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Noncommutative Logic Systems with Applications in Management and Engineering

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Abstract

Zadeh's (min-max, standard) fuzzy logic and various other logics are commutative, but natural language has nuances suggesting the premises are not equal, with premises contributing to the conclusion according to their prominency. Therefore, we suggest variants of salience-based, noncommutative and non-associative fuzzy logic (prominence logic) that may better model natural language and reasoning when using linguistic variables. Noncommutative fuzzy logics have several theoretical and applicative motivations to be used as models for human inference and decision making processes. Among others, asymmetric relations in economy and management, such as buyer-seller, provider-user, and employer-employee are noncommutative relations and induce noncommutative logic operations between premises or conclusions. A class of noncommutative fuzzy logic operators is introduced and fuzzy logic systems based on the corresponding noncommutative logics are described and analyzed. The prominence of the operators in the noncommutative operations is conventionally assumed to be determined by their precedence. Specific versions of noncommutative logics in the class of the salience-based, noncommutative logics are discussed. We show how fuzzy logic systems may be built based on these types of logics. Compared with classic fuzzy systems, the noncommutative fuzzy logic systems have improved performances in modeling problems, including the modeling of economic and social processes, and offer more flexibility in approximation and control. Applications discussed include management and engineering problems and issues in the field of firms' ethics or ethics of AI algorithms.

Keywords: fuzzy logic, fuzzy logic system, noncommutative logic, modeling, control, ethics, psychology, decision making.

1 Introduction

Building useful predictive knowledge structures requires suitable models and an understanding of human reasoning under various circumstances. It seems that human thinking is flexible and sensitive to the order of facts presentation with a rich set of aggregators between statements. Taking a standpoint from the theory of organization mission [3] and of public value management, one has to accept "*value rationality first and instrumental rationality second*" [9], a difference appears between the preeminence of some values over others, making the first more impactful on rational construction. The "public value paradigm", which is "*a new paradigm for thinking about government activity, policy-making and service delivery*" [23] in strategic public management views "*citizens as shareholders in how their tax is spent*", where "*values* [may be] *economic prosperity, social cohesion, cultural development, . . . better services, enhanced trust or social capital, or social problems diminished or avoided*" [23].

There are reasons to believe that premises and conclusions in human thinking are not always aggregated in a commutative manner and, in general, that noncommutative logic (NCL) is deeply entrenched in human thinking [1]. Zadeh proposed "*a perception-based theory of probabilistic reasoning*" [42]. Klir suggested that an "*arbitration principle*" should "*select the meaningful alternatives*" from a set of solutions, thus establishing their preeminence [17]. Yager advocated a "*hierarchical prioritized structure*" as a framework for representing rules in systems of inference rules [39]. The order of exposing ideas matters and the conclusions someone derives from a set of exposed facts (premises) may be affected by the order of the presented facts. Also, it appears that in some cases the effects of the order of presentation of the facts may be increased by the quantity or quality of the facts, while for other quantities or qualities the effect of the order may be null. All these indicate that the aggregation of statements in premises and in conclusions in reasoning may use in some circumstance commutative aggregators, while in other cases noncommutative connectives seem more suitable to describe inferences.

2 Noncommutativity in applications

The concept of noncommutativity in decision making and in subjective valuation became recently operational in marketing in the form of expert systems for trip planning and advising [24, 38, 41]. The expert system trip planner developed by Vansteenwegen et al. [24] "*provides personalised interest estimation*". Yu et al. [41] compares path planning "*for packet transmission over a network*" , which is efficiently solved with commutative operators, with the touristic tour planning; they recall that in travel planning "*ordinary evaluation functions, especially commutative ones, usually fail to resolve the issue properly*"; then, these authors propose "*an un-conventional noncommutative path planning strategy*".

In another research direction, AI has produced a new branch of ethics, the ethics of algorithms, and potentiated the interest in ethics and in fairness criteria in the broader sense [10]. Several mathematical criteria have been proposed, but they have been proved incompatible with each other and all imperfect. The consensus today is that no ethical criterion is perfect [30, 36]. An answer to the quest for acceptable criteria for ethical algortihms could be a combination of criteria with weights, or the use of a noncommutative aggregation of criteria, based on their preeminence as perceived by a certain social group or companies in an industry.

Also in the ethical realm, consider the 'Value statements' of companies, which is an abstracted version of the ethical values, a *credo* of the firms. Every major firm has such a statement. It turns out that the pubic, where the 'public' may be the employees of a company or the candidates to a job, may be sensitive to the order of criteria aggregation. This is apparent from the carefully chosen wording of the statements (and criteria) in the value statements of companies, where the exact wording is a problem of priorities, and priorities are not commutative. According to [44], "*Value statement [is] what's important to your company, what it prioritizes, and how it conducts itself*" and they are important because "*Value statements are good guidelines for culture, marketing, and more*". Value statement may target the employees in the first place, or only the customers, or the investors, or all of them. The target impacts the prioritization of values. Schwartz [26] talks about value priorities, while Clarke et al. [6] bring experimental evidence that values are dynamic and may change depending on the economic circumstances. Sagiv and Schwartz [25] decisively prove that the "value priorities" determine many human daily decisions, such as out-group social contacts. A similar approach based on prioritization may be useful in determining the "degree of ethical value" of an algorithm.

It is clear that a priority, a rank is assigned to the list of values: no company puts them in alphabetical order. A reason for the ranking is the impact the firms believe the values' mentioning may have on the employees, the customers, and the shareholders, with the values with higher impact placed in a predominant position. Management may be assuming that the first ranked value has the highest impact, the second a very high impact, the third a high impact, and so on; importantly, the impact of the most important is diminished when it is swapped with the second, yet the impact of the second is not increased to a level comparable with the impact the first had on the first position. Thus, ranking with noncommutativity may be a way of modeling the impact of the rank-aggregated values list, with the aggregation noncommutative. Firms wishing to optimize the effect of their value statements should carefully choose the elementary values, moreover the ranking of the values in the value statements.

Assuming that the aggregation of the elementary values in the order $ABCD$ has the highest impact, the order $BACD$ a very large impact, the order $CBAD$ a large impact, and so on, one can write $\theta(A\&B\&C\&D) > \theta(B\&A\&C\&D) > \theta(C\&B\&A\&D) > \ldots$, where θ is the impact-value function attributing an impact to each ordered list of values. Further, assume that the impact refers to the subjective degree of approval or appreciation of the firm by the "*stakeholders like employees, customers, suppliers, and communities*" [14]. Suppose that value A has a very high impact on the appreciation shown by the targeted population of employees, value B has a high impact and so on. Building a simple model based on the literature [3], [7, 27], assume that the subjective degree of approval may affect the retention of the employees, namely the very high impact determines a very high retention rate of employees, a high impact produces a high retention rate etc. This order-dependent model fits well one of the noncommutative fuzzy models described in the subsequent sections. If the retention rate increases (approximatively) linearly, then the line segment approximation problem described in Section 6 applies. If the increase is faster than linearly, for example exponentially, the second example in Section 6 is suitable to exemplify the use of noncommutative operators.

Not only in texts related to ethics, but in common language too, when forming complex propositions such as "it is rainy and cold", people tend to use the rule "first things first", that is, to place the most important part on the first place. As discussed, the precedence of an elementary proposition in a composed one seems to affect the meaning and the truth degree of the composed proposition. This is equivalent with saying that the order of elementary propositions connected with logic connectives matters, with the first elementary proposition more prominent than the second. This is also the typical manner the legislators write the laws and regulations, doctors present symptoms, and people present wishes, desires, imperatives, and decisions. Ethical constraints and rules are usually given in the same way. Consider that the position of an elementary sentence in a compound sentence p determines the salience of its parts $p = p_1\&p_2\&p_3$ and the logic is multivalued. Then p has or may have a different truth value $\theta(p)$ than the compound proposition constituted of the same elementary propositions but in a different order, $\theta(p')$, $p' = p_2\&p_1\&p_3$ and $\theta(p) \neq \theta(p')$ if $\theta(p_1) \neq \theta(p_2)$ and $\theta(p_1)\theta(p_2) \neq 0$, whatever is the third proposition $\forall p_3$, when $\theta(p_3) \neq 0$.

The predominance of a specific value of an attribute in a set of values of the attribute is often implied in discourse and assessments. For example, when assessing the potential of a student in mathematics, the highest grades are more important. In contrast, when assessing the limits of an employee, the lower ratings count more. The importance of the values may be substantiated using arithmetic weights, as in weighted averages, or may be substantiated using a combination of rank in enumeration and noncommutative connectives that favor the higher ranks, as shown in Section 3. This may lead to a difference between a set of rules under commutative and noncommutative logics. In case of commutative logics, including fuzzy logic, premises in different rules act independently. Yet, in case of the assessment above described, two elementary rules such as "If grade is high (A or B) then student's potential is high OR If grade is low (D or F) student's potential is low" make little sense even if high grade and low grade are fuzzified; a student with a combination of high and low grades still have high potential. Arithmetic averages is also unjustified in this case. The salience of high grades can be ensured by assuming either an effect of the premise ”grade is high” on the premise ” grade is low”, when both have a non-zero degree of truth (both situations occur for a student), or a noncommutative OR between the rules, with a preeminence given to the first rule.

In a further example of potential usefulness of noncommutative logics, the connective AND in enumerations is not compatible with commutativity. When one says ”I love miso soup AND sushi” one does not understand the minimum (as in fuzzy logic), or the product (as in probabilistic logic) of the two truth degrees. In a sense, AND in enumerations is not compatible with commutative logics, because these unintuitively and unjustifiably diminish the truth degree in logic aggregation. A solution might be the use of noncommutative logics.

Expectedly, under noncommutative logics (NCLs), several properties of commutative logics do not hold, as discussed in the next sections.

In this paper, without disregarding important theoretical foundations, we are mainly interested in the instrumental capabilities of noncommutative logics; consequently, the treatment is kept at an elementary level.

3 Noncommutative operations based on t-norms and s-conorms

We say that two functions form a hard-gentle (h, g) pair of functions if they satisfy the conditions: (i) they are defined on $[0, 1]$, continuous on $(0, 1)$, and $h(u) \in [0, 1]$, $g(u) \in [0, 1]$ for $u \in [0, 1]$; (ii) $h(0) = g(0) = 0$ and $h(1) = g(1) = 1$; (iii) $h(u) \geq u$ and $g(u) \leq u$ on $[0, 1]$.

A stronger condition than (iii), tacitly verified in all examples, is that h and g are monotonic. In some situations, we will add the property (iv) h is invertible and its inverse is g, $h(g(u)) = u, \quad g(h(u)) = u, \ \forall u \in [0, 1]$.

For some choices of h, $\lim_{n\to\infty} h^{(n)}(x) = 1$ on $(0, 1]$, where $h^{(n)} = h(h(\ldots(h)\ldots)$, n times, and $\lim_{n\to\infty} g^{(n)}(x) = 0$ on $[0, 1)$ (the zero function). It is convenient that the functions h, g are infinitely derivable, but this condition is not enforced here.

Without risk of confusion, as the meaning will result from the context, we use the same notation (h, g) for a pair of functions satisfying: (i) $h, g : [0, 1] \times [0, 1] \to [0, 1]$, continuous on $[0, 1] \times [0, 1] - \{(0, 0), (1, 1)\}$; (ii) $h(1, v) = v \ \forall v \in [0, 1]$, $g(0, v) = v \ \forall v \in [0, 1]$; (iii) $h(u, v) \geq v \ \forall u$ and $g(u, v) \leq v \ \forall u$; (iv) $h(0, 0) = g(0, 0) = 0$, $h(1, 1) = g(1, 1) = 1$.

Removing the points $(0, 0), (1, 1)$ from the continuity condition offers more flexibility in the choice of the functions. Condition (iv) is introduced for idempotency. The pair $h(u, v)$, $g(u, v)$ above serves for building right noncommutative operations. For left noncommutativity, condition (ii) is replaced by ii)’ $h(u, 1) = u \ \forall u \in [0, 1]$, $g(u, 0) = u \ \forall u \in [0, 1]$. Next, we recall the notions of t-norm and conorm.

t-norms and s-conorms

Many classes of logics, fuzzy logic(s) included, are based in triangular norms and their co-norms used as connectives (aggregators). The name ’triangular’ norm comes from the analogy with the triangle, where the sum of the lengths of any two edges is larger or equal than the lengths of the third edge. We use the term triangular norm in the general sense of a function $t : [0, 1]^2 \to [0, 1]$ satisfying the condition $t(x + y, z) \leq t(x, z) + t(y, z)$, with the additional property that it has the unit 1, $t(x, 1) = t(1, x)$. The generally accepted definition of triangular norms, shortly t-norms is:

A triangular norm is a function $t : [0, 1]^2 \to [0, 1]$ satisfying the conditions: i) commutativity, $t(x, y) = t(y, x) \ \forall (x, y)$; ii) associativity $t(x, t(y, z)) = t(t(x, y), z) \ \forall (x, y, z)$; iii) 1 is unity (neutral) of the t-norm, $t(x, 1) = t(1, x) = x \ \forall x$; iv) it is monotonic, non-decreasing on $[0, 1]^2$ in both variables, that is $t(x, y) \leq t(w, z)$ if $x \leq w, y \leq z$ [13, 16, 21]. The last defining condition is equivalent with $t(t(x, y), z) \leq t(x, z) + t(y, z)$, hence the name of triangular norms (analogous to $d(A, B) \leq d(A, C) + d(B, C)$ where A,B,C are the vertices of a triangle).

t-norms serve as a definition of the AND aggregator. Frequently, t-norms that are continuous in both variables are used in applications, yet sometimes the continuity condition is relaxed to left-or right-continuity [16].

An s-conorm is the dual of the t-norm and has the same properties, except that its neutral element is the null. A t-norm and its s-conorm are dual in the sense that $s(x, y) = 1 - t(1 - x, 1 - y)$. The conorm has also the triangular property, $s(x + y, z) \leq s(x, z) + s(y, z)$. Notice that the definition does not require that $t(x, x) = x$ (idempotency) [21]. In most examples. Idempotency is satisfied, except in Section 7. A norm is said to be Archimedean iff (i) for any element $t(x, x) < x$, and (ii) the norm is continuous [21]; it is clear that t should be decreasing.

Forming connectives

With a pair (h, g) and given a t-norm and s-conorm , noncommutative operations are defined as follows (we use the framework of propositional logic for explanations, with θ the truth valuation function). The left-noncommutative operator AND, denoted by $\cap^h$, is defined as

$$\theta(p_1 \cap^h p_2) = t(h(\theta(p_1)), \theta(p_2)),$$

where p_1, p_2 are propositions.
The right-noncommutative AND operator, $\cap_h$, is defined as

$$\theta(p_1 \cap_h p_2) = t(\theta(p_1), h(\theta(p_2))).$$

We use throughout the paper only noncommutativity in the right-hand sense; left-noncommutativity is obtained from the right-noncommutativity by a change of order of the propositions. Whenever $\theta(p_1) \neq \theta(p_2)$ and with $h(x)$ different from the identity function, $t(\theta(p_1), h(\theta(p_2))) \neq t(\theta(p_2), h(\theta(p_1)))$ and the operation is noncommutative.

Zero remains the null value in the sense that $\theta(0 \cap_h p_2) = t(\theta(0), h(\theta(p_2))) = 0$ and $\theta(p_1 \cap_h 0) = t(\theta(p_1), h(0)) = 0$ because $h(0) = 0$, but 1 is not preserved as unit for $\cup_h$:

$$\theta(1 \cap_h p_2) = t(\theta(1, h(\theta(p_2))) = h(\theta(p_2)) \neq \theta(p_2).$$

Here, we used the same notation '0' for propositions with zero truth degree, $\theta(0) = 0$, and for the number zero.

Because $h(u) \geq u$ and the t norm is increasing, $t(x, h(x)) \geq t(x, x)$ and $t(x, h(y)) \geq t(x, y)$. When $t(x, y) = \min(x, y)$, because $h(u) \geq u$, idempotency results, $\theta(p \cap_h p) = \theta(p)$. The "middle excluded principle" in binary logic becomes, for $\neg x = 1 - x$, $t(x, h(1 - x)) \geq t(x, 1 - x)$; when t is min, $t(x, h(1 - x))$ may have values larger than 0.5, which is a strong departure from classic fuzzy logic.

A stronger version of the AND operator is defined by:

$$\theta(p_1 \cap_h p_2) = t(\theta(p_1), h(\theta(p_2)) \quad if \quad \theta(p_1) \neq 1, \quad \theta(p_2) \quad else.$$

The strong AND definition preserves the role of the unity. However, the strong operation $\cap_h$ is no more continuous in both variables at $\theta(p_1) \to 1$; for example, with $\theta(p_2) = 0.8$ and $h(x) = x^{0.5}$ with t interpreted as min, $t(x, y) = \min(x, y)$, the right limit at 1 of $\theta(p_1) \cap_h p_2) = 0.64$, while at $\theta(p_1) = 1$, the value is 0.8. Therefore, the strong noncommutative AND is not even right-continuous at 1. This affects noncommutative fuzzy logic systems with trapezoidal membership functions.

Neither the weaker version of noncommutative AND nor the stronger one are distributive.

The join (OR) operation, denoted by, $\cup_g$, is defined as

$$\theta(p_1 \cup_g p_2) = s(\theta(p_1), g(\theta(p_2))$$

The properties are derived as for $\cap_h$. Because $g(u) \leq u$, $s(x, g(x)) \leq s(x, x)$ and $s(x, g(y)) \leq s(x, y)$. When s is max, $s(x, g(x)) = \max(x, g(x)) = x$ and $\cup_g$ is idempotent. In particular, we will use the functions $h(u) = u^a$, $0 \leq a \leq 1$, $g(u) = u^{(1/a)}$.

The choice of the definitions of the connective AND as $\theta(p_1 \cap_h p_2) = t(\theta(p_1), h(\theta(p_2))$ seems unsuitable because h increases $\theta(p_2)$, not the more prominent $\theta(p_1)$. However, recalling that the α-set, S_α of a function $f : \mathbf{R} \to \mathbf{R}$ is the set of all points where f takes values larger or equal with α, $S_\alpha = \{x | f(x) \geq \alpha\}$, the effect of AND through t and h is to increase the α-sets of $\theta(p_1 \cap_h p_2)$, where $\theta(p_1 \cap_h p_2) = \theta(p_1)$. Hence, the effect is to increase the contribution of $\theta(p_1)$ in the result, which justifies the choice. We agree, however, that in some applications it may be justified to use an alternative definition, changing the variable to which h, g are applied in the t norm.

The (h, g) based intersection and union operations are non-associative; in this article, we will use the convention:

Convention. In a sequence of noncommutative logic operations, we assume the usual convention of operation order imposed by the brackets. When brackets are missing, we assume that each operand has effect only on the one immediately at the right side and operations are performed from left to right. Thus, by convention, $x \cap_h y \cap_h z$ reads $(x \cap_h y) \cap_h z$.

The (h, g) operations can be mixed, without confusion, with commutative operations when brackets are used; for example, $p \cap_h (q \cap r)$ and $(p \cap_h q) \cap r$ are well defined. In addition, we will extend the above convention to mixed operators; thus, $p \cap_h q \cap r$ is equivalent with $(p \cap_h q) \cap r$.

A stronger version of the OR operator is defined by:

$$\theta(p_1 \cup_g p_2) = s(\theta(p_1), g(\theta(p_2))) \quad if \quad \theta(p_1) > 0, \quad \theta(p_2) \quad else.$$

The strong OR definition is required for intuitive reasons and for avoiding 0^0 in some specific cases of the functions h, g, see Section 6. The intuitive reason for the strong OR definition is that a null element (zero-valued proposition), even when in a predominant position, should have no effect. This definition preserves the role of 0. When $\theta(p_1) = 0$, the strong OR is equivalent with the usual OR in fuzzy logic. This simplifies the programming for simulations of noncommutative fuzzy systems. We will use in applications only the stronger version of OR for the above reasons. However, using the stronger definition makes $\theta(p_1 \cup_g p_2)$ discontinuous at $\theta(p_1) = 0$. Recall that in establishing the above forms of the operators we considered that the preeminence of one of the variables in the t norm implies that its influence on the result should be extended beyond the space where it dominates (is larger). In other words, we impose the condition $t(x, h(y)) \geq t(x, y)$, where y is dominated; this condition is satisfied when $h(x) \geq x \quad \forall x \in [0, 1]$, due to the monotonic increase of t. Similarly, for the conorm, we impose the condition $s(x, g(y)) \leq s(x, y)$ to enforce the role of x over y. Again the condition is satisfied when $g(x) \leq x$, because of the monotony (decrease) of g. The conditions can be interpreted as ensuring that marginal, non-essential dominance of y in $t(x, y)$ or in $s(x, y)$ is replaced by the enforced dominance of x. For example, when the variable x is considered essential, $x = 0.4$ and $y = 0.41$, the difference of 0.01 can be considered unessential because the variable y is given less importance, and the result of the modified t-norm is still given by y. Similarly, when $x = 0.4$ and $y = 0.39$, because x is the salient variable, the result in $s(x, y)$ will be due to x. The functions $h(u) = u^a,\ 0 < a \leq 1$. $g(u) = u^b,\ b \geq 1$, in particular $b = 1/a$, satisfy these requirements on the interval $[0, 1]$.

Some desirable properties are lost for the noncommutative operations. Fig. 1 illustrates the non-associativity for two values of a, for $h(u) = u^a,\ g(u) = u^{1/a}$.

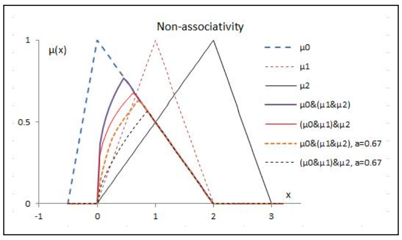

Figure 1: Non-associativity

The proposed operations can be further refined imposing a "threshold of prominence", γ, as a minimal value of truth of the prominent proposition for its prominence becomes active,

$$\theta(p_1 \cap_{h,\gamma} p_2) = t(\theta(p_1), h(\theta(p_2))) \quad if \quad \gamma < \theta(p_1) < 1, \quad \theta(p_2) \quad else,$$

$$\theta(p_1 \cup_{g,\gamma} p_2) = s(\theta(p_1), g(\theta(p_2))) \quad if \quad \theta(p_1) > \gamma > 0, \quad \theta(p_2) \quad else.$$

Rules with noncommutative operations

In an inference based on a set of rules, it is not necessary that all rules use noncommutative connectives. Also, it is not required that always some variable is dominant. In fact, human thinking seems to grant prominence to a variable in some rules and only when that variable has a specific value. For example, in the rule:

If the velocity is high and the car weight is "overloaded", start applying the brakes from a very large distance,
the dominant variable might be "the car is overloaded", because drivers seem to know less or take less into account the overcharging (weight) of the car, being more aware about the velocity – hence the need to emphasize the role of the overloading. But the car weight is not salient when the weight is normal or light (underloaded). Therefore, for modeling the reasoning, only in the rule with order of premises as below:

If the car weight is "overloaded" and the velocity is high, start applying the brakes from a very large distance,
the variable "weight" will be salient and a noncommutative AND is used between the premises, while in the rule

If the car weight is "usual" and the velocity is medium, start applying the breaks from a medium distance,
the AND is commutative. Numerous other examples can be found where humans seem to endow one variable with more importance when the variable has a specific value. Medical doctors talk about key symptoms and secondary symptoms, and usually a symptom becomes "key" when it reaches a specific intensity. Similar language is used in various technologies when describing good recipes and procedures. For an extended discussion of cases where noncomutative logics may be of interest see [28].

In terms of standard (min-max Zadeh's) fuzzy logic, preeminence of a variable in aggregation with another would mean that a membership function will be potentiated toward another one in a rule when connecting two premises, if one premise is given more salience, while it will act as usual in another rule. For example, in one rule it will act as $\min(\mu_A(x), \mu_B^a(y))$, but it will act as $\min(\mu_A(x), \mu_C(z))$ in some other rule, where A (and μ_A) is not prominent.

NB. The discussion in this paper is not connected with the proposal [22] of using noncommutative operators in probability theory; the authors' name similarity with the cited author is a coincidence.

4 Noncommutative operators based on min-max

In this section and in the remaining part of the paper, the noncommutative weak operator AND, denoted by $\cap_a$, is defined as

$$\theta(p_1 \cap_a p_2) = \min(\theta(p_1), \theta^a(p_2)), \quad a \leq 1$$

where the value of a less than 1 of a produces the salience of p_1 and $\theta^a(x)$ denotes $(\theta(x))^a$. The join weak operation, denoted by, $\cup_a$, is defined as

$$\theta(p_1 \cup_a p_2) = \max(\theta(p_1), \theta^{(1/a)}(p_2)), \quad a \leq 1.$$

The operations preserve some of the desirable properties:

$$\theta(\underbrace{p_1 \cap_a (p_2 \cap_a (p_2 \cap_a (\ldots \cap_a p_2)}_{n} \ldots) = \theta(p_1 \cap_a p_2) \quad \forall n > 1,$$

$$\theta(p_1 \cup_a p_2 \cup_a \ldots \cup_a p_2) = \theta(p_1 \cup_a p_2).$$

The following properties result directly

$$\lim_{a \to 0} \theta(p_1 \cap_a p_2) = \theta(p_1) \quad \text{for} \quad \theta(p_2) \neq 0,$$

$$\lim_{a \to 0} \theta(p_1 \cup_a p_2) = \theta(p_1) \quad \text{for} \quad \theta(p_2) \neq 0.$$

The condition $a \neq 0$ is required to avoid cases such as 0^0, $1/0$. One can generalize the definitions using two constants, a and $b \neq 1/a$ in the exponents.

The property of idempotency of the two operators is preserved,

$$\theta(p_1 \cap_a p_1) = \min(\theta(p_1), \theta^a(p_1)) = \theta(p_1), \quad 0 < a \leq 1,$$

as $\ln(\theta(p_1) \leq a\ln(\theta(p_1)$ for $a < 1$ because $\theta(p_1) \leq 1$ and thus $\ln\theta(p_1)$ is negative or null. Similarly, $\theta(p_1 \cup_a p_1) = \max(\theta(p_1), \theta^{(1/a)}(p_1)) = \theta(p_1), \quad 0 < a \leq 1$.

For the weak version of the join operation, union with 0 is noncommutative :

$$\theta(0 \cup_a p) = \max(0, \theta^{(1/a)}(p)) = \theta^{(1/a)}(p) \leq \theta(p \cup_a 0) = \theta(p), \quad a \leq 1,$$

and intersection with 1 is noncommutative,

$$\theta(p \cap_a 1) = \min(\theta(p), 1) = \theta(p) \neq \theta(1 \cap_a p) = \min(1, \theta^a(p)) = \theta^a(p), \quad a \leq 1.$$

Zero is null for the intersection, $\theta(p \cap_a 0) = \theta(0 \cap_a p) = 0$, and 1 is unity for union,

$$\theta(p \cup_a 1) = \max(\theta(p), 1) = \max(1, \theta(p)^{1/a}) = \theta(1 \cup_a p) = 1.$$

These properties can be summarized as:

$$0 = \theta(p \cap_a 0) = \theta(0 \cap_a p) = 0 \leq \theta(p_1 \cap p_2) \leq \theta(p_1 \cap_a p_1),$$

$$\theta(p_1 \cap_a p_1) \leq \theta(p_1 \cup p_2) \leq \theta(p_1 \cup_a p_2) \leq \theta(p \cup_a 1) = \theta(1 \cup_a p) = \theta(1 \cup p) = 1.$$

Above, $\cup$ is the usual OR. The associativity is not preserved; in general,

$$\theta((p_1 \cup_a p_2) \cup_a p_3) \neq \theta(p_1 \cup_a (p_2 \cup_a p_3)).$$

Indeed, for $\theta(p_1) \neq 1, \theta(p_2) \neq 1, \theta(p_3) \neq 1$,

$$\theta((p_1 \cup_a p_2) \cup_a p_3) = \max((\theta(p_1 \cup_a p_2), \theta^{(1/a)}(p_3)) = \max[\max[(\theta(p_1), \theta^{(1/a)}(p_2)], \theta^{(1/a)}(p_3)],$$

while

$$\max(\theta(p_1), (\max(\theta(p_2), \theta^{(1/a)}(p_3)))^{(1/a)}) = \theta(p_1 \cup_a (p_2 \cup_a p_3).$$

As discussed, when there is no reason to assign a specific order (precedence) or preeminence to any fuzzy set or fuzzy proposition, the standard operators should be applied. In addition, a mixture of standard and noncommutative operators may be used, depending on the salience of variables. For example, when two factors, y and z have secondary but equal preeminence, while a third variable, x, has precedence over them, the interpretation of their joining should be understood as $x \cup_a (y \cup z)$. Similar considerations can be used for intersection. Denote for sake of brevity $x = \theta(p_1)$, $y = \theta(p_2)$, $z = \theta(p_3)$. As an example,

$$\theta(p_1 \cup_a (p_2 \cup_a p_3) = \begin{cases} x & \text{if } x^a > \max(y, z^{1/a}), \\ y & \text{if } y > z^{(1/a)} \text{ and } y > x^a, \\ z^{(1/a)} & \text{else } (z > y^a, z > x^{a^2}) \end{cases}$$

The connectives are continuous in the variables, but they do not have continuous derivatives, which change at the points on the curve $x = y^a$, respectively on $y = x^a$. Compare with the case for min and max operators, which change the derivative along the first diagonal. The implication is defined in the usual sense, based on minimum. The property $(A \to B) \to ((B \to C) \to (A \to C))$ is hereditary preserved from standard fuzzy logic (FL) because implication is defined in a similar manner (essentially, truncation in applications with membership functions).

A fuzzy logic (FL) with noncommutative operations (NCO) as above does not satisfy several properties known from the usual fuzzy logics (FLs). Among others, the weak version of the principle *tertium non datur* (law of excluded middle) in Zadeh's FL

$$\theta(p_1 \cup_a \neg p_1) \leq 0.5, \quad \theta(p_1 \cup_a \neg p_1) \geq 0.5 \quad \forall p_1$$

is not valid. In addition, $x \cup \neg x \neq x \cup_a \neg x \neq \neg x \cup_a x$ and $x \cap \neg x \neq x \cap_a \neg x \neq \neg x \cap_a x$ for $a \neq 1$.

From this point, we focus on noncommutative fuzzy logic and systems; therefore, we will transit from the use of the truth valuation function θ to the membership functions μ. For example, we will write

$$\mu_0(x) \cap_a (\mu_1(x) \cap_a \mu_2(x)) = \min[\mu_0(x), [\min(\mu_1(x) \cap \mu_2^a(x))]^a]$$

For the case of isosceles triangular membership functions (m.f.s) with unitary semi-base (along the [0,1] interval), (0,1,2), on the ascending slope:

$$\min[\mu(x), (1-\mu(x))^a] = \mu(x) \quad \text{for} \quad x \le x_0, \quad (1-\mu(x))^a \quad \text{else}$$

where x_0 is solution of the equation $x = (1-x)^a$, or $\ln \mu_0(x) = a \ln(1-\mu(x_0))$, and a similar solution for the descending slope. For the value x_0, the minimum is higher than 0.5 when $a < 1$, with the precise value of the minimum imposed by a, see Fig. 2.

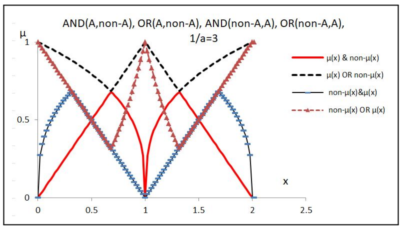

Figure 2: Invalidity of the weaker version of the principle of excluded middle in standard FL. Operations with left-precedence, $a = 1/3$.

5 Rules with NCOs and fuzzy logic systems with NCOs

5.1 Systems of rules and the use of noncommutative operations

In FL, in a system of rules connected by a commutative OR operation, the order of rules plays no role. In a system of rules connected by OR in a noncommutative logic, using noncommutative AND in the premises and an implication compatible with the noncommutative AND and OR, the order of elementary premises in the rules and the order of rules matter. In noncommutative FL there is a difference between the rule

R_{xy}: *If* x *is* A *and* y *is* B *than* z *is* C

and the rule with reversed order of the premises,

R_{yx}: *If* y *is* B *and* x *is* A *than* z *is* C*.*

The difference is materialized by using a noncommutative AND operator between the premises, assuming that one of them is more salient and dominates the other. As a consequence, a construction such as $(xy)(yx)$ makes sense and the rule

$R_{((xy)(yx))}$: *If (*x *is* A *and* y *is* B*) and (*y *is* B *and* x *is* A*) than* z *is* C

differs from the rules R_{xy} and R_{yx}. Also, the following two rules are different,

$R_{xy \wedge yx}$: *If (*x *is* A *and* y *is* B*) and (*y *is* B *and* x *is* A*) than* z *is* C*.*

$R_{xy \vee yx}$: *If (*x *is* A *and* y *is* B*) or (*y *is* B *and* x *is* A*) than* z *is* C*.*

Consider as an example five triangular membership functions (m.f.s.), $\mu_0(x), \ldots, \mu_4(x)$, with μ_0 not symmetrical, and $\mu_k(x),\ k > 0$ isosceles triangular with the maximum at $x = k$, as in Fig. 3. This set of functions is also used in Section 6 as input m.f.s. When the order of rules is assumed to determine their salience, with different results of the inference when the order is changed, the rules are connected with a noncommutative OR. However, in daily life and natural language, one or a few rules are prominent, while the others have the same secondary importance. The context of prominent rules is important because of the non-associativity; therefore, we need conventions for applying the noncommutative join operators. Among the several possible conventions, we exemplify two:

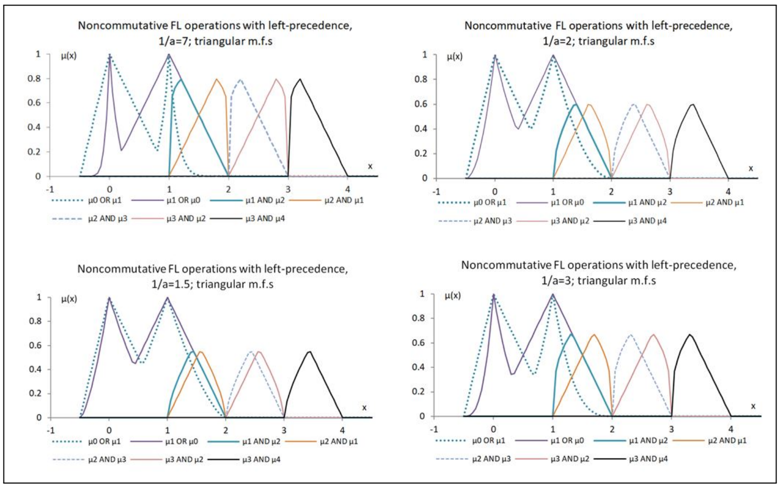

Figure 3: Operations with left-precedence, for various values of a.

a) When a single rule dominates all the others whatever is its context (position), it is placed first and operated with the result of all other rules. Denoting the result of the k^{th} rule by $Y_k(y)$, one defines the overall result as $Y = Y_1 \cup_a (Y_2 \cup \ldots \cup Y_n)$,

$$\mu_Y(y) = \max(\mu_{Y_1}(y), (\max(\mu_{Y_2}(y), \ldots, \mu_{Y_n}(y))^a),$$

b) When a single rule dominates and its context (position) matters, its position being at place j, the result is computed as

$$\mu_Y(y) = \max(\mu_{Y_1}(y), \ldots, \mu_{Y_{j-1}}(y), \max(\mu_{Y_j}(y), [\max(\mu_{Y_{j+1}}(y), \ldots, \mu_{Y_{n]}}(y))]^a).$$

The prominence of rules establishes a partial order over the set of rules and thus determines the aggregation manner. A special case occurs when two or several rules have the same conclusion Then, we have to aggregate by OR several instances of the same original membership function, $\mu_k(y)$, truncated by different truncation values, $\mu_j(x_0)$ from the premises of the rules. Denote the truncated output m.f.s by $\mu_{k;j}$. If there are r rules with the same conclusion and a single rule, k_1, out of the r rules is prominent, then we perform the aggregation according to the prominence, with the other conclusions operated under commutative OR, as $\mu_k^*(y) = \max[\mu_{k_1}(y), \max_{i=1,\ldots,r,i\neq 1}^{1/a}\{\mu_i(y)\}_i]$. When several rules are prominent with the same rank, they are aggregated with commutative OR. When several rules are prominent with different ranks, they are aggregated according to the rank etc.

In case of single input single output systems (i.e., based on sets of rules with a single premise), except when several rules have the same conclusion, specifying the order of conclusions (or rules) uniquely determines the order of noncommutative aggregation. For example, specifying $R1 = R2 = R4 = R5 < R3$ means that only $R3$ is prominent and that the other rules should be aggregated commutatively. When the output membership functions are isosceles triangles as frequently in applications, $(a_{i-1}, a_i, a_{i+1}), (a_i, a_{i+1}, a_{i+2})$, under the above description only two such m.f.s are not null for a specified value of the input. If the specified prominence is according to $R1 < R2 < \ldots < Rr$, then the result of aggregation has the form $\max(\mu_i(y; x_0), \mu_{i+1}^{1/a}(y; x_0))$. In case of systems with two inputs, that is, with two premises in the rules, assuming again isosceles triangular m.f.s for both the inputs and the output, as above, four rules can be active at a time. The order of premises determines the truncation value for the conclusions. For the aggregation of the conclusions, a convention can be established for the prominence for the four active rules; for example, $R_{h,j} = R_{h+1,j+1} > R_{h+1,j} = R_{h,j+1}$. Then, the aggregation is according to: $(C_{h,j} \cup C_{h+1,j+1}) \cup_a (C_{h+1,j} \cup C_{h,j+1})$ where $C_{\cdot,\cdot}$ denotes the respective conclusions.

Finally, we emphasize that in some problems, such as modeling the customers' behavior, the prominence (rank) is empirically imposed, while in other problems, such as the optimization of a fuzzy logic system (FLS) for approximation or control, the ranking and possibly the constants in the definition of the aggregators have to be determined during the FLS adaptation process.

5.2 An detailed example of NCO use

Subsequently, we assume that a single rule has prominence and the union is taken in the strong sense. This determines that the noncommutative union acts 'locally', that is, only when it respective rule is 'fired' ('active'). An example for this case is discussed next. Consider a noncommutative FLS (NCFLS) with the rules

If x is A_h AND_{nc} *y is* B_j *then z is* C_{hj}, $h = 1, \dots, m,\ j = 1, \dots, n$

where A_h and B_j are fuzzy sets and where AND_{nc} denotes a noncommutative AND with the preeminence of x over y. Let the actual values of the variables be $x = x_0$, $y = y_0$. In addition, we will assume that, in the conclusions of the rules, the conclusion with $h = 2, k = 3$ has a 'local' preeminence, meaning that the preeminence acts only if the membership function $\mu_{C_{23}}(z; x_0, y_0)$ has values larger than 0. The composed premises have the truth values

$$\theta_{hj}(x = x_0, y = y_0) = \min[\mu_{A_h}(x_0), \mu^a_{B_j}(y_0)], \quad if\ 0 < \mu_{A_h}(x_0) < 1,\ 0 < a \leq 1.$$

The value $\theta_{hk}(x_0, y_0)$ truncates C_{hk} according to

$$\mu_{C_{hj}}(z; x_0, y_0) = \begin{cases} \min[\mu^a_{C_{hj}}(z), \min[\mu_{A_h}(x_0), \mu^a_{B_j}(y_0)]], & \text{if } \theta_{hj}(x_0, y_0) > 0, \\ 0 & \text{else} \end{cases}$$

The overall output membership function, obtained by the reunion of all partial conclusions C_{hj} is computed by first applying the locally preeminent but truncated m.f., denoted by $C_{23}(x_0, y_0)$. Using a similar notation for all truncated fuzzy sets C_{hj}, one determines

$$C_{23}(x_0, y_0) \cup_a [C_{11}(x_0, y_0) \cup C_{12}(x_0, y_0) \cup C_{22}(x_0, y_0) \cup C_{24}(x_0, y_0) \cup \ldots \cup C_{mn}(x_0, y_0)]$$

with only the first join operator noncommutative. This expression has the equivalent form

$$\mu_{out}(z; x_0, y_0) = \begin{cases} \max[\mu_{C_{23}}(z; x_0, y_0), [\max_{(hj)\neq(23)}(\mu_{C_{hj}}(z; x_0, y_0)]^{1/a}], & \text{if } \mu_{C_{23}}(z; x_0, y_0) > 0 \\ \max_{(hj)\neq(23)}\{\mu_{C_{hj}}(z; x_0, y_0)\} & \text{else.} \end{cases}$$

6 Mamdani-type FLS with noncommutative operations

One of the limits of the common Mamdani systems with triangular m.f.s is that they approximate poorly straight line segments, because of their essentially rational expression of the characteristic function. For testing the modeling power of the suggested NCFLS, we show first how to approximate a line segment with a NCFLS and compare the modeling error with the error of standard Mamdani FLSs with c.o.g. defuzzification.

As discussed, systems with mixed operators, that is, with one or several logic connective commutative used together with one or several instances of noncommutative operators (making the logic noncommutative) are conceivable. Regarding the implication, one can argue that the value truncating the membership functions in the conclusions should dominate, and thus should have higher salience than the truncated membership function. Therefore, the rule

If x is A then y is B,

where A and B are fuzzy sets, will produce for $x = x_0$ in the premise the membership function $\mu_{(A\to B)}(y) = \mu_{B|x_0}(y) = \min(\mu_A(x_0), \mu^a_B(y)),\ a \leq 1$.

The non-associativity of the operations implies that the order of applying the operators, that is, of associating the variables should be defined in advance. Consider the natural language sentences from the example already discussed

If velocity of the car is high and the mass m is low, the brakes should be applied from a moderate distance in advance, but if the car is overloaded and the velocity is high, the braking should start from a long distance.

Because no brackets are used, we will assume that the order of premises determines their prominence and thus the order of the AND operators. For example, in the rule

If velocity v is high (H) AND_{nc} the mass m is low (L), the brakes should be applied from a long (L) distance d,

the AND_{nc} is applied to the premises as $\mu_{Hl} = \min[\mu_H(v), \mu_l^a(m)]$ (when $\mu_H(v) \neq 0, \mu_H(v) \neq 1$) and the result truncates the m.f. for "long distance".

Remark 1. The effect of the above definition is to use modified membership functions (locally, in this operation), where the original $\mu_B(y)$ is transformed into $\mu_B^a(y)$.

Using several rules and assuming the usual (max) commutative connective OR between them, one obtains results dependent on the choice of a, as in Fig. 4.

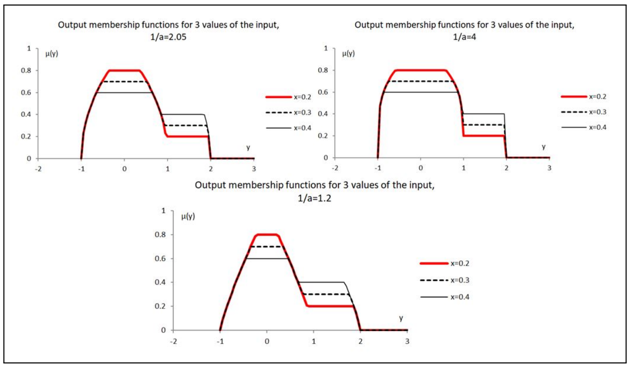

Figure 4: Results of the rules, using operations with left-precedence, for various values of the constant a and the same input value.

When a mixture of noncommutative and standard operations is used, it is worth noting that
Remark 2. When the OR between rules is defined by the classic operator max, the resulted non-commutative FLS (NCFLS) is equivalent with a min-max FLS with modified output m.f.s., where the output m.f.s are obtained from the original ones as $\mu_{A_j}^{(1/a)}(x)$.

Rephrasing, the remark says that applying only implication based on noncommutative operations does not bring much novelty to the FLS construction and that the same results can be obtained by the change of the output m.f.s in a common FLS.

Consider two rules with a single antecedent and a single consequent, *If A1 Then B1*, *If A2 Then B2*, where A1, A2, B1, B2 are triangular fuzzy sets, $A1 = (0, 1, 2)$, $A2 = (1, 2, 3)$, $B1 = (1, 2, 3)$, and $B2 = (2, 3, 4)$ and the antecedents and the consequents representing different variables. Assume that the prominence according to the indices, $A1 > A2, B1 > B2$. For a specified value of the input variable, $x_0 \in (1, 2)$, both consequents are activated. Using the definitions $\mu_{A\cap_a B}(u) = \min[\mu_A(u), \mu_B^a(u)]$ and $\mu_{A\cup_a B}(u) = \max[\mu_A(u), \mu_B^b(u)],\ b = 1/a$ the result is:

$$\mu_{out}(y; x_0) = \max[\min(\mu_{A1}(x_0), \mu_{B1}^a(y)), [\min(\mu_{A2}(x_0), \mu_{B2}^a(y))]^{1/a}].$$

For y values where the first element in the max operation is larger and $\mu_{B1}^a(y) < \mu_{A1}(x_0)$, the result is $\mu_{out}(y; x_0) = \mu_{B1}^a(y)$. When the second element in the max operation is larger (which happens, for example, when $\mu_{B1}(y)$ is close to 0),

$$\mu_{out}(y) = [\min(\mu_{A2}(x_0), \mu_{B2}^a(y))]^{1/a}.$$

In this case, when further $\mu_{B2}^a(y)) \leq \mu_{A2}(x_0)$, the result is $\mu_{out}(y; x_0) = \mu_{B2}(y)$. This shows that for some values of y the results is given by the m.f. of B_1 at power a, $\mu_{B1}^a(y)$; but, for other values of y, the result is due to the m.f. $\mu_{B2}(y)$, without exponent. Using the definitions of the operations with the condition that the second operand is not affected when the first is null will produce a slightly different situation, see the corresponding definitions.

The above discussion leads to the following property:

Property. Consider a single input single output NCFLS (Mamdani type) with all input and output m.f.s overlapping two by two, each input and output pair pair intersecting in a single point, and with right noncommutative operations defined by $h(u) = u^a,\ g(u) = u^b,\ 0 < a < 1,\ b > 1$. Consider that two rules attribute $A1 \to B1, A2 \to B2$, A1, A2, B1, B2 triangular m.f.s, A1 at the left of A2, B1 at the left of B2. Then, the c.o.g. defuzzified output is less than or equal with the output of a common FLS (with commutative min-max logic) with the same m.f.s.

The proof results from the above discussing noticing that on any interval where two m.f.s (denoted by $A1$ and $A2$) overlap, the value of x defined by

$\min(\mu_{A1}(x_1), \mu^a_{B1}(y)) = [\min(\mu_{A2}(x_1), \mu^a_{B2}(y))]^{1/a}$

is unique. For that interval and for $x < x_1$, there are values of x where $\mu_{out}(y; x_0) = \mu^a_{B1}(y) \geq \mu_{B1}(y)$ and where the output of the equivalent FLS would be $\mu_{B1}(y)$. For $x > x_1$, the output of both systems is $\mu_{B2}(y)$. As a result, $\int y\mu_{out}(y; x_0)dy / \int \mu_{out}(y; x_0)dy$ is less or equal than the same expression for the corresponding classic FLS.

Example 1: Approximation of a line segment

As an example of capabilities of the noncommutative operations, we applied Mamdani-type systems with c.o.g. defuzzification to the approximations of a line segment and of an exponential. For the line segment approximation, three approximating FLSs were compared. The first was a standard Mamdani FLS, the second a Mamdani NCFLS where only the implication was according to the NCFL, and the third was a Mamdani NCFLS with NCFL implication and the third output m.f. assumed the single preeminent m.f. salient and with the constants a and b not necessarily equal. For the straight line

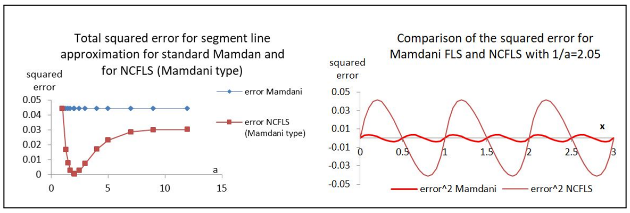

Figure 5: Comparison of a line segment approximation with standard Mamdani and NCFLS Mamdani.

segment approximation with a standard Mamdani FLS with c.o.g. defuzzification, with the input m.f.s (-0.5, 0, 1), (0, 1, 2), (1,2,3), (2,3,4), (3,4,5) and the output m.f.s (-1,0,1), (0,1,2), (1,2,3), (2,3,4), (3,4,5), associated one-to-one input to output in the order they are listed, the squared error was $\epsilon^2 = 0.04397$. With the same membership functions, the NCFLS produced the best approximation for $1/a = 2.05$, with the minimal error more than two orders of magnitude lower $\epsilon^2 = 0.00037$. The NCFLS was defined using a Mamdani-type NCFLS with the implication only defined using a non-commutative AND, that is, a basic solution; the approximation was on the interval $x \in [0, 2.45]$ and the line was the first diagonal. Further refinements can be applied to further reduce the error.

Notice that in both cases (FLS and NCFLS) elementary m.f.s were chosen. The variation of the squared error, for various values of a is shown in Fig. 5, compared with the standard Mamdani error (which is constant, hence the horizontal line). Also shown in Fig. 5 is the local variation of the two approximations with respect with x, $\epsilon^2(x)$, where the error of the NCFLS is for the optimal value $a = 2.05$.

Example 2: Approximation of an exponential

Models based on power laws and exponential models abound in economics and management science. An example of power law is the gravitational model, which is a product of powers model with the general form $z = k \times \frac{x^\alpha y^\beta}{d^\gamma}$. The model describes commercial exchanges [2], knowledge spillovers [37], and it is now used in various process models beyond trade economics. For example, Lewer and Van den Berg (2008) developed a gravity model of immigration [18]. Without changing the key components of the gravity equation, specifically the two measures of 'mass' (x and y) and the measure of 'distance' d, numerous alterations to the model are possible. The derivation of the linear regression model by taking the logarithm of the two sides of the equality encourages wide applications of the gravity model in economics. Exponential models are also used in survival analysis, utility function of investors [19], and various other applications. This interest justifies the next example of monotonic function approximation with NCFLSs.

For the approximation of the exponential function, we compared the results obtained with a Mamdani NCFLS, the results with standard Mamdani, and with the approximation with line segments through the points defined by the vertices of the output membership function (the choice of the linear piecewise function ensures the compatibility in the comparison).

In the approximation of the exponential function, see Fig. 6, the square error for Mamdani FLS with a basic choice of the m.f.s was $\epsilon^2 = 1.731$; for Mamdani FLS with a better choice of the m.f.s, the square error was $\epsilon^2 = 0.092$, the approximation with linear segments through the vertices of the basic m.f.s had an error of $\epsilon^2 = 0.133$, and for the approximation with NCFLS with $a = 1.2, b = 0.4$, the square error was $\epsilon^2 = 0.071$. The approximation with the NCFLS and the local error are shown in Fig. 6. Computation details are available from the second author.

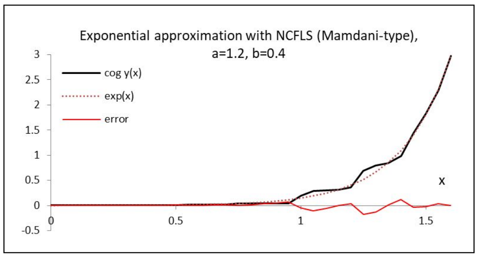

Figure 6: Results of the rules, using operations with left-precedence, for various values of the input x

The above results are expected, as NCFLSs are parametric systems, with the parameters a and b easy to adjust for adaptation of the approximating function (represented by the input-output function of the NCFLS).

7 Variable exponent, context dependent aggregation

In this section we briefly discus the use of functions (h, g) of two variables applied to the definition of the logic operators, as specified in Section 3.

In some applications it may be justified to use context-dependent, variable exponent aggregators as defined below. The right-noncommutative variable exponent AND operator, $\cap_h$, is defined as

$$\theta(p_1 \cap_h p_2) = t(\theta(p_1), h(\theta(p_1), \theta(p_2)) \text{ if } \theta(p_1) > 0, \ 0 \ else,$$

with $h(u, v) = v^{au} \ for \ u \neq 0, \ 0 < a \leq 1$. For $t(x, y) = \min(x, y)$,

$$\theta(p_1 \cap_h p_2) = \min(\theta(p_1), \theta^{a\theta(p_1)}(p_2)) \text{ if } \theta(p_1) > 0, \ 0 \text{ else.}$$

For meet operation of x with its negation, one has $x \cap_h \neg x : \ \min(x, (1-x)^{ax})$. '1' remains unit for the operation only if $a = 1$. The operation is idempotent, $x \cap_h x : \ \min(x, x^{ax}) = x$ because $ax < 1$.

Similarly, for $s(x, y) = \max(x, y)$ and for $g(u, v) = v^{(a/u)}, \ u > 0$,

$$\theta(p_1 \cup_g p_2) = \max(\theta(p_1), \theta^{b/\theta(p_1)}(p_2)), \text{ if } \theta(p_1) > 0, \ \theta(p_2) \text{ else.}$$

For join operation of x with its negation, one has $x \cup_h \neg x : \ \max(x, (1-x)^{a/x})$. '0' remains null for the operation by definition. For $b \leq 1$, including $b = 1/a$, $b/\theta(p_1) > 1$ and the join is idempotent.

Examples of results are illustrated in Fig. 7. In applications, it may be useful to have independent constants, a and $b \neq 1/a$, in the definitions of the two operations.

Various other noncommutative operations can be introduced in a similar way, when they seem suitable for a specific application. For example, an operand (fuzzy set) may become preeminent only when its truth value is larger than a threshold, yet behave as a non-priority operand else; such a noncommutative operation seems justified in some human reasoning cases.

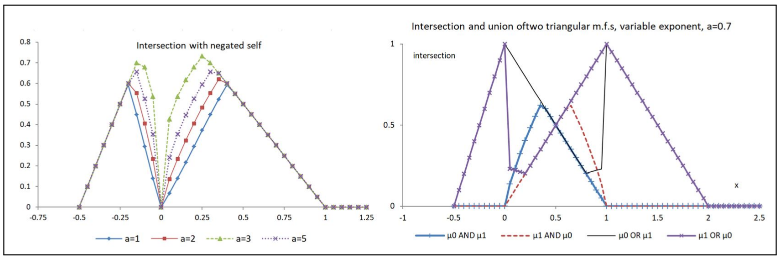

Figure 7: Examples of variable exponent operations (variable exponent logic).

There are intuitive hints that, in probabilistic reasoning, the human inference may be sensitive to knowledge about the derivative value of the probability distribution at the point of interest; in this respect, hints come from the behavior of stock, currency, and futures trading. That would lead to definitions of the operators as:

$$\theta(x(u_0) \cap_h y(v_0)) = t(\theta(x(u_0)), \theta^{a\frac{d\theta(y)}{dv}(v_0)}) \text{ if } \theta'(v_0) > 0,\ 0 \text{ else},$$

or, possibly,

$$\theta(x(u_0) \cap_h y(v_0)) = t(\theta(x(u_0)), \theta^{a\theta''(v_0)}) \text{ if } \theta''(v_0) > 0,\ 0 \text{ else}.$$

and similarly for the union. These definitions open a different investigation problem to be dealt with elsewhere. Notice that for these cases with t defined as min, finding the point where $\theta(x(u_0)) = \theta^{a\theta''(v_0)})$ requires solving a differential equation, a problem connected with various interesting issues, see [15] as an example.

Sugeno-type FLSs based on NCFLs are built similarly with the Mamdani ones, as explained above, but the noncommutative operators are effective only at the level of premises and implication. Finally, we mention that FLSs based on NCFLs are generalizations of standard FLS, and hence are universal approximators; this derives directly from the fact that using the constants $a = 1$ and $b = 1$ in the definitions of the operators produce standard FLSs, which are known to be universal approximators [4, 33]. Yet, the versatility of standard fuzzy logic methods in representation (approximation) is counterbalanced by the need of sometimes tedious tweaking of the membership functions and rules [34].

8 Conclusions and future work

The interest in applied NCLs based on t-norms and conorms and in particular of NCFLs is multifaceted. In the first place, they may help better modeling preferences [40, 43] priorities, dynamics [31] and ranking of motives in human thinking; they may help to better represent human thinking in decision-making and expert systems, with numerous applications in marketing and services planning. Second, NCLs and NCFLs may be a more appropriate theoretical tool in representing strategic management processes and ethical assessments. Third, NCFLSs represent a more flexible instrument in approximations, allowing better approximations than standard FLSs when the same membership functions are used, meaning that NCFLS may be applied to refine the use of FLSs in a wide range of control and modeling applications.

We have proposed several variants of NCLs based on t-norms and conorms, applied them to develop NCFLs, and studied their basic properties. Then, we have shown how to apply NCFL in for developing Mamdani NCFLSs. We exemplified the enhanced flexibility of these systems in approximation problems and in decision-making descriptions, with direct applicability in modeling and assessment of ethical reasoning. Further discussion of these topics for the cases where the operator OR is defined as $s(u, v) = 1 - t(1 - u, 1 - v^a)$, as $s(u, v) = 1 - t(1 - u, (1 - v)^a)$, as $s(u, v) = 1 - t(1 - u, 1 - v^{1/a})$ etc. will appear in another study. Also, Sugeno-type systems will be dealt elsewhere.

The domain of NC logic systems and in particular of NCFLSs is potentially vast and has numerous applications in linguistics, economy, management, and engineering; these applications remain to be investigated. The deeper theoretical aspects of NCLs were left apart in this semi-empirical paper; much has to be done in line with the theoretical aspects presented in works as [1, 5, 8], [11, 12, 20]. Further combining NCLs and NCFLs with hesitant sets in modeling opinions, paralleling [29] and investigating how approximate reasoning models [32] can be developed based on NCLs may be another research directions in the future.

Applying NCL and NCLSs requires an increased computational effort and may need machine learning procedures [35] adapted to the task of extracting information from human thinking and of automatically adapting the parameters of the logic model. However, as machine learning becomes less and less a mysterious thing and more of a regular topic, the increased complexity of the problems described using NCLs should not be an issue.

Numerous other versions of noncommutative operations may be devised. For example, the definition of AND as $min(x, y \cdot (1-x))$ and the (pseudo-)conorm for OR, $s(x, y) = 1-min(1-x, 1-y(1-x))$ seem to have justifications in human reasoning. This and other variants of the definitions of the logic connectives will be studied elsewhere.

A difficulty in using NCFLs and NCFLSs is determining when they are appropriate models of human thinking. In addition, while the mixtures of commutative and NCLs seem justified under several circumstances, how to choose the appropriate mixture is a further difficulty in applications related to modeling human thinking, for example in ethical assessments. Yet, NCLs, NCFLs, and NCFLSs represent a progress over their standard counterparts and should be applied whenever their use seems reasonable and possible. We believe this is the first study on NCLSs and NCFLSs and the first to apply forms of NCFLs to various applications.

Authors' contributions

The paper stems from MT's reflections on ethical and managerial issues in algorithmics. MT suggested the use of priorities and preeminence of criteria in the ethics of algorithms and in several applications in management and proposed several desirable properties of the NC operations. HNT has developed the technical and engineering aspects in the paper and wrote most of it with feedback from MT.

Conflict of interest

The authors declare no conflict of interest.

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