms [21].
3) The Łukasiewicz Delta axiom:

$$\Delta(P \to_L Q) \to_L (P \to_\Pi Q) \tag{12}$$

4) The Product Delta axiom:

$$\Delta(P \to_\Pi Q) \to_L (P \to_L Q) \tag{13}$$

5) The axiom:

$$R \otimes (P \ominus Q) \leftrightarrow_L (R \otimes P) \ominus (R \otimes Q) \tag{14}$$

The rules of inference are:

1) Modus ponens.
2) Product necessitation (infer ΔQ from P).

Reference [40] includes several theorems that follow from the definition of ŁΠ propositional fuzzy logic. In the next section, we define ŁΠ first order predicate fuzzy logic (ŁΠ∀).

3.2 First Order Predicate Fuzzy Logic

Following classical logic, ŁΠ first order predicate fuzzy Logic, referred to as ŁΠ∀, extends ŁΠ propositional fuzzy logic. The primitives include constants, variables, arbitrary arity functions and arbitrary arity predicates. Formulae are constructed using the basic connectives defined in table 1, derived connectives such as the connectives presented in table 2, the truth-constants, the quantifier ∀ and the identity sign =.The quantifier ∃ can be used to abbreviate formulae derived from the basic primitives and connectives. A fuzzy interpretation, of a proposition $P(x_1, ..., x_n)$

over a domain M is a mapping that assigns a fuzzy truth-value to each n-tuple of elements of M. As in the case of ŁΠ, we closely follow the system used in ref. [40].

Assuming that y can be substituted for x in P and x is not free in Q the following axioms are used.

1) Instances of the axioms of ŁΠ obtained through substitution.

2) Universal axiom I:

$$(\forall x)P(x) \to P(y) \tag{15}$$

3) Universal axiom II:

$$(\forall x)(P \to_L Q) \to (P \to_L (\forall x)Q) \tag{16}$$

4) Identity axiom I:

$$x = x \tag{17}$$

5) Identity axiom II:

$$(x = y) \to \Delta(P(x) \leftrightarrow P(Y)) \tag{18}$$

Modus ponens, product necessitation, and generalization are used for inference. In the next section, we define propositional and first order predicate CFL.

3.3 Propositional and First Order Predicate Complex Fuzzy Logic

As noted, the main motivation for developing CFL is twofold: First, we have observed that many multidimensional processes cannot be adequately represented as a combination of single dimension processes expressed via single dimension fuzzy logic. Second, we are interested in an axiomatic based formal system that can be used for a rigorous definition of complex fuzzy classes.

A complex fuzzy proposition P is a composition of two propositions each of which can accept a truth-value in the interval [0,1]. In other words; the interpretation of a complex fuzzy proposition is a pair of truth-values from the Cartesian interval [0,1] × [0,1]. Alternatively, the interpretation can be formulated as a mapping to the unit circle. Formally a fuzzy interpretation of a complex fuzzy proposition P is an assignment of fuzzy truth-value of the form $i(p_r) + j \cdot (p_i)$, or of the form $i(r(p))e^{j\sigma i(\theta(p))}$, where σ is a scaling factor in the interval (0,2Π], to P.

For example, consider a proposition of the form "x ... A ... B ...," along the definition of linguistic variables and constants. Namely, a linguistic variable is a variable whose domain of values is formal or natural language words [1]. Generally, a linguistic variable is related to a fuzzy set such as {*very young male, young male, old male, very old male*} and can get any value from the set. A linguistic constant has a fixed and unmodified linguistic value i.e., a single word or phrase from formal or natural language.

Thus, in a proposition of the form "x ... A ... B ...," where A and B are linguistic variables, i(p_r)(i(r(p))) can be assigned to the term A and i(p_i)(i(θ(p))) can be assigned to term B.

Propositional CFL extends the definition of propositional fuzzy logic and first order predicate CFL extends the notion of first order predicate fuzzy logic. Nevertheless, since propositional CFL is a special case of first order predicate CFL we only present the formalism for first order predicates CFL here. Reference [19] presents propositional CFL. Hence, this section augments ref. [19] with first order predicate CFL. It is the first work that provides this formalism.

Tables 3 and 4 present the basic and derived connectives of ŁΠ∀ CFL. In essence, the connectives are symmetric with respect to the real and imaginary parts of the predicates.

Operation	Interpretation
Ł-Implication	i(P$\rightarrow_L$ Q) $=\min(1,1-i(p_r)+i(q_r))+j\cdot\min(1,1-i(p_i)+i(q_i))$
Π-Implication	i(P$\rightarrow_\Pi$ Q) $=\min(0,i(p_r)/i(q_r))+j\cdot\min(0,1(p_i)/i(q_i))$
Π-Conjunction	i$(P\otimes Q)=i(p_r)\cdot i(q_r)+j\cdot(i(p_i)\cdot i(q_i))$

Table 3: Basic ŁΠ∀ CFL Connectives

Operation	Interpretation
Ł-Negation	i(¬P) = 1 + j1 - i(P)
Π-Delta	Δ(i(P)) = 1 if i(P)=1 + j1; else Δ(i(P))=0 + j0
Equivalence	$i(P\leftrightarrow Q)=i(P_r\rightarrow_L Q_r)\otimes i(Q_r\rightarrow_L P_r)+j\cdot(i(P_i\rightarrow_L Q_i)\otimes i(Q_i\rightarrow_L P_i))$
P⊖ Q	i(P ⊖ Q) $=\max(0,1(p_r)-i(q_r))+j\cdot\max(0,i(p_i)-i(q_i))$

Table 4: Derived ŁΠ∀ CFL Connectives

Following classical logic, ŁΠ∀ CFL extends, ŁΠ CFL. The primitives include constants, variables, arbitrary arity functions and arbitrary arity predicates. Formulae are constructed using the basic connectives defined in table 3, derived connectives such as the connectives presented in table 4, the truth-constants, the quantifier ∀ and the identity sign =. The quantifier ∃ can be used to abbreviate formulae derived from the basic primitives and connectives. A fuzzy interpretation of a proposition $P(x_1,...x_n)=P_r(x_1,...x_n)+j\cdot P_i(x_1,...x_m)$ over a domain M is a mapping that assigns a fuzzy truth-value to each (n-tuple) × (m-tuple) of elements of M. As in the case of ŁΠ fuzzy logic, we closely follow the system used in ref. [40].

The same axioms used for first order predicate fuzzy logic are used for first order predicate complex fuzzy logic, and Modus ponens as well as product necessitation, and generalization are the rules of inference.

Complex Fuzzy Propositions and Connectives Examples.

One form of a fuzzy proposition is: "x ... A ...;" where A is a linguistic variable such as "young male" or "tall person" and '...' denote natural language constants such as "Moses." "is," "portfolio," "mutual fund," etc. For example, under one interpretation, the fuzzy truth-value associated with the fuzzy proposition: P = "Moses is *young male*" can be 0.3, and under another interpretation, the fuzzy truth-value associated with the proposition P can be 0.9. In this case, the linguistic variable is "young male," and it is distinguished from the Fuzzy constants "Moses" and "is" by its Italics font.

One form of a complex fuzzy is: "x ... A ... B ..." where A and B are values assigned to linguistic variables and '...' denotes natural language constants. For example, under one interpretation, the complex fuzzy truth-value associated with the complex fuzzy proposition: "x is a *volatile stock in a strong-portfolio*," can be 0.1 + j0.5. Alternatively, in another context, the same proposition can be interpreted as having the complex truth-value $0.3e^{j0.2}$. Consider the following propositions(P and Q respectively):

1) "x is a *volatile stock in a strong-portfolio.*"

2) "x is a stock in a *decline-trend in a strong-portfolio*"

Hence, P is of the form: "x is a A in a B" and Q is of the form "x is a stock C in B" In this case, "*volatile stock*," "*a strong-portfolio*," and "*a decline-trend*" are values assigned to the linguistic variables {A,B,C}. Assume that the complex fuzzy interpretation (i.e., degree of confidence or complex fuzzy truth-value) of P is $i(p_r)+j\cdot(p_i)$, while the complex fuzzy interpretation of Q is $i(q_r)+j\cdot(p_i)$. Thus, the truth-value of "*x is a volatile stock*" is $i(p_r)$, the truth-value of "*x is in a strong-portfolio*" is $i(p_i)$, and the truth-value of "*x is in a decline-trend*" is $i(q_r)$ Suppose that the term "*non*-volatile" stands for "*not volatile*" the term "*weak*" stands for "*not strong*," and the term "*rising*" stands for "*not declining*," Then, the complex fuzzy interpretation of the following composite propositions is:

1) i(¬P)= $(1 - i(p_r)) + j \cdot (1 - i(p_i))$.

That is, $\neg P$ denotes the proposition "x is a non-volatile stock in a weak portfolio." The confidence level in $\neg P$ is $(1 - i(p_r)) + j(1 - i(p_i))$; where the fuzzy truth-value of the term "x is a non-volatile stock" is $(1 - i(p_r))$ and the fuzzy truth-value of the term "weak portfolio" is $(1 - i(p_i))$.

2) i(¬P $\rightarrow_L$ ¬Q) $= min(1, 1 + i(p_r) - i(q_r)) + j \cdot min(1, 1 + i(p_i) - i(q_i))$.

Thus, (¬P$\rightarrow_L$ ¬Q) denotes the proposition "If x is a non-volatile stock in a weak portfolio, THEN x is a stock in a rising-trend in a strong-portfolio." The truth-values of individual terms, as well as the truth-value of ¬P$\rightarrow_L$ ¬Q are calculated according to table 3.

3) i(P ∨ Q) $= max(i(p_r) \cdot i(q_r)) + j \cdot max(i(p_i) \cdot i(q_i))$.

That is, (P∨¬Q) denotes the proposition "x is a volatile stock in a strong-portfolio" OR "x is a stock in a rising-trend in a week portfolio." The truth-values of individual terms, as well as the truth-value of P∨¬Q are calculated according to tables 3 and 4.

4) i(¬P ⊗ Q) $= min(1 - i(p_r), i(q_r)) + j \cdot min(1 - i(p_i), i(q_i))$.

That is, (¬P ⊗ Q) denotes the proposition "x is a volatile stock in a strong-portfolio" AND "x is a stock in a rising-trend in a strong-portfolio." The truth-values of individual terms, as well as the truth-value of ¬P ⊗ Q are calculated according to table 3.

Complex Fuzzy Inference Example.

Assume that the degree of confidence in the proposition R = ¬P (P is defined above) is $i(r_r) + j \cdot (r_i)$ let S = ¬Q and assume that the degree of confidence in the fuzzy implication T = R $\rightarrow_L$S is i(t_r) + j · (t_i). Then, using Modus ponens:

R

$\underline{R \rightarrow S}$

S

one can infer S with a degree of confidence min(1,i(r_r) · i(t_r)) + j · min(1,i(r_i) · i(t_i)).

In other words one using:

"x is a *non-volatile stock* in a *weak portfolio.*"

IF

"x is a *non-volatile* stock in a weak portfolio," THEN

"x is a stock in a *rising-trend* in a *weak-portfolio.*"

"x is a stock in a *rising-trend* in a *weak-portfolio*"

then, one can infer "x is a stock in a *rising-trend* in a *weak-portfolio*" with a degree of confidence min(1,i(r_r) · i(t_r)) + j · min(1,i(r_i) · i(t_i)).

4 Complex Fuzzy Class Theory

Many natural phenomena are complex and cannot be modeled using one dimensional classes and / or one dimensional variables. For example, in pattern recognition, objects can be represented by a set of measurements and are considered as vectors in a multidimensional space. These patterns might be clustered into several clusters where different clusters relate to different subsets of features. Often, it is not practical to assume that this multidimensional information can be represented via a one-dimensional combination of variables and operators. As a relatively simple example consider the stock market; at any given time each stock can be evaluated by its current value and trend. Each of these parameters can be represented by a fuzzy set and inference can be carried by fuzzy logic. Nevertheless, ignoring the current state of the entire market and many other parameters provides a very limited capability to make sound decisions. Moreover, often the relations between the current value of the stock and the current market trends cannot be adequately represented via simple fuzzy logic operations over individual parameters. Other examples for the need in a mechanism that can represent complex objects and complex sets occur in signal processing where many signals are represented using complex variables. In a noisy environment these signals might form a “complex fuzzy class." Finally, consider a set of values where each value is a member of a fuzzy set. This set referred to as fuzzy set of type-2 cannot be represented by basic operations on fuzzy sets of type-1 [11, 12]. This type of sets however, can be represented via complex classes. Motivated by these considerations Ramon et al., and Dick propose complex fuzzy sets. The limitation of their approach is described in section 2. Tamir et al. introduce a new interpretation of complex fuzzy grade of membership and derive the concept of complex fuzzy classes using the framework of a complex fuzzy grade of membership [18]. This form is limited since it cannot be easily axiomatized and rigorously formalized. In this paper we provide an alternative formulation for complex fuzzy classes which is based on first order predicate complex fuzzy logic (ŁΠ∀ CFL). Based on the logic system presented in section 3.1, the next section introduces the axiomatic based formulation of fuzzy class theory (FCT) as developed in ref. [40]; section 4.2 provides the extension of FCT to complex fuzzy class theory (CFCT).

4.1 Axiomatic Based Fuzzy Class Theory

The axiomatic fuzzy logic can serve as a basis for establishing an axiomatic FCT. Similarly, axiomatic based complex fuzzy logic can serve as the basis for formal definition of complex fuzzy classes. Several variants of FCT exists, most of them use a similar approach and mainly differ in the selection of the logic base. Another difference between various approaches is the selection of class theory axioms. Běhounek, et al., present and analyze a few variants of FCT. In ref. [40] they present a ŁΠ∀ based FCT. Their formalism serves as the starting point for the complex FCT presented in this paper. Next, we provide an overview of the ŁΠ∀ based FCT.

The main components of FCT are:

1) Variables.

a. Variables denoting objects;

b. Variables denoting crisp sets, i.e., a universe of discourse and its subsets;

c. Variables denoting fuzzy classes of order 1;

d. Variables denoting fuzzy classes of order n, that is, fuzzy classes of fuzzy classes of order n-1.

We use the following notation for variables: objects are denoted by lower case Latin letters, crisp sets are denoted by upper case Latin letters, fuzzy classes of order n are denoted by upper case Greek letter of the form Γ^n (n is omitted when n=1).

2) The ŁΠ∀ fuzzy logic system along with its variables, connectives, predicates, and axioms defined in section 3.1 and ref. [40].

3) Additional predicates.

a. A binary predicate ‘x $\in \Gamma$’ denoting membership of objects in fuzzy classes.

4) Additional Axioms.

a. Instances of the comprehension schema (further explained below)

$$(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow p(X)) \tag{19}$$

b. The axiom of extensionality

$$(\forall x)\Delta(x \in \Gamma \leftrightarrow x \in \Psi) \rightarrow \Gamma = \Psi \tag{20}$$

Note that a grade of membership is not a part of the above specified terms; yet it can be derived or defined using these terms.

The comprehension schema is used to "construct" classes. It has the basic form of: $(\forall X)(x \in \Gamma \leftrightarrow P(x))$. Intuitively, this schema refers to the class Γ of all the objects x that satisfy the predicate P(). Instances of this scheme have the generic form: $(\exists\Gamma)(\forall\ \mathrm{x})(\mathrm{x} \in \Gamma \leftrightarrow \mathrm{P(x)})$. Associated with this schema are comprehension terms of the form: $\mathrm{y} \in \{x|P(x) \leftrightarrow \mathrm{P(y)}\}$. The Δ operation introduced in equation 20, is used to produce precise instances of the extensionality schema and ensure the conservatism of comprehension terms.

Fixing a standard model over the FCT, enables the definition of commonly used terms, set operations, and definitions, as well as proving FCT theorems [45]. Some of these elements are listed here:

1) The characteristic function $\chi_{x\in\Gamma} \equiv \chi_\Gamma$ and the grade of membership function $\mu_{x\in\Gamma} \equiv \mu_\Gamma$.

2) Class constants, α-cuts, iterated complements, and primitive binary operations such as union and intersection etc. These operations are constructed using the schema $O_p(\Gamma) \equiv \{x|P(x \in \Gamma)\}$. Table 5 lists some of these elements.

3) Uniform and supreme relations defined in ref. [40] enable the definition of fuzzy class relations such as inclusion.

4) Theorems on axioms, primitive fuzzy class operations, and fuzzy class relations [40].

Term	Symbol	P	Comments
Empty Class	Θ	0	
Universal Class	Φ	1	
Strict Complement	$\backslash\Gamma$	$\sim$	$\sim$ stands for Gödel(G) negation
Class Intersection	$\cap$	$\oplus$	$\oplus$ stands for a G, Ł, or Π conjunction T-norm
Class Union	$\cup$	$\vee$	$\vee$ stands for a G, Ł, or Π disjunction

Table 5: Primitive Fuzzy Class Operations

4.2 Axiomatic based Complex Fuzzy Class Theory

The axiomatic fuzzy logic can serve as a basis for establishing an axiomatic FCT. Similarly, axiomatic based complex fuzzy logic can serve as the basis for formal definition of complex fuzzy classes. In this section we provide a formulation of complex fuzzy class theory (CFCT) that is based on the logic theory presented in section 3.2. This is the first work that provides this formalism.

A new concept that is introduced in this section is the concept of complex objects. Intuitively this may sound odd since we are used to think about set objects as one-dimensional or dimensionless. Nevertheless, in reality, most of the objects of interest in many applications and research projects are represented by a set of measurements and are considered as multidimensional. In the context of this paper, a multi-dimensional object is an object that is associated with (or perceived by) more than one measurement or stimuli. A simple example is human color perception which associates the "qualities" of hue and saturation with many colors. A complex object is represented by two measurements or stimuli also referred to as features.

The main components of FCT are:

1) Variables.

a. Variables denoting objects (potentially complex objects);

b. Variables denoting crisp sets, i.e., a universe of discourse and its subsets;

c. Variables denoting complex fuzzy classes of order 1;

d. Variables denoting complex fuzzy classes of order n, that is, complex fuzzy classes of complex fuzzy classes of order n-1.

3) Additional predicates.

a. A binary predicate ‘x $\in \Gamma$’ denoting membership of objects in complex fuzzy classes.

To elaborate, assume that Γ is a complex class, and let x be a complex object of the form $x = x_r + j \cdot x_i$, then the predicate x $\in \Gamma$ yields a complex truth-level of the form x $\in \Gamma =$ $[x_r \in \Gamma + j \cdot x_i \in \Gamma]$. Intuitively this truth-value represents fuzzy validity of the compounded assertion “$[x_r, x_i]$ belongs to Γ," or “$[x_r, x_i]$ is a member of Γ”.

4) Additional Axioms.

a. Instances of the comprehension schema (further explained below)

$$(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow P(x)) \tag{21}$$

Where x is a complex fuzzy object, Γ is a complex fuzzy class, and P() is a complex fuzzy predicate.

b. The axiom of extensionality

$$(\forall x)\Delta(x \in \Gamma \leftrightarrow x \in \Psi) \to \Gamma = \Psi \tag{22}$$

Again, x is a complex fuzzy object, Γ is a complex fuzzy class, and P() is a complex fuzzy predicate.

Note that a grade of membership is not a part of the above specified terms; yet it can be derived or defined using these terms.

The comprehension schema is used to ”construct" classes. It has the basic form of: $(\forall x)(x \in \Gamma \leftrightarrow P(x))$. Intuitively, this schema refers to the class Γ of all the objects x that satisfy the predicate P(). Instances of this scheme have the generic form: $(\exists\Gamma)(\forall x)(x \in \Gamma \leftrightarrow P(x))$. Associated with this schema are comprehension terms of the form: $y \in \{x|P(x) \leftrightarrow P(y)\}$. The Δ operation introduced in equation 22, is used to produce precise instances of the extensionality schema and ensure the conservatism of comprehension terms.

Fixing a standard model over the CFCT enables the definition of commonly used terms, set operations, and definitions, as well as proving CFCT theorems. Some of these elements are listed here:

1) The complex characteristic function $\chi_{x\in\Gamma} \equiv \chi_\Gamma$ and the complex grade of membership function $\mu_{x\in\Gamma} \equiv \mu_\Gamma$.

2) Complex class constants, α-cuts, iterated complements, and primitive binary operations such as union and intersection etc. These operations are constructed using the schema $O_p(\Gamma) \equiv \{x|P(x \in \Gamma)\}$. Table 6 lists some of these elements.

3) Uniform and supreme relations defined in ref. [40] enable the definition of fuzzy class relations such as inclusion.

4) Theorems on axioms, primitive fuzzy class operations, and fuzzy class relations [40].

Following the axiomatic based definition of grade of membership equations (9-11) can be used as a basis for the definition of “membership-grade based” complement, union, and intersection.

We are currently performing research aimed to provide a comprehensive list of theorems, complex class operations, and relations.

Complex Fuzzy Propositions and Connectives Examples.

In order to provide a concrete example we define the following complex fuzzy classes using the comprehension schema. Let the universe of discourse be the set of all the stocks that were

Term	Symbol	P	Comments
Empty Complex Class	Θ	0	
Universal Complex Class	Φ	1	
Strict Complement	$\backslash\Gamma$	$\sim$	$\sim$ stands for Gödel(G) negation
Complex Class Intersection	$\cap$	$\oplus$	$\oplus$ stands for a G, Ł, or Π conjunction T-norm
Complex Class Union	$\cup$	$\vee$	$\vee$ stands for a G, Ł, or Π disjunction

Table 6: Derived Primitive Class Operations

available for trade on the opening of the New York stock exchange (NYSE) market on January 3, 2011 along with a set of attributes related to historical price performance of each of these stocks. Consider the following complex propositions:

1) P(x)≡ "x is a volatile stock in a strong-portfolio,"

2) Q(X)≡ "x is a stock in a decline-trend in a strong-portfolio,"

Then, the proposition: $(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow (P(x) \otimes Q(x))$ where x is any member of the universe of discourse defines a complex fuzzy class Γ that can be "described" as the class of "*volatile stocks in a decline-trend in strong-portfolios*," On the other hand, the proposition: $(\exists\Gamma)\Delta(\forall x)(x \in \Gamma \leftrightarrow (\neg P(x) \vee Q(x))$ where x is any member of the universe of discourse, defines a complex fuzzy class Γ that can be "described" as the class of "*non-volatile stocks in a decline-trend in strong-portfolios.*"

5 Conclusions and Directions for Further Research

A new and innovative formal definition of complex fuzzy logic (CFL), referred to as Ł$\Pi\forall$ CFL as well as a formalism of complex fuzzy class theory (CFCT) that is based on CFL, is presented in this paper. The new form significantly improves the expressive power and inference capability of CFL and CFCT based systems. In addition, it enables axiomatic and rigorous development of the relevant theory. In the future, we plan to extend the theory. In addition, we plan to apply the new formalism to multidimensional fuzzy propositional and predicate logic; and further explore its potential for usage in advanced complex fuzzy logic systems as well as inference with type-2 (or higher) fuzzy sets.

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