INTERNATIONAL JOURNAL OF COMPUTERS COMMUNICATIONS & CONTROL
ISSN 1841-9836, 12(1):26-40, February 2017.

A Similarity Measure-based Optimization Model for Group Decision Making with Multiplicative and Fuzzy Preference Relations

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Abstract: Group decision making (GDM) problem based on different preference relations aims to obtain a collective opinion based on various preference structures provided by a group of decision makers (DMs) or experts, those who have varying backgrounds and interests in real world. The decision process in proposed question includes three steps: integrating varying preference structures, reaching consensus opinion, selecting the best alternative. Two major approaches: preference transformation and optimization methods have been developed to deal with the issue in first step. However, the transformation processes causes information lose and existing optimization methods are so computationally complex that it is not easy to be used by management practice. This study proposes a new consistency-based method to integrate multiplicative and fuzzy preference relations, which is based on a cosine similarity measure to derive a collective priority vector. The basic idea is that a collective priority vector should be as similar per column as possible to a pairwise comparative matrix (PCM) in order to assure the group preference has highest consistency for each decision makers. The model is computationally simple, because it can be solved using a Lagrangian approach and obtain a collective priority vector following four simple steps. The proposed method can further used to derive priority vector of fuzzy AHP. Using three illustrative examples, the effectiveness and simpleness of the proposed model is demonstrated by comparison with other methods. The results show that the proposed model achieves the largest cosine values in all three examples, indicating the solution is the nearest theoretical perfectly consistent opinion for each decision makers.
Keywords:group decision making; multiplicative preference relations; fuzzy preference relations; similarity measure; optimization model.

1 Introduction

Group decision making (GDM) aims to obtain a solution alternative(s) to a given question based on the opinions provided by a set of experts. When comparing alternatives in real word problem, experts may use any of the following preference structures: preference orderings, utility functions, multiplicative preference relations, and fuzzy preference relations (Herrera et al.[1], Herrera-Viedma et al.[2]). The process is composed of the three parts. 1) Integrating the different preference structures provided by varying decision makers [1,2,3,4,5,6,7,8,9]. 2) Reaching consensus based on modified the preference information when the collective opinion cannot be accepted by non-cooperative decision makers [10,11,12,13,14]. 3) Selecting the best alternative from the collective opinion.

In first step of previous question, two main groups of techniques have been developed for integrate different preference relations in GDM problem. The first class of techniques (Chiclana et al.[3,5]; Delgado et al.[6]; Herrera et al.[1]; Herrera-Viedma[2]) transforms different preference structures into uniform formats, after which a selection operator is implemented to rank alternatives on the basis of a fuzzy majority. The second class of techniques (Fan et al.[7]; Ma et al.[8]; Xu et al.[9]) uses optimization models to directly obtain a collective priority vector instead of using transform functions to obtain uniform preference information. The limitations of these two classes of approaches are: 1) the transformation functions may cause information loss during the conversions, and 2) existing optimization methods are so complex to obtain solution that it is not easy to be used to management practice.

The goal of this study is to propose a cosine similarity measure-based optimization method for deriving a collective priority vector in decision making problem with multiplicative preference relations and fuzzy preference relations. The basic idea of the model is that the collective priority vector should be as similar per column as possible to a pairwise comparative matrix (PCM) from each decision maker so that the solution is the nearest theoretical perfectly consistent opinion for each decision makers. The proposed model can be implemented in four steps with simple computation for integrating multiplicative preference relations and fuzzy preference relations. Moreover, the proposed model can also be used to derive priority vector from PCM in fuzzy AHP. Furthermore, the model can be extended to other preference structures by transforming other preference formats into a mixture of multiplicative preference relations and fuzzy preference relations, and used the consensus reaching methods by iterative modification based on our proposed priority vector obtaining method to improve consensus degree (such as Chen et al.[15];Wu and Xu[16]; Dong et al[17]; Xu et al[18,19], Yu et al.[20], etc.).

The remainder of this study is organized as follows. Section 2 reviews related work and provides some preliminaries. Section 3 describes the proposed cosine similarity measure-based optimization model. Section 4 uses three numeric examples to illustrate the implementation of the proposed model and compare with some well-known methods. Section 5 concludes the study.

2 Related work and preliminaries

2.1 Related work

The group decision making problem with different preference structures, which was also called multi-person decision making in their papers, has a long research history and been defined as a decision-making process by different DMs or experts with different decision information presented in terms of utility functions, preference orderings, or preference relations with multiplicative or fuzzy formats (Arrow [21]; Sen[22]; Kickert et al.[23]; Saaty[24]; Kacprzyk and Fedrizzi [25]; Chiclana et al.[3,5]; Herrera et al.[1], etc.). A utility function comprises real numbers for alternatives to represent the utility evaluations provided by DMs. A preference ordering is a ranking of alternatives in best-to-worst order, with no additional information. A preference relation is based on pairwise comparison. It is a simple process in which experts provide ratios that compare one alternative to another. For a multiplicative preference relation, the ratio can be a real number in the set $\{1/9, 1/8, ..., 1/2, 1, 2,...,8 ,9\}$. For a fuzzy preference relation, the ratio is provided as a fuzzy number.

An important issue in this problem is the derivation of a collective opinion on the basis of different preference formats provided by each decision maker (Chiclana et al.[3,5]; Delgado et al.[6]; Herrera et al.[1]; Herrera-Viedma[2]). Various approaches have been developed to solve above problems in following steps: (1) convert different preference information into uniform structures, (2) aggregate uniform preference structures using selection operators, and (3) rank the alternatives and make a decision.

Delgado et al. [6] developed a fusion operator for integrating numerical and linguistic information using transformation functions between the numerical and linguistic values. Chiclana et al. [3] proposed a general model for integrating different preference formats using fuzzy preference relations. They aggregate uniform preference information using an ordered weighted averaging (OWA) geometric operator with a fuzzy quantifier. Chiclana et al.[5] discussed the properties of different preference formats and multiplicative preference relations and proposed transformation functions that can convert varying preference information into uniform representations and then used an OWA geometric operator to rank alternatives. The operator, which includes two degree choices, is a multi-criteria decision-making method (MCDM). The transformation functions proposed in Chiclana et al.[26] have been used in many studies as a benchmark for comparing different methods. Herrera et al.[1] defined a transformation function between the multiplicative and fuzzy preference relations. They also adopted fuzzy preference relations as a uniform preference format. The OWA geometric operator was also used to rank alternatives. Chiclana et al. [26] developed a fuzzy MCDM model to derive a collective opinion when preferences were presented using preference orderings, utility functions, and fuzzy preference relations and studied the internal consistency of the model. Herrera-Viedma et al. [2] proposed a consensus model for varying preference information in GDM questions. Their model can automatically obtain a consensus. Xu [27] studied different interval preference relations and proposed a model for deriving overall weights in multi-attribute decision making.

Another class of methods for obtaining a collective opinion is optimization-based. One strength of this kind of approach is that it can avoid information loss during the conversion of preference relations because there is no need to transform different preference formats into a uniform structure. Fan et al. [7] constructed a goal-programming model to reduce differences in collective opinions and each decision maker's PCMs, thereby demonstrating that the collective opinion must be close to perfectly consistent. Ma et al. [8] proposed an optimization model for integrating four different preference formats without unifying them into one structure. They analyzed perfectly consistent conditions in four preference relations and constructed an objective function on the basis of the idea that the derived priority vector has the least distance from each decision maker's preference. Xu et al.[9] introduced a nonlinear programming method for handling decision makers coming from different areas and providing varying preference information, including utility values, preference orderings, and multiplicative, incomplete multiplicative, fuzzy, and incomplete fuzzy preference relations. Wang et al. [28] proposed a chi-square model to derive priority methods wherein the goal is to develop an optimal priority vector as close to each decision maker's opinion as possible.

2.2 Preliminaries

The GDM problem with different preference formats can be divided into two stages; first aggregating the varying preference information and determining a collective group preference and then selecting a ranking of alternatives using a previous collective group preference.

Assume that $\Omega = \{A_1, A_2, ..., A_n\}$ is a finite set containing n alternatives and $\Pi = \{DM_1, DM_2, ..., DM_n\}$ is a finite set including K decision makers.$\Lambda = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ is a finite set containing the degrees of importance of decision makers. In general, $\sum_{i=1}^{n} \sigma_i = 1$. with each σ_i being prespecified by decision makers. The collective group preference $w = \{w_1, w_2, ..., w_n\}$ indicates the degree of importance and best-to-worst selection order ranking the alternatives. The preference relations can be represented in different formats with orderings, utility values, and multiplicative and fuzzy preference relations. Herein, we discuss the aggregation of multiplicative and fuzzy preference relations, and other preferences can be transformed into these two types of preference relations (Chiclana et al.[26]).

Multiplicative preference relations are the most widely used in decision-making situations. A multiplicative preference relation is essentially a binary relation $f = \Omega \times \Omega \to \{1/9, 1/8, \cdots, 8, 9\}$ represented as a PCM matrix $(a_{ij})_{n\times n}$, i.e. $f(A_i, A_j) \mapsto a_{ij}$.The entry $a_{ij}$of the PCM denotes the relative degree of importance of alternative A_i with respect toA_j. A_i and $A_j$are equally important if $a_{ij} = 1$, and a higher a_{ij} value indicates a greater weight for the alternative.When $a_{ij} = 9$, A_i is preferred absolutely to A_j, and A_i is inferior to every A_j if $a_{ij} = 1/9$.It is evident that the PCM of a multiplicative preference relation is positive reciprocal, i.e. $a_{ij} \times a_{ji} = 1$ for arbitrary $i, j \in \{1, 2, \cdots, n\}$. A PCM is called perfectly consistent if $a_{ij}a_{jk} = a_{ik}$for arbitrary $i, j, k \in \{1, 2, \cdots, n\}$.In practice, the PCM is not always perfectly consistent. Saaty [24][29] proposed the consistency ratio index, which must be less than 0.1 when consistency can be accepted. Aguarón et al. [30] and Escobar et al. [31] proposed a geometric consistency index for AHP. Many other methods for measuring consistency have also been proposed Ergu et al. [32]; Lin et al.[33,34]; Lin and Kou[35].etc). If consistency is present, there are many derivation methods for priority vectors, including the eigenvector method (Saaty [36]), weighted least squares method (Chu, Kalaba, and Spingarn [37]), logarithmic least squares method (Crawford and Williams [38]), a heuristic approach (Lin et al.[39]), and the cosine maximization method (Kou and Lin[40], etc). Moreover, Gomez-Ruiz [41] proposed a estimation method using neural network.

Fuzzy preference relations have been extensively studied (Tanino [42]; Chiclana et al.[3]; Herrera-Viedma et al.[43]) and have many practical applications in the decision-making process owing to the difficulty of comparing two alternatives using a crisp real number. A fuzzy preference relation is a binary fuzzy membership function $g : \Omega \times \Omega \to [0, 1]$, i.e., $g(A_i, A_j) \mapsto p_{ij}$, where p_{ij} is the fuzzy degree of importance of A_i with respect to A_j. The fuzzy pairwise comparative matrix $(p_{ij})_{n\times n}$ is constructed as a real number from 0 to 1. if $p_{ij} = 0.5$, A_i has the same preference as those of A_j, $p_{ij} = 1$ and $p_{ji} = 0$ indicates that A_i is unanimously preferred to A_j. $p_{ij} \in (0.5, 1)$ and $p_{ji} \in (0, 0.5)$ indicate that A_i is preferred over A_j. There is an additive reciprocal relation in a fuzzy pairwise comparative matrix $(p_{ij})_{n\times n}$, i.e., $p_{ij} + p_{ji} = 1$ for arbitrary $i, j \in \{1, 2, \cdots, n\}$. The definition of perfectly consistent for a fuzzy pair-wise comparative matrix is different from that for a multiplicative preference relation: $(p_{ij})_{n\times n}$ is considered to be perfectly consistent, if $p_{ij}p_{jk}p_{ki} = p_{ji}p_{kj}p_{ik}$ for arbitrary $i, j, k \in \{1, 2, \cdots, n\}$,

Herrera-Viedma et al.[43]summarized eight consistency properties of fuzzy preference relations and proposed a method for testing and improving the consistency of a fuzzy preference relation. Xu et al. [18] studied the ordinal consistency of a fuzzy preference relation in terms of three cycles in a directed graph. Zhang et al.[44] discussed consistency issues in group decision making with fuzzy preference relations. Mikhailov L. [45] introduced linear fuzzy preference programming for deriving a priority vector for a fuzzy preference relation. Xu and Da [46] proposed a least-deviation method for obtaining a priority vector of a fuzzy preference relation. Wang et al.[28] used a chi-square method for deriving a priority vector on the basis of fuzzy and multiplicative preference relations. Other methods for priority derivation of fuzzy AHP can be found in Kacprzyk[47], Roubens[48], and Chiclana et al.[3].

A similarity index in mathematics is used to measure the similarity of two vectors. Many similarity measurement indices, including similarity coefficient and distance function methods, have been introduced and employed in many domains, such as machine learning and decision science. The most widely used similarity measure is the cosine similarity index. For two vectors $\vec{r}_i = \{r_{i1}, r_{i2}, \cdots, r_{in}\}$ and $\vec{r}_j = \{r_{j1}, r_{j2}, \cdots, r_{jn}\}$ the cosine similarity measure is a binary function defined as

$$\langle \vec{r_i}, \vec{r_j} \rangle = \frac{\vec{r_i} \cdot \vec{r_j}}{\|\vec{r_i}\| \|\vec{r_j}\|} = \frac{\sum_{k=1}^{n} r_{ik} r_{jk}}{\sqrt{\sum_{k=1}^{n} r_{ik}^2} \sqrt{\sum_{k=1}^{n} r_{jk}^2}} \in [0,1] \tag{1}$$

Kou and Lin [40] used this similarity measure for AHP and constructed a cosine maximization model to derive a priority vector in AHP. The idea is that the derived priority vector is to be most similar to each column of a PCM according to the cosine similarity measure.

Inspired by the work of Kou and Lin [40], this study extends the similarity measure method to group decision-making by applying the cosine similarity measure to the GDM problem with multiplicative and fuzzy preference relations.

3 Cosine similarity measure maximization model for GDM problem with multiplicative and fuzzy preference relations

Assume there are two classes of preference structures: multiplicative preference relations and fuzzy preference relations. let $(a_{ij}^{(k)})_{n\times n}$, $k = 1, 2, \cdots, k_m$ and $(p_{ij}^{(k)})_{n\times n}$, $k = k_{m+1}, k_{m+2}, \cdots, K$ be multiplicative and fuzzy preference relations, respectively.

Existing results show that the PCM is perfectly consistent when it satisfies the following conditions:

$$a_{ij}^{(k)} = \frac{w_i^k}{w_j^k}, i = 1, 2, \cdots, n; j = 1, 2, \cdots, n. \tag{2}$$

$$p_{ij}^{(k)} = \frac{w_i^k}{w_i^k + w_j^k}, i = 1, 2, \cdots, n; j = 1, 2, \cdots, n. \tag{3}$$

Let $\vec{a_j}^{(k)} = (a_{ij}), a_{2j}, \cdots, a_{nj})^T$ be the column vector of $(a_{ij}^{(k)})$ and $\vec{w} = (w_1, w_2, \cdots, w_n)^T$. Kou and Lin [40] proved that the existing similarity relation between each column of a PCM and a derived priority vector-particularly, the cosine similarity measure-is equal to one if and only if the PCM is perfectly consistent.

The higher the degree of consensus, the closer the cosine similarity measure is to 1. That is

$$\langle \vec{a_j}^{(k)}, \vec{w_j} \rangle = \frac{\sum_{k=1}^{n} a_{kj} w_k}{\sqrt{\sum_{k=1}^{n} a_{kj}^2} \sqrt{\sum_{k=1}^{n} w_k^2}} = \frac{\sum_{k=1}^{n} \frac{w_k}{w_j} w_k}{\sqrt{\sum_{k=1}^{n} \frac{w_k}{w_j}^2} \sqrt{\sum_{k=1}^{n} w_k^2}} = \frac{\sum_{k=1}^{n} w_k^2}{\sqrt{\sum_{k=1}^{n} w_k^2} \sqrt{\sum_{k=1}^{n} w_k^2}} = 1 \tag{4}$$

For fuzzy preference relations, we can show that there exists a similarity relation between a derived priority vector and each column of the PCM after the transformation:

$$b_{ij}^{(k)} = \frac{p_{ij}^{(k)}}{1 - p_{ij}^{(k)}} \tag{5}$$

Let $\vec{b_j} = (b_{1j}, b_{2j}, \cdots, b_{nj})^T$,$j = 1, 2, \cdots, n$. When the PCM is perfectly consistent, i.e., $b_{ij} = \frac{w_i}{w_j}$, the following relation can be obtained:

$$\langle \vec{b_j}^{(k)}, \vec{w_j} \rangle = \frac{\sum_{k=1}^{n} b_{kj} w_k}{\sqrt{\sum_{k=1}^{n} b_{kj}^2} \sqrt{\sum_{k=1}^{n} w_k^2}} = \frac{\sum_{k=1}^{n} \frac{p_{kj}}{1-p_{kj}} w_k}{\sqrt{\sum_{k=1}^{n} \frac{p_{kj}}{1-p_{kj}}^2} \sqrt{\sum_{k=1}^{n} w_k^2}} = \frac{\sum_{k=1}^{n} \frac{w_k}{w_j} w_k}{\sqrt{\sum_{k=1}^{n} \frac{w_k}{w_j}^2} \sqrt{\sum_{k=1}^{n} w_k^2}} = 1 \tag{6}$$

Therefore, it is evident that the cosine similarity measure is equal to 1 if the multiplicative and fuzzy preference relations of decision makers are perfectly consistent. If an entry in a PCM is 1 or 0 when a preference relation is fuzzy, one alternative is unanimously preferred over the others. In this case,$b_{ij}$is 0 or does not exist. Now, b_{ij} is defined as the limit of p_{ij}close to one. If we assume, without loss of generality, that $p_{kj} = 1$, then

$$\langle \vec{b_j}^{(k)}, \vec{w_j} \rangle = \frac{b_{1j}w_1 + b_{ij}w_2 + \cdots + b_{kj}w_k + \cdots + b_{nj}w_n}{\sqrt{b_{1j}^2 + b_{2j}^2 + \cdots + b_{kj}^2 + \cdots + b_{nj}^2}\sqrt{\sum_i w_i^2}} = \frac{\frac{b_{1j}}{b_{kj}}w_1 + \cdots + w_k + \cdots + \frac{b_{nj}}{b_{kj}}w_n}{\sqrt{\frac{b_{1j}^2}{b_{kj}^2} + \cdots + 1 + \cdots + \frac{b_{nj}^2}{b_{kj}^2}}\sqrt{\sum_i w_i^2}} \tag{7}$$

Since $b_{ij} = \lim_{x \to 1^-} \frac{p_{ij}}{1-P_{ij}} \to +\infty$, the cosine similarity measure yields

$$\lim_{b_{kj} \to +\infty} \langle \vec{b_j}^{(k)}, \vec{w_j} \rangle = \lim_{b_{kj} \to +\infty} \frac{\frac{b_{1j}}{b_{kj}}w_1 + \cdots + w_k + \cdots + \frac{b_{nj}}{b_{kj}}w_n}{\sqrt{\frac{b_{1j}^2}{b_{kj}^2} + \cdots + 1 + \cdots + \frac{b_{nj}^2}{b_{kj}^2}}\sqrt{\sum_i w_i^2}} = \frac{w_k}{\sqrt{\sum_i w_i^2}} \tag{8}$$

Letting $\lim_{b_{kj} \to +\infty} \langle \vec{b_j}^{(k)}, \vec{w_j} \rangle = 1$, we obtain $w_k = 1$ and $w_i = 0$, $i = 1, 2, \cdots, k-1, k+1, \cdots, n$, which is nonsensical in terms of economic and management theories and practices. This shows that the PCM is not perfectly consistent in this case. Therefore, in practice, we assume that $p_{ij}^{(k)} \approx 1$(e.g.$p_{ij}^k = 0.9999$) and $p_{ij}^{(k)} \approx 0$(e.g.$p_{ij}^k = 0.0001$)when DMs apply the cosine similarity measure to GDM problem. Including one in each column and row of a fuzzy PCM directly yields a preference ordering from the fuzzy PCM.

In the GDM problem, the derived collective priority vector should be most largely consistent for each decision makers. Therefore, the group preference should have the highest similarity measure between the derived collective priority and each column of decision makers' PCMs. Inspired by this idea, we construct a cosine similarity measure maximization optimization model as follows to optimize the maximization similarity measure:

$$\begin{aligned} &\max\ C = \sum_{k=1}^{K}\sum_{j=1}^{n} \sigma_k C_j^{(k)} = \sum_{k=1}^{k_m}\sum_{j=1}^{n} \sigma_k \frac{\vec{w}\vec{a_j}^{(k)}}{\|\vec{w}\|\|\vec{a_j}^{(k)}\|} + \sum_{k=k_{m+1}}^{K}\sum_{j=1}^{n} \sigma_k \frac{\vec{w}\vec{b_j}^{(k)}}{\|\vec{w}\|\|\vec{b_j}^{(k)}\|} \\ &s.t. \quad \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \qquad i = 1, 2, \cdots, n \end{cases} \end{aligned} \tag{9}$$

Where C is the total similarity measure and $C^{(k)}$ is the similarity measure between the collective priority vector and the PCM of the kth decision maker.

To simplify the computing process, we normalize the vectors and denote $\vec{w}'$, $\vec{a_j}'^{(k)}$ and $\vec{b_j}'^{(k)}$ as follows:

$$\vec{w}' = (w_1', w_2', \cdots, w_n')^T = \frac{\vec{w}}{\|\vec{w}\|} = (\frac{w_1}{\|\vec{w}\|}, \frac{w_2}{\|\vec{w}\|}, \cdots, \frac{w_n}{\|\vec{w}\|})^T \tag{10}$$

$$\vec{a_j}'^{(k)} = (a_{1j}'^{(k)}, a_{2j}'^{(k)}, \cdots, a_{nj}'^{(k)})^T = \frac{\vec{a_j}^{(k)}}{\|\vec{a_j}^{(k)}\|} = (\frac{a_{1j}^{(k)}}{\|\vec{a_j}^{(k)}\|}, \frac{a_{2j}^{(k)}}{\|\vec{a_j}^{(k)}\|}, \cdots, \frac{a_{nj}^{(k)}}{\|\vec{a_j}^{(k)}\|})^T \tag{11}$$

$$\vec{b_j}'^{(k)} = (b_{1j}'^{(k)}, b_{2j}'^{(k)}, \cdots, b_{nj}'^{(k)})^T = \frac{\vec{b_j}^{(k)}}{\|\vec{b_j}^{(k)}\|} = (\frac{b_{1j}^{(k)}}{\|\vec{b_j}^{(k)}\|}, \frac{b_{2j}^{(k)}}{\|\vec{b_j}^{(k)}\|}, \cdots, \frac{b_{nj}^{(k)}}{\|\vec{b_j}^{(k)}\|})^T \tag{12}$$

This transforms the optimization model into the following model:

$$\begin{aligned} &\max\ C = \sum_{k=1}^{k_m}\sum_{j=1}^{n}\sigma_k \vec{w}' \vec{a_j}'^{(k)} + \sum_{k=k_{m+1}}^{K}\sum_{j=1}^{n}\sigma_k \vec{w}' \vec{b_j}'^{(k)} \\ &s.t. \quad \begin{cases} \sum_{i=1}^{n} w_i'^2 = 1 \\ w_i\prime > 0, \qquad i = 1, 2, \cdots, n \end{cases} \end{aligned} \tag{13}$$

For the sake of simplicity, the above optimal model naturally can be denoted by vector in space analytic geometry. The objective function can be rewritten as follows:

$$C = \vec{w}' \cdot (\sum_{k=1}^{k_m}\sum_{j=1}^{n}\sigma_k \vec{a_j}'^{(k)} + \sum_{k=k_{m+1}}^{K}\sum_{j=1}^{n}\sigma_k \vec{b_j}'^{(k)}) \tag{14}$$

Denoting the $\vec{v} = (v_1, v_2, \cdots, v_n)^T = (\sum_{k=1}^{k_m}\sum_{j=1}^{n}\sigma_k \vec{a_j}'^{(k)} + \sum_{k=k_{m+1}}^{K}\sum_{j=1}^{n}\sigma_k \vec{b_j}'^{(k)})^T$. It is evident that the $\vec{v}$ must exist and be uniqueness. Therefore, the objective function is

$$\max\ C = \vec{w}' \cdot \vec{v} \tag{15}$$

Since the $\vec{w}'$ and $\vec{v}$ are normalized vectors, the C essentially is cosine similarity value of two vectors. Therefore, we can draw the conclusion that the maximum value of (15) must be 1. In this time, the solution $\vec{w}'$ must be $\vec{v}$ or $-\vec{v}$ and the latter is rounded off in real life questions.

From the constraint condition $\sum_{i=1}^{n} w_i'^2 = 1$, it follows that $\vec{w}'$ should be normalize and the solution is (16).

$$\vec{w}' = \frac{\vec{v}}{\|\vec{v}\|} \tag{16}$$

Now, it is evident that following condition exists by (9) and (13):

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} w_i' \|\vec{w}\| = \frac{\sum_{i=1}^{n} v_i}{\|\vec{v}\|}\|\vec{w}\| = 1 \tag{17}$$

This means that

$$\|\vec{w}\| = \frac{\|\vec{v}\|}{\sum_{i=1}^{n} v_i} \tag{18}$$

We can obtain the unique solution

$$\vec{w} = \vec{w}'\|\vec{w}\| = \frac{\vec{v}}{\|\vec{v}\|}\|\vec{w}\| = \frac{\vec{v}}{\|\vec{v}\|}\frac{\|\vec{v}\|}{\sum_{i=1}^{n} v_i} = \frac{\vec{v}}{\sum_{i=1}^{n} v_i} \tag{19}$$

Remark1: It is evident that total cosine value $C = nK$ is optimal if the decision makers' PCMs are all perfectly consistent. In this case, the priority vector has highest cosine value to each column vectors of PCMs. For single PCM, the cosine value $C = n$ which is order of the PCM if the priority vector has perfectly consistency, owing to the cosine value is 1 between each column vector of the PCM and priority vector in this case.

In order to use the calculation process, the proposed model can be implemented using the following algorithm in four steps with simple mathematical operations.

Algorithm:
Step 1: Transform fuzzy PCMs (p_{ij}), $k = k_{m+1}, k_{m+2}, \cdots, K$ into (b_{ij}), $k = k_{m+1}, k_{m+2}, \cdots, K$ by (5);
Step 2: Calculate $\vec{a_j}'^{(k)}$, $k = 1, 2, \cdots, k_m$ and $\vec{b_j}'^{(k)}$,$k = k_{m+1}, k_{m+2}, \cdots, K$ by (11) and (12);
Step 3: Calculate transformed weights coefficients vector $\vec{w}'$ by (16);
Step 4: Calculate the collective priority vector $\vec{w}$ by (19)
Remark2:In step 2, a column including 1 in the fuzzy PCM is not considered if $b_{ij} \leq 10^4$, in the calculation. Letting $p^k_{hl} = 1$, without loss of generality, $\sigma_l \langle \vec{w}, \vec{b_l}^{(k)} \rangle$ in optimization model (13) cannot be considered, and in this case, the transformed weights coefficient vector $\vec{v}$ is

$$\vec{v} = (\sum_{k=1}^{K} \sum_{j=1}^{n} \sigma_k \vec{a_j}'^{(k)} + \sum_{k=k_{m+1}}^{K} \sum_{j=1, j\neq l}^{n} \sigma_k \vec{b_j}'(k))^T \tag{20}$$

Since the column including $b_{hl} = 1$, approximate to $(0, 0, \cdots, 1_h, \cdots, 0)^T$ and $\lim p_{kl} \to 1 b_{kl} = 0$, $k \neq h$ cannot provide any effective preference information. In this case, this column will be removed to avoid a large deviation in the priority vector.

4 Illustrative examples

This section describes the implementation of the proposed model and presents three numerical examples to compare the proposed model with some existing methods in same problems.

Example 1 Consider the following multiplicative and fuzzy preference structures, which comes from Chiclana et al.[5]; Fan et al.[7]; Wang et al.[28]:

An investment company wishes to obtain the best option for investing a sum of money among four alternatives Car Company, Food Company, Computer Company, and Arms Company. The decision makers come from four consultancy departments of the investment company, and they have the same degrees of importance. Each department provides its preference information, with DM_1 and DM_2 providing the following multiplicative preference relations:

$$DM_1 = \begin{pmatrix} 1 & 1/7 & 1/3 & 1/5 \\ 7 & 1 & 3 & 2 \\ 3 & 1/3 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{pmatrix} \text{ and } DM_2 = \begin{pmatrix} 1 & 3 & 1/4 & 5 \\ 1/3 & 1 & 2 & 1/3 \\ 4 & 1/2 & 1 & 2 \\ 1/5 & 3 & 1/2 & 1 \end{pmatrix}$$

DM_3 and DM_4 express their preference information in terms of fuzzy preference relations as

$$DM_3 = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.1 & 0.5 \end{pmatrix} \text{ and } DM_4 = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0.1 \\ 0.5 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.5 & 0.8 \\ 0 & 0.4 & 0.2 & 0.5 \end{pmatrix}$$

It is evident that $p^{(4)}_{14} = 1$ and $\lim_{p^{(4)}_{14} \to +\infty} b^{(4)}_{14} = 1$. In this case,$b^{(4)}_{24} = 0.15 \times 10^{-9} \leq 10^{-4}$,$b^{(4)}_{34} = 0.40 \times 10^{-9} \leq 10^{-4}$ and $b^{(4)}_{44} = 0.1 \times 10^{-9} \leq 10^{-4}$; moreover, we can calculate $\vec{v}$ by (20).

The collective priority vector by the cosine similarity measure optimization model is found to be $(w_1, w_2.w_3, w_4)^T = (0.2075, 0.3882, 0.2124, 0.1919)^T$, with the ranking of alternatives $A_2 \succ A_3 \succ A_1 \succ A_4$, which is the same as the results provided by Chiclana et al. [5]. Chiclana et al. [26] select the best alternative using the weighted average operator on the basis of uniform preference formats using the transformation function

$$p^k_{ij} = \frac{1}{2}(1 + \log_9 a^k_{ij}), i = 1, 2, \cdots, n; j = 1, 2, \cdots, n \tag{21}$$

Then, they aggregate uniform fuzzy information into a collective opinion using the OWA operator ϕ_Q as

$$p_{ij}^* = \phi_Q(p_{ij}^1, p_{ij}^2, \cdots, p_{ij}^n) = \sum_{h=1}^{n} v_h d_{ij}^h, i = 1, 2, \cdots, n; j = 1, 2, \cdots, n \tag{22}$$

Where $d_{ij}^h =$ is the hth largest value among the collection of $p_{ij}^1, p_{ij}^2, \cdot, p_{ij}^n$ and $v_1, v_2, \cdots, v_n$is a weight vector with $v_h = Q(\frac{h}{n}) - Q(\frac{h-1}{n})$, $h = 1, 2, \cdots, n$. $Q(\cdot)$is a fuzzy quantifier with the membership function

$$Q(r) = \begin{cases} 0 & r < \gamma, \\ \frac{r-\gamma}{\mu-\gamma} & \gamma < r < \mu \\ 1 & r < \mu. \end{cases} \tag{23}$$

In practice, the fuzzy quantifier (γ, μ) is always selected as (0.3,0.8) ,(0,0.5) and (0.5,1), representing "most", "at least half", and "as much as possible, respectively. The OWA operator ϕ_Q is used to rank the alternatives or select the most desirable alternative(s) through the same calculation as that in the previous process.

Fan et al. [7] established a goal-programming model for GDM problem with two preference information based on the idea that the priority vector must be closest to each column of the PCMs provided by the DMs. Their model does not need to transform different preference formats into a uniform structure. It is constructed as

$$\begin{aligned} &\max \sum_{k=1}^{k_m}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_k |w_i - a_{ij}^{(k)} w_j| + \sum_{k=k_{m+1}}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_k |w_i - p_{ij}^{(k)}(w_i + w_j)| \\ &s.t. \quad \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \qquad i = 1, 2, \cdots, n \end{cases} \end{aligned} \tag{24}$$

Wang et al. [28] constructed a chi-square optimization model on the basis of the same idea as that of Fan et al. [7]. They differentiate the distance using chi-square optimization

$$\begin{aligned} &\max \sum_{k=1}^{k_m}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_k \left[\frac{(a_{ij}^{(k)} - w_i/w_j)^2}{w_i/w_j}\right] + \sum_{k=k_{m+1}}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n} \left[\frac{(a_{ij}^{(k)} - w_i/(w_i + w_j))^2}{w_i/(w_i + w_j)}\right] \\ &s.t. \quad \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \qquad i = 1, 2, \cdots, n \end{cases} \end{aligned} \tag{25}$$

Table 1 shows the results as compared with these three existing methods. The model proposed in this paper achieves the greatest cosine similarity measure. There is a small difference in the ranking of alternatives. All four methods provide the same best and second-best alternatives, but they disagree on the last two alternatives.

Example 2 Consider single fuzzy PCM to derive priority vector, which is used in Xu and Da[46]; Wang et al.[28]:

This example was used by Xu and Da [46] and Wang et al. [28] to derive a priority vector in a fuzzy preference relation. The following matrix is the fuzzy PCM in their example.

$$DM = \begin{pmatrix} 0.5 & 0.7 & 0.6 & 0.8 \\ 0.3 & 0.5 & 0.4 & 0.6 \\ 0.4 & 0.6 & 0.5 & 0.7 \\ 0.2 & 0.4 & 0.3 & 0.5 \end{pmatrix}$$

Table 1: Comparative results with existing methods for GDM with multiplicative and fuzzy preference relations

Approaches	w_1	w_2	w_3	w_4	Ranking of alternatives	Cosine value(C)
Chiclana et al. [5]	0.2452	0.2972	0.2487	0.2092	$A_2 \succ A_3 \succ A_1 \succ A_4$	12.6642
Fan et al. [7]	0.1280	0.4301	0.2515	0.1903	$A_2 \succ A_3 \succ A_4 \succ A_1$	12.4711
Wang et al.[28]	0.1697	0.3376	0.2741	0.2184	$A_2 \succ A_3 \succ A_4 \succ A_1$	12.5692
Our model	0.2075	0.3882	0.2124	0.1919	$A_2 \succ A_3 \succ A_1 \succ A_4$	**12.8284**

Xu and Da[46] constructed a least-deviation method for priority vector derivation in fuzzy AHP. A transformation function and rank transitivity property were used in the iteration algorithm.

$$\begin{array}{ll} \min & \sum_{i=1}^{n}\sum_{j=1}^{n}\left[9^{2p_{ij}-1}(w_i/w_j)+9^{2p_{ij}-1}(w_j/w_i)-2\right] \\ s.t. & \begin{cases} \sum_{i=1}^{n} w_i = 1 & \\ w_i > 0, & i=1,2,\cdots,n \end{cases} \end{array} \tag{26}$$

Table 2 summarizes the results of the two above mentioned methods and the proposed model. The rankings of alternatives by these three methods are the same. The cosine similarity measure value generated by our model is slightly greater than those by the other approaches.

Table 2: Comparative results with existing methods for GDM with multiplicative and fuzzy preference relations

Approaches	w_1	w_2	w_3	w_4	Ranking of alternatives	Cosine value(C)
Xu and Da[46]	0.4297	0.1784	0.2769	0.1150	$A_1 \succ A_3 \succ A_2 \succ A_4$	3.998186
Wang et al. [28]	0.4284	0.1286	0.2755	0.1159	$A_1 \succ A_3 \succ A_2 \succ A_4$	3.998209
Our model	0.4300	0.1800	0.2479	0.1151	$A_1 \succ A_3 \succ A_2 \succ A_4$	**3.998231**

Example 3 Consider four different preference formats by transformation functions, which is investigated by Ma et al. [8] and Xu et al.[9]:

This example includes four different preference formats, i.e., preference ordering, utility values, multiplicative preference relations, and fuzzy preference relations.

$$DM_1 = \begin{pmatrix} 1 & 1/7 & 1/3 & 1/5 \\ 7 & 1 & 3 & 2 \\ 3 & 1/3 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{pmatrix} \text{ and } DM_2 = \begin{pmatrix} 1 & 3 & 1/4 & 5 \\ 1/3 & 1 & 2 & 1/3 \\ 4 & 1/2 & 1 & 2 \\ 1/5 & 3 & 1/2 & 1 \end{pmatrix}$$

DM_3 and DM_4 express their preference information in terms of fuzzy preference relations as

$$DM_3 = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.1 & 0.5 \end{pmatrix} \text{ and } DM_4 = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0.1 \\ 0.5 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.5 & 0.8 \\ 0 & 0.4 & 0.2 & 0.5 \end{pmatrix}$$

$DM_5 = \{u_i | i = 1, 2, 3, 4\} = \{3, 1, 4, 2\}$, $DM_6 = \{u_i | i = 1, 2, 3, 4\} = \{2, 3, 1, 4\}$

$DM_7 = \{0_i | i = 1, 2, 3, 4\} = \{0.5, 0.7, 1.0, 0.1\}$ and $DM_8 = \{o_i | i = 1, 2, 3, 4\} = \{0.7, 0.9, 0.6, 0.3\}$.

where DM_1 and DM_2 provide multiplicative preference relations, DM_3 and DM_4 express their preference information in terms of fuzzy preference relations, DM_5 and DM_6 utilize preference orderings, and DM_7 and DM_8 use utility values.

We can convert preference orderings and utility values to fuzzy preference relations using the following transformation functions (Chiclana et al.[5]; Herrera et al.[1]; Ma et al.[8]):

$$p_{ij} = 0.5(1 + \frac{o_j}{n-1} - \frac{o_i}{n-1}) \tag{27}$$

$$p_{ij} = \begin{cases} \frac{u_i^2}{u_i^2+u_j^2} & (u_i, u_j) \neq (0,0), \\ 0.5 & (u_i, u_j) = (0,0) \end{cases} \tag{28}$$

By (27) and (28), the multiplicative and fuzzy preference relations can be obtained as follows:

$$\overline{DM_5} = \begin{pmatrix} 0.5 & 0.167 & 0.667 & 0.333 \\ 0.833 & 0.5 & 0.1 & 0.667 \\ 0.333 & 0 & 0.5 & 0.167 \\ 0.667 & 0.333 & 0.833 & 0.5 \end{pmatrix}, \overline{DM_6} = \begin{pmatrix} 0.5 & 0.667 & 0.333 & 0.833 \\ 0.333 & 0.5 & 0.167 & 0.667 \\ 0.667 & 0.833 & 0.5 & 1 \\ 0.167 & 0.833 & 0 & 0.5 \end{pmatrix}$$

$$\overline{DM_7} = \begin{pmatrix} 0.5 & 0.338 & 0.200 & 0.962 \\ 0.833 & 0.5 & 0.329 & 0.980 \\ 0.800 & 0.671 & 0.5 & 0.990 \\ 0.038 & 0.020 & 0.010 & 0.5 \end{pmatrix} \text{ and } \overline{DM_8} = \begin{pmatrix} 0.5 & 0.377 & 0.576 & 0.845 \\ 0.623 & 0.5 & 0.692 & 0.900 \\ 0.424 & 0.308 & 0.5 & 0.800 \\ 0.155 & 0.100 & 0 & 0.5 \end{pmatrix}$$

Then, we can derive a priority vector $(w_1, w_2.w_3.w_4)^T = (0.2006, 0.3811, 0.2886, 0.1297)^T$ using the cosine similarity measure optimization model, with the ranking of alternatives $A_2 \succ A_3 \succ A_1 \succ A_4$.

From Table 3, we observe that the four methods generate the same ranking (Chiclana et al.[3]; Ma et al.[8]; Xu et al.[9]), and that our model again achieves the highest cosine value, indicating that the similarity is most similar to each column of the PCMs of the decision makers.

Table 3: Comparative results with existing methods for four different preference formats in GDM

Approaches	w_1	w_2	w_3	w_4	Ranking of alternatives	Cosine value(C)
Chiclana et al. [3]	0.5651	0.7826	0.6619	0.4973	$A_2 \succ A_3 \succ A_1 \succ A_4$	-
Ma et al. [8]	0.2210	0.3426	0.2755	0.1159	$A_2 \succ A_3 \succ A_4 \succ A_1$	26.0116
Xu et al.[9]	0.2210	0.3426	0.2827	0.1537	$A_2 \succ A_3 \succ A_4 \succ A_1$	25.9763
Our model	0.2006	0.3811	0.2886	0.1297	$A_2 \succ A_3 \succ A_1 \succ A_4$	**26.0279**

Furthermore, though the models developed by Ma et al.[8] and Xu et al.[9] are optimization-based, which are similar to the model proposed in this study, their computational complexities are higher than our model. The former obtain results by transforming preference into certain formats which are suitable to calculate solution of programming, the latter need to solve nonlinear programming using genetic algorithm. Their models are presented as follows:

Ma et al. [8] proposed an optimization model for different formats. They optimize the minimum distance between the priority vector and a perfectly consistent PCM. Their model is given by

$$\begin{aligned} \min & \sum_{k=1}^{m_1}\sum_{i=1}^{n}\sum_{j=1}^{n}(w_i u_j^{(k)} - w_j u_i^{(k)})^2 + \sum_{k=m_1+1}^{m_2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[w_i(n-o_j^{(k)}) - w_j(n-o_i^{(k)})\right]^2 \\ & + \sum_{k=m_2+1}^{m_3}\sum_{i=1}^{n}\sum_{j=1}^{n}(w_i - w_j a_{ij}^{(k)})^2 + \sum_{k=m_3+1}^{m_4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[w_i - (w_i+w_j)p_{ij}^{(k)}\right]^2 \\ s.t. & \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \qquad i = 1, 2, \cdots, n \end{cases} \end{aligned} \tag{29}$$

Xu et al.[9] constructed a nonlinear optimization model to handle a GDM problem with missing values in the PCM. Their model is given as

$$
\begin{aligned}
&\min\, [\sum_{k=1}^{m_1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|w_i - u_i^{(k)}/\sum_{j=1}^{n}u_j^{(k)}\right|^p + \sum_{k=m_1+1}^{m_2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|w_i - \frac{n-o_i^{(k)}}{n-1}/\sum_{j=1}^{n}\frac{n-o_j^{(k)}}{n-1}\right|^p \\
&+\sum_{m_2+1}^{m_3}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}^{(k)} - w_i/w_j\right|^p + \sum_{m_3+1}^{m_4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|p_{ij}^{(k)} - w_i/(w_i+w_j)\right|^p]^{1/p} \qquad (30)\\
&s.t. \quad \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ w_i > 0, \qquad i=1,2,\cdots,n \end{cases}
\end{aligned}
$$

Both models (Ma et al.[8] and Xu et al. [9]) needed to solve complex optimization models. Particularly, model (30) uses a genetic algorithm to obtain a solution and involves high computational complexity. However, the model proposed herein can be solved using a Lagrangian approach and can be directly used to obtain a collective priority vector by following four simple steps.

5 Discussion and Conclusion

How to aggregate different preference formats is an important question in group decision making problem because it's natural for DMs with different backgrounds to represent their preferences using different formats. In this paper, an optimization model was developed on the basis of the cosine similarity measure to deal with multiplicative preference relations and fuzzy preference relations. The basic idea of the model is that the collective priority vector should be as similar per column as possible to a pairwise comparative matrix (PCM) from each decision maker in order to guarantee the priority vector is nearest to perfectly consistency for each decision makers. Compared with existing optimization-based methods, the proposed model is computationally simple, because it can be solved using a Lagrangian approach and obtain a collective priority vector by following four simple steps. The proposed method can also be used to derive priority vector of fuzzy AHP and provided to reach consensus by mans of existing iterative modification methods.

Three previously published examples were used to compare the proposed model with some existing approaches. The results show that the rankings generated by different approaches are similar and that the proposed model achieves the greatest cosine values in all three examples, indicating that the proposed model achieves the nearest theoretical perfectly consistent opinion for each decision makers. Furthermore, since the proposed model can be adapted to GDM problems with various preferences formats using transformation functions, it provides an efficient and simple way to handle different preference structures.

Acknowledgment

This research was supported in part by grants from the National Natural Science Foundation of China (#71222108, #71325001 and #71471149) and Major project of the National Social Science Foundation of China (# 15ZDB153).

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