Mathematical Proof and Discovery *Reductio ad Absurdum*

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Abstract: The uses and interpretation of *reductio ad absurdum* argument-ation in mathematical proof and dis-covery are examined, illustrated with elementary and progressively sophisti-cated examples, and explained. Against Arthur Schopenhauer's ob-jections, *reductio* reasoning is defen-ded as a method of uncovering new mathematical truths, and not merely of confirming independently grasped ma-thematical intuitions. The application of *reductio* argument is contrasted with purely mechanical brute algo-rithmic inferences as an art requiring skill and intelligent intervention in the choice of hypotheses and attribution of contradictions deduced to a particular assumption in a contradiction's deri-vation base within a *reductio* proof structure.

Resumé: On examine, illustre avec des exemples élémentaires et pro-gresssivement sophistiqués, et expli-que les usages et les interprétations du raisonnement *reductio ad absurdum* dans les preuves et les découvertes mathématiques. Afin de répondre aux objections d'Arthur Schopenhauer contre ce raisonnement, on le présente comme une méthode qui confirme des intuitions mathématiques et qui dé-voile des nouvelles vérités mathé-matiques. On compare l'application des arguments *reductio* aux inférences purement algorithmiques et méca-niques pour souligner que l'usage des ces arguments est un art qui exige de l'habileté et de l'intelligence dans la sélection d'hypothèse à réfuter et dans la déduction d'une contradiction.

Keywords: algorithm; argumentation; Euclidean, non-Euclidean geometry; mathematics; methodology; proof; Schopenhauer, Arthur

1. A useful method

It is one of the most time-honored and powerful methods of mathematical reasoning. *Argumentum reductio ad absurdum* is an indispensable mode of inference to which, with a bit of careful reformulation, every logical and mathematical argument can be reduced. Relatively easy to master in its simplest applications,

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reductio style reasoning poses a number of heuristic problems that have not always been appreciated in discussions of its role in mathematical discovery and proof.

The discussion to follow explains the logical structure and illustrates the utility of *reductio* arguments in mathematics. It highlights the fact that *reductio ad absurdum* needs to be understood as an art, rather than a brute algorithm. The method of reducing an assumption to an absurdity requires judgment and finesse in order to be applied successfully, in contrast with such comparatively more pedestrian rules of inference as *modus ponendo ponens* and *tollendo tollens*. When these aspects of *reductio* reasoning are explained, it becomes possible to recognize what is truly remarkable and distinctive about the unlimited potential of indirect proof.

We reason by *reductio ad absurdum* when we assume the negation of what we are trying to prove, and then, from this assumption, together with other explicit or background assumptions, we are able to deduce an explicit contradiction. The assumptions in that case are said to have been reduced to an absurdity. Since the definition of a deductively valid inference prohibits the valid deduction of a false conclusion from true assumptions, and since a contradiction is always necessarily or logically false, it follows that at least one of the assumptions from which the contradictory false conclusion is deduced must itself be false. If the responsibility for the deduction of a false proposition falls squarely on the hypothesis made for purposes of indirect proof or *reductio ad absurdum*, which is generally the negation of the proposition to be proved, then that hypothesis must be false. This, finally, implies without further ado that the hypothesis, the negation of the target proposition, is true. The target proposition, the negation of the *reductio* hypothesis is thereby proved on the strength of the contradiction that has been produced. It is a contradiction that would otherwise obtain, in that event, were the hypothesis true. The hypothesis in this way is shown by the method not to be true; for, if it were, then it would imply a logical contradiction. Proving by *reductio ad absurdum* that the hypothesis of the *reductio* argument is false is logically equivalent to demonstrating by indirect proof that the target proposition is true, which is the purpose of the argument all along.[1]

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[1] I have been influenced throughout much of this discussion by the dispute concerning traditional methods of construing the logical form of *reductio* inferences, as in Copi's popular textbook, *Symbolic Logic, 2ⁿᵈ ed. (1955)*, begun by Scherer (1971) and Lambros (1973.

2. Breviary of *reductio*-style arguments

Indirect proofs or *reductio ad absurdum* arguments often offer the best, most compact and convincing demonstrations of a conclusion. This is particularly so where the conclusion is a necessary truth, the denial of which is self-contradictory, or where the denial of the conclusion can easily be shown to contradict the argument's assumptions in light of reasonably accepted background assumptions.

The use of *reductio* reasoning is seen in this simple inference:

1. Susan will not run for President. [Hypothesis for *reductio*]
2. If Susan will not run for President, then Mark will not run for Vice-President. [Background assumption]
3. Mark will run for Vice-President. [Background assumption]
   _____

4. Mark will not run for Vice-President.(1,2,) [Contradiction with (3)]
5. Susan will run for President. [Negation of assumption (1)]

The task is to prove the conclusion in proposition (5), that Susan will run for President. We reason indirectly by assuming that the conclusion is false in assumption (1). Along with the assumption in (2) that if Susan will not run for President, then Mark will not run for Vice-President, and the assumption that Mark will run for Vice-President, we conclude in (4) that Mark will not run for Vice-President—a proposition that contradicts the assumption we are independently prepared to accept as true in (3). If we are certain of (2) and (3), then we can only suppose the contradiction results from assumption (1). As a consequence, we reject assumption (1) and deduce its negation by indirect proof or *reductio ad absurdum* in conclusion (5). The deductive grounds for (5) are that if assumption (1) were true, it would lead to the outright logical contradiction in (3) and (4), and that therefore (1) is not true.

More interesting examples occur throughout all of mathematics. Again, we can begin with a relatively simple illustration. Suppose that I want to prove that there is no greatest even number. I assume the opposite of the conclusion I hope to establish, by assuming on the contrary that there is a greatest even number, which I call 'N'. If N is an even number, no matter how

great, then I can always obtain an even number greater than N by adding 2 to N. Thus, I have reduced the assumption that there is a greatest even number to an absurdity in the form of an outright logical contradiction, that N is the greatest even number (by assumption), and that N is not the greatest even number (from the fact that N + 2 is necessarily an even number greater than N). The argument can be reconstructed in this way:

1. There is a greatest even number, N. [Hypothesis
   for *reductio*]
2. If N is an even number, then N + 2 is an even
   number greater than N. [Concept of even number]
   _____

3. N + 2 is an even number greater than N. (1,2)
4. N is the greatest even number.
   [Contradiction with (5)]
5. N is not the greatest even number. [Contradiction
   with (4)]
6. There is no greatest even number. [Negation of
   assumption (1)]

A related but trickier and to that extent more interesting example is discussed by the Cambridge University philosopher Frank Ramsey in his posthumous book, *Foundations of Mathematics and Other Logical Essays*. Ramsey's argument, which he probably did not originate, demonstrates that there are at least two persons living on Earth who have precisely the same number of hairs on their heads from the assumption that there are at least two more persons living on Earth than there are hairs on any one person's head (see Ramsey 1931, 35).

The reasoning is by *reductio ad absurdum*. We assume, contrary to the conclusion, that it is false or that it is not the case that there are two persons living on Earth who have precisely the same number of hairs on their heads, but that every person has a different number of hairs on his or her head than every other person. The hypothesis quickly leads to a logical contradiction. The assumption, which states that the conclusion of the argument is false, is reduced to an absurdity. If the hairiest person has N hairs, and if, as seems reasonable, there are at least two more persons living on Earth than there are hairs on any one person's head, then there are at least N + 2 people living on Earth. If the person with the fewest hairs is totally bald, then that person has 0 hairs on his or her head, and the hairiest person by the assumptions we have already made can have at most (N + 2) − 1 or N + 1 hairs on his or

her head. If every person has a different number of hairs, as we are supposing in assuming for *reductio* purposes that the conclusion of Ramsey's argument is false, then one and only one person living on Earth, the hairiest, has precisely N + 1 hairs. This flatly contradicts the assumption that the hairiest person has N hairs. Assuming that the conclusion of Ramsey's argument is false, that no two persons have precisely the same number of hairs on their heads, but that every person living on Earth has a different number of hairs, reduces in stages to the logical absurdity that some person has both precisely N and N + 1 hairs, or that some person both has precisely N and does not have precisely N hairs, on the assumption that there are at least two more persons living on Earth than there are hairs on any one person's head. The argument can be reconstructed in this way for comparison with the previous example:

1. The hairiest person living on Earth has N hairs on his or her head.
2. There are at least N + 2 persons living on Earth.
3. It is not the case that at least two persons living on Earth have precisely the same number of hairs on their heads; everyone has a different number of hairs.
   [Hypothesis for *reductio*]
   _____

4. If there are at least N + 2 persons living on Earth, and if they all have a   different number of hairs on their heads, then, if a totally bald person has 0 hairs, then the hairiest person living on Earth has at least (N + 2) − 1 or N + 1 hairs.
5. The hairiest person living on Earth has precisely N hairs on his or her head and the hairiest person living on Earth does not have precisely N (but at least N + 1) hairs on his or her head.
   [Contradiction between (1) and (4)]
6. At least two persons living on Earth have precisely the same number of hairs on their heads.
   [Negation of assumption (3)]

     Reductio arguments can sometimes appear superficially to be logically inconsistent. This is because they draw inferences from false assumptions, and, in fact, from assumptions that the arguments themselves are designed to prove false. Such a characterization is nevertheless a misdescription of how *reductio* arguments work. *Reductio* arguments merely hypothesize propositions that later in the proof are shown to be false, in order to expose their falsehood by using them to derive an absurdity or contradiction. Whereas in non-*reductio* arguments the assumptions

with which the argument begins remain propositions to the truth of which the argument is committed throughout, or at least conditionally to the proposition that if the assumptions are true, as logically they (might be), then so is the conclusion, in *reductio* arguments, false assumptions are rejected as part of the argument's conclusion, and their negations are deduced instead.[2]

Arthur Schopenhauer provides a useful illustration of *reductio* reasoning in mathematics in his 1847 work, *On the Fourfold Root of the Principle of Sufficient Reason*. The context is one in which Schopenhauer is considering the limitations of rigorous mathematical demonstration, and proposes to rephrase the sixth proposition in Book I of Euclid's *Elements* as a *reductio* argument requiring a somewhat imaginative construction. Schopenhauer criticizes Euclid's style of mathematical proof for its lack of intuitive insight into what he refers to as the *ground of being* (*ratio essendi*) of mathematical theorems. He maintains that *reductio* reasoning in mathematics offers only conviction (*convictio*) based on reasoning (*Vernunft*) without understanding (*Verstand*). The latter epistemic state he believes can only result from a perceptual grasp of the basis for a mathematical truth, which *reductio* thinking never affords. Proof by contradiction, according to Schopenhauer, at most convinces us *that* a propositions is true without offering any satisfactory insight into *why* it is true, a position he develops at greater length in the first edition and first volume of later editions of his treatise, *The World as Will and Representation* (see Jacquette, forthcoming). In the *Fourfold Root*, Schopenhauer nevertheless presents a faithful reconstruction of Euclid's *reductio* proof, when he writes:

> Let *abc* be a triangle having the angle *abc* equal to the angle *acb*, then the side *ac* is equal to the side *ab*.
> For if *ac* is not equal to *ab*, then one of them is greater than the other. Let *ab* be greater; from *ba* cut off *bd* equal to *ac*, and draw *dc*. Now since (in the triangles *dbc*, *abc*), *db* equals *ac*, and *bc* is common to both, the two sides *db* and *bc* are equal to the two sides *ac* and *cb*, each taken separately, and the angle *dbc* is equal to the angle *abc* is equal to the triangle

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[2] Here we might further distinguish between an arguer's versus an argument's commitments. The former is a subjective matter of what an individual advancing an argument happens to believe, whereas the latter is a more objective matter of what the argument states and of what must be true in order for the argument to be sound or at least to be intelligibly propounded. The present context requires only argument commitments, although in many instances an arguer's commitments will naturally coincide with those of the arguer's argument.

> *dcb*, the greater is equal to the smaller, which is
> absurd. Therefore *ab* is not equal to *ac*,
> consequently *ab* is equal to *ac*. (Schopenhauer 1974,
> 201)

Another remarkable example of *reductio* reasoning appears again in Book X of Euclid's *Elements*. Bertrand Russell provides a useful paraphrase of Euclid's reasoning in his (1919) book, *Introduction to Mathematical Philosophy*, to show that there exists no fraction m/n = square root of 2, which is to say that the square root of 2 must be irrational. The argument proceeds by *reductio*, involving the deduction of a contradiction from the assumption that the conclusion is not true. Russell reasons that if there did exist a fraction equal to the square root of 2, then it would have to be true that $(m/n)^2 = 2$. From this conclusion, it would further follow that $m^2/n^2 = 2$, whereby both m and n *per impossibile* would both need to be simultaneously even and odd. This, however, we know, is impossible. Indeed, it is an outright logical contradiction on the further undeniable assumption that a number is even if and only if it is not odd. Whatever m/n is, it will be reducible to an equivalent fraction in lowest terms; suppose, then, that m/n is already in lowest terms. It follows by definition in that case that m and n cannot both be even, or it would not be in lowest terms. Yet m and n *must* both be even. If $m^2/n^2 = 2$, then $m^2 = 2n^2$, which is to say that m^2 is divisible without remainder by 2, something that is possible only if m^2 is itself an even number, on the grounds that no odd number multiplied by itself is equal to an even number. It further follows then that $n^2 = m^2/2$, which is to say that n^2 is equivalent to a number that is divisible without remainder by 2. This is possible once again, for the same reason as before, only if n is itself an even number (Russell 1919, 67). Thus, m and n must and cannot possibly both be even numbers. The argument upholds a useful lemma in Socrates' discussion with the slave boy in Plato's early aporetic dialogue, the *Meno*, where Socrates presupposes but does not try rigorously to demonstrate that no rational number between 2 and 3 exactly measures the length of the diagonal of a square with a side length of 2. Socrates, instead, proceeds entirely intuitively, a practice Schopenhauer would commend, drawing diagrams of the square and its diagonal in the sand with a stick (*Meno* 82ᵉ).

As a final example, consider another of Euclid's proofs in his writings on number theory to show that there is no greatest prime number. The argument is similar in structure to that offered above to show that there is no greatest even number. A prime number is a whole number that is evenly divisible (which is to say, without remainder) only by 1 and itself. We might visualize a number line stretching out infinitely and containing prime numbers

scattered about its length. As we go further and further toward higher and higher numbers, we might imagine that eventually, with so many lower numbers to serve as possible even divisors that eventually the primes will give out and there will occur a greatest prime number. Imagination or conceivability by itself is thus no sure guide to the question as to whether there is or is not a greatest prime number. It is precisely here that *reductio ad absurdum* steps in to answer the question definitely in favor of the conclusion that there is no greatest prime. The argument proceeds as follows. Assume as a hypothesis for *reductio,* much as before, that there is a greatest prime number, which we may call N. Now, however, no matter how large N is, we can always produce a number greater than N by multiplying together all the primes less than N and adding 1. The reason why this operation produces a prime number is a more complicated matter. For present purposes, it is enough to appreciate the *reductio* structure of the reasoning involved in reaching the conclusion. We assume the negation of a proposition we believe to be true, and whose negation would therefore be false. Then we derive a contradiction from the hypothesis. The contradiction in this application is that N is the greatest prime number and N is not the greatest prime number, on the grounds that the above formula provides a way of using the value of N to construct a prime number greater than N, for any prime number N, no matter how large. The method in this case parallels very closely the previous example in which we assume that N is the greatest even number, for which we have an even simpler and more obvious method of constructing from N an even number larger than it, for any even number N.

3. Mathatical discovery *reductio ad absurdum*

The above illustrations of *reductio* reasoning in mathematics suggest a somewhat misleading interpretation of the method. It can appear when *reductio* arguments are presented in textbook cases that the inference pattern of indirect proof generally requires that we somehow already know what conclusion to aim at, and then we cast about for a way to prove the proposition that has been independently projected, finding a route to the conclusion by assuming the conclusion's negation and generating a contradiction. It is precisely this sort of apparent limitation that Schopenhauer exploits in defending his view of the importance of mathematical intuition versus formal demonstration. That the reality of working with *reductio ad absurdum* is also a mode of mathematical discovery rather than routine proof of foregone intuited conclusions is clear from several considerations.

First, however trustworthy our intuitions may seem even to professional practitioners with a good grasp of mathematical relations and a proven track record of accurate apprehensions of mathematical truths, it always remains an open question whether or not the intuition of a mathematical conclusion is true unless or until a formal proof can be rigorously presented. This is an aspect of the use of *reductio* reasoning in mathematics that Schopenhauer completely overlooks. Schopenhauer is right to credit mathematical intuition with important insight into where mathematical truth *might* be found, but not yet to where it actually exists in the absence of exact demonstration. It is better, all things considered, as a result, to regard mathematical intuitions as something more like the hypotheses and educated guesses that scientists make in trying to establish logically contingent truths about physical phenomena and laws of nature. The mathematician is guided by instinct and experience and possibly an *a priori* grasp of essential relations to be tested beyond mere psychological certainty by the experiment of trying to see whether or not a rigorous formal proof in support of such intuitions is actually forthcoming. The point is rather a general one, affecting the nature of all mathematical methodology, and not just *reductio* reasoning; yet the particular implication holds as well for *reductio* proof, thereby categorizing it also as a species of mathematical discovery in every practical application.[3]

We must try to imagine what any honest mathematician is compelled to conclude from the fact, if it turns out to be a fact, that a *reductio* proof is simply not available for a particular strongly intuited conclusion concerning the truth of a mathematical proposition. The situation is presumably precisely the same as with respect to the observationally or experimentally unconfirmable truth of a proposition in natural science, despite its initial plausibility. The mathematician similarly relies heavily on intuition of what looks as though it must be the case where mathematical properties and relations are concerned, but then subjects such intuitions, no matter how firm or psychologically deep-rooted, to the test of rigorous proof. If proof can be provided, well and good, and *reductio ad absurdum*, as one of the most powerful proof methods in mathematics, plays an important role in this stage of mathematical discovery. If no proof is found, the mathematician confronts the merely negative fact that the guiding intuition may or may not be true, but in any event cannot as yet be acknowledged as definitely true. More work needs to be done in that case, just as in

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[3] The analogy between hypothetical reasoning in mathematics and the natural sciences is often associated with the work of Imre Lakatos. See especially Lakatos (1976, 1978a, 1978b). See also Kampis, Kvasz, and Stoltzner (Eds.) (2002).

the parallel circumstances facing the natural scientist proposing and testing hypotheses by conjecture and refutation. A judicious mathematician might continue to express belief in the proposition's truth, but in lieu of proof it would not be responsible mathematical practice to label such a proposition that anything more than a provisionally unsubstantiated even if widely accepted supposition.

Second, as another way in which *reductio* reasoning in mathematics constitutes a method of discovery rather than mere confirmation of prescient truths, *reductio* argumentation can sometimes lead to entirely unexpected mathematical consequences. Here an appropriate analogy with experimental and experiential thinking in the natural sciences is suggested. The phenomenon in question is sometimes called serendipity. The discovery of penicillin is a shopworn but still illuminating example. Generally, what happens in serendipitous discovery is that an investigator is trying to achieve a certain result that does not work out quite as expected, but along the way, something unanticipated occurs that leads to new truths. Again, the analogy between mathematics and the natural sciences is striking. The fungus penicillium, commonly found in bread mold, was known in some form from ancient times. It was not until 1929, however, that Alexander Fleming noticed that penicillium accidentally occurring on a Petri culture dish inhibited the spread of a Staphylococcus bacterial colony. Only then was it conjectured that the mold might be producing a substance capable of checking the growth of bacteria. The important point is that there was no prior hint that bread mold might be antibacterial. If the mold and bacterium had not through sheer accident appeared on the agar-agar culture in the Petri dish, and if Fleming had not noticed the effect of penicillium on Staphylococcus, the discovery might have awaited many years or never been made. Other examples of serendipity in the natural sciences abound. The chemical synthesis of Urea, the invention of Teflon, and even the discovery of America by Christopher Columbus, if the history books have their facts straight, were all accidental discoveries of this type. Search for a sea route west to the Indies in the fifteenth century, and you might find something very different than you envisioned. The same can be true even in the formal sciences, as epitomized by the use of *reductio* reasoning in the discovery of elliptic and hyperbolic non-Euclidean mathematics.

Two nineteenth-century mathematicians, the Russian Nikolai Ivanovich Lobachevsky and the German Bernhard Riemann, discovered internally consistent systems of geometry by means of *reductio* mathematical reasoning that were different from Euclid's classical geometry. The fifth or so-called parallels postulate in Euclid's *Elements* states: 'If a straight line crossing two straight lines makes the interior angles on the same side less than

two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are less than the two right angles'. In effect, the parallels postulate says that for a line and a point in the same plane, exactly one line parallel to the line can be drawn through the point.

It had long been a question among geometers whether Euclid's fifth postulate was independent of the other four axioms in Euclid's system of twenty-three definitions, five common notions, and five axioms or postulates. A standard technique for testing the independence of any member of a set of mathematical axioms is to deny the truth of the axiom whose independence is being examined, and then check to see whether a contradiction results. The method can thus be understood as an application of *reductio*, where a proposition otherwise thought to be true is denied and a contradiction is sought as an implication of the hypothesis along with other background assumptions. In this case, the hypothesis is the negation of Euclid's fifth or parallels postulate, and the background assumptions are the remaining four axioms or postulates of Euclid's geometry.

The strategy in this application is one in which, if a contradiction obtains then, it follows the fifth or parallels postulate is proved *reductio ad absurdum*, and, as a dividend, that therefore the postulate must be independent of the other four postulates. If the fifth postulate were not independent, then putting its negation together with the remaining four postulates would not generate a contradiction. For in that event, the fifth postulate would have been proven *reductio ad absurdum* from, and hence could not be independent of, the other four postulates.

What Riemann and Lobachevsky soon discovered was that no contradiction whatsoever obtains in a *reductio* argument combining the negation of Euclid's fifth postulate with the other four. This is a negative result, and as such does not definitively establish that the fifth postulate is dependent on or independent of the first four postulates. The method at such an impasse leaves open the possibility in principle that a contradiction might still be derived within the framework consisting of the *reductio* hypothesis and other assumptions, but that no such contradiction has yet been deduced.

Previously, the Italian mathematician, Giovanni Girolamo Saccheri, had arrived at the same conclusion in his 1733 treatise, *Euclides ab Omni Naevo Vindicatus* (translatable as *Euclid Liberated from All Flaws*). Saccheri went on to explore the possibilities of elliptic and hyperbolic geometries, based on a denial of Euclid's fifth postulate. In the case of elliptic geometries, denying the fifth postulate takes the form of supposing that instead of there being just one line passing through a point outside of and parallel to a given line, there are none at all; whereas hyperbolic

geometries suppose that there are infinitely many lines parallel to the given line. Saccheri rejected elliptic geometry on the grounds that other Euclidean postulates must be revised in order to preserve consistency, and instead carried forward a kind of hyperbolic geometry up to the point where he concluded that the mathematics was logically inconsistent.

The problem of whether Euclid's fifth postulate was truly independent of the other four as a result was not entirely satisfactorily resolved. Thus the question persisted until late into the nineteenth century. Where Riemann and Lobachevsky improved upon Saccheri's discovery was in the development of alternative non-Euclidean geometries based on the failure of the *reductio* effort to prove Euclid's fifth postulate from the first four. They persevered in hammering out the problems encountered in denying the fifth postulate in order to extend new systems of geometry in which there could be zero (Lobachevsky) or infinitely many (Riemann) parallel lines through a point outside a given line. As is well known, these elegant geometries were admired but assumed to lack application until Albert Einstein found a use for them in describing the pathways of light rays passing through immense distances in space and traversing large gravitational fields.[4]

The moral of this episode in the history of mathematics for present purposes is that *reductio* reasoning in mathematics serves in at least a second important way as a mode of discovery of previously unsuspected mathematical structures, properties, relations, and theorems. Were it not for the availability of *reductio* proof as a method of establishing the independence of one axiom from a set of axioms to which it belongs, as in the case of the accidental revelation of penicillin as an antibacterial agent, there might have been no discovery of the formally remarkable and scientifically useful non-Euclidean geometries. Applying *reductio* experimentally, as in the serendipitous discovery of non-Euclidean geometries, to test for an axiom's independence from other axioms, is only one use of the method of indirect proof in mathematics. The example illustrates the general point that mathematicians use *reductio* reasoning, contrary to Schopenhauer's complaint, not merely to establish truths that are pre-demonstratively intuited, but also to discover new truths. The early mathematicians who denied Euclid's fifth or parallels postulate in order to see whether or not the axiom was provable together with the remaining four postulates by the deduction of a contradiction may have had their suspicions, but they evidently did not know with certainty in advance what the

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[4] Among many valuable sources, see, in particular, Bonola (1955), Rosenfeld (1987) and Trudeau (1987).

answer would be. As far as their prior beliefs were concerned, until they completed the experiment, the outcome could have been the very opposite. In that case, the fifth postulate would have been proven *reductio ad absurdum*, from which the postulate's logical dependence on the other four axioms would also have been discovered. Since the postulate was not, and, in retrospect, could not have been proved in this way, the failure of the *reductio* opened the door to the discovery of a field of an indefinitely large family of non-Euclidean geometries representing an important advance in mathematical knowledge.

4. Where blame for contradiction falls

It is a curiosity about *reductio* reasoning generally, and one rich with philosophical significance, especially for mathematical applications, that the contradiction obtaining in an indirect proof framework can be attributed to any of the assumptions on which the contradiction rests. Logicians sometimes speak of 'blaming' the contradiction on a particular chosen assumption in the argument, thereby reducing it to an absurdity and supporting the deduction of its negation.

Ordinarily, one expects that the contradiction in a *reductio* argument is to be blamed on the *reductio* hypothesis, but there is no imperative necessitating this choice. Thus, in the example given above, proposing to prove that there is no greatest even number N, it is equally open to a logician, upon validly deriving an explicit contradiction within the proof, to blame the contradiction on the hypothesis that there exists a greatest even number N, or on any of the other assumptions in the argument. We could in principle alternatively attribute the contradiction instead to the assumption that if N is an even number, then N + 2 is an even number greater than N. This would clearly be a difficult way to go, given the fact that the assumption reflects the meaning of the concept of an even number as a fundamental arithmetical truth. Still, it remains a possibility as far as the logical structure of *reductio ad absurdum* is concerned; blaming the contradiction on the *reductio* hypothesis is not dictated by logical principle.

In other applications of *reductio* reasoning, both in and outside of mathematics, there may be other equally acceptable choices. Take, for example, the first *reductio* argument represented above, in which the conclusion is ostensibly that Susan will not run for President. Here the intent of the argument does not remotely approach the inevitability of the *reductio* designed to prove that there is no greatest even number. As far as the internal content of the argument and one's general background knowledge is concerned, there is no compulsion to blame the contradiction on the

assumption that Susan will run for President. We could with equal *a priori* justification blame the contradiction on the background assumption (2) that if Susan will not run for President, then Mark will not run for Vice-President, or (3) that Mark will run for Vice-President. Either of these assumptions could presumably be held responsible for the contradiction rather than the hypothesis. Merely referring to this proposition as the *reductio* hypothesis and to the other premises of the argument as background assumptions cuts no logical ice. The occurrence of the contradiction shows only that the premises considered conjointly imply a contradiction. Since the contradiction is eliminated by the negation of any of the premises, whether labeled as the *reductio* hypothesis or as background assumption, the contradiction, to continue the culpability metaphor, could be considered to arise as the fault of any one of the premises.

The general methodological problem to be considered with respect to *reductio* reasoning generally and in mathematics more particularly is whether a given use of *argmentum reductio ad absurdum* by itself can ever be properly understood to determine the expected result of its application. Do we really know that Susan will not run for President, or only that either Susan will not for President or that Mark will not run for Vice-President, or that it is not the case that if Susan will not run for President, then Mark will not run for Vice-President? Moving beyond the transparency of the *reductio* proof to show that there is no greatest even number to the *reductio* proof that there is no greatest prime number, how do we know, short of further appeal to mathematical truths, themselves standing in need of formal demonstration, that there is no greatest prime number N, or merely that either there is no greatest prime number or that it is not the case that the product of all the primes less than N plus 1 is a prime number greater than N? If such a number is prime, then it is certainly greater than N, but we might be in doubt as to whether the number is prime, and whether such an algorithm is guaranteed in absolutely every case in every position along the number line to produce a prime number. The proposition can of course be independently proved, although it might be wondered even so whether the proof of this key proposition does not somehow depend on the assumption that there is no greatest prime number, thereby involving both demonstrations in a significance-depriving vicious circularity.

We leave these questions of number theory aside in order to concentrate instead on a more philosophically interesting implication. The lesson to be learned from the multiple potential blame-ability of any of the premises on which the contradiction in a *reductio* inference depends is that *reductio ad absurdum* is not what I propose to call brute-algorithmic. Rather, the application of *reductio* proof methods, wherever there is more than one premise

as hypothesis of the argument, requires decision and proficiency, and ultimately a capacity for justifying the choice of a particular assumption on which to blame a resulting contradiction that is not automatically entailed by the fact that a contradiction is validly derivable.

Effective use of *reductio ad absurdum* in this very broad sense is an art. The hypothesis for a *reductio* argument must be chosen, along with the background assumptions, and these must admit of rationalization if the proof is to hold any weight. We must know what we want to prove and assume its negation, and we must know what additional assumptions may need to be adduced in order to generate a logical contradiction. We must be prepared in a good application of *reductio* reasoning to defend these choices, and to argue that we are not overlooking an explicit or implicit assumption other than the hypothesis that may be essential to the contradiction. The method does not simply grind along mechanically, although it depends on deductive canons. It requires intelligent decision and intervention at the beginning and end, in setting up the argument at the outset and in choosing an assumption afterward to hold responsible for any contradictions derived in the course of transacting the proof.

Again, there is an instructive analogy to be made out here between the structure of *reductio* reasoning and the possibility of alternative explanations and the search for hidden factors in the interpretation of observational phenomena and scientific experimentation. The question always remains, when an experiment has been performed or an empirical observation explained, whether the implications have been rightly understood, or whether there exists another less obvious explanation lurking behind the scenes of what immediately occurs to the investigator that would have invalidated a preferred surmise. The successful application of *reductio ad absurdum* similarly requires a skillful navigation among alternative explanations of how a contradiction arises, on precisely which premises a logical inconsistency with a *reductio* argument structure ought to be attributed. The proponent of an application of *reductio* reasoning must always be prepared to argue for these choices in defense of the argument's conclusion, which in practice as in principle might be differently handled by different reasoners, should the inference come under challenge.

The art rather than brute algorithmic stature of *reductio* reasoning is reinforced by the fact that there is no formal reduction of *reductio ad absurdum* to any logical inference rule or combination of inference rules in propositional logic supportable by truth table analysis. We can formalize the logical structure of the natural deduction inference rule of *reductio* reasoning in this way:

$$\alpha \vdash \beta \wedge \neg\beta$$

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¬α

The trouble is that the inference indicator, ⊢, is not a truth functional operator. If we try to reduce the inference rule to a truth functional relation, the best we can do is something like: $[\alpha \rightarrow [\beta \wedge \neg\beta]] \rightarrow \neg\alpha$.[5] This proposition, although it is a tautology of propositional logic, does not adequately represent the logic of *reductio* reasoning. In order to detach the conclusion of a *reductio* inference, symbolized here as ¬α, the antecedent of the main conditional would need to be true. That is, it would need to be true that $\alpha \rightarrow [\beta \wedge \neg\beta]$. This subconditional, however, is only true when α is false. This requirement implies a further problem which can be expressed in two related ways. First, the formula above reduces logically to $\neg\alpha \rightarrow \neg\alpha$, in which nothing distinctive of *reductio* reasoning remains. Second, a *reductio* proof as reconstructed above is supposed to prove that ¬α is true. We have just seen that the subconditional $\alpha \rightarrow [\beta \wedge \neg\beta]$ that needs to be true in order validly to detach ¬α is itself true only when α is false. If this is the only circumstance under which ¬α can be detached, then the derivation of the contradiction is altogether irrelevant, since a conditional with a false antecedent is true by default, according to truth table definition, regardless of the form, content, or truth value of the consequent.

The implication is that truth table analysis by itself does not support the special form of reasoning and corresponding inference rule of *reductio ad absurdum*. The further implication once again is that *reductio* reasoning is a method of craft rather than a matter of brute algorithm. To the extent that correct and effective use of *reductio* inference can be mechanized or computationalized, it must simulate through expert system programming the same kind of informal judgments that intelligent logically competent human reasoners make in choosing an hypothesis together with background assumptions, formally drawing a contradictory inference from this foundation, and blaming the inference on the hypothesis, to the exclusion of any other ground of contradiction.

5. Triviality of *reductio* constructability

A final problem to be addressed in understanding *reductio* argumentation in mathematical reasoning is that of the triviality of

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[5] Lewis Carroll makes a similar point in 'What the Tortoise Said to Achilles' (1895).

reductio ad absurdum inference constructibility. Any deductively valid inference whatsoever can be reconstructed as a deductively valid *reductio* argument. What does this imply with respect to the significance of offering a *reductio* proof in support of a conclusion? Is there anything distinctive or unique about the structure of *reductio* reasoning, particularly in mathematics, as a consequence?

To appreciate the fact, first of all, that any valid inference can be properly reconstructed as a valid *reductio*, we shall not try to argue definitively by mathematical induction, or the like, but simply point toward this conclusion by studying a few characteristic examples. Consider inferences by *modus ponendo ponens*, *modus tollendo tollens*, and hypothetical syllogism. The argument that from P and if P then Q we can infer that therefore Q can be restructured as a *reductio* in this way. Suppose as a hypothesis for *reductio* that not-Q. It follows from if P then Q that therefore not-P, which contradicts the assumption that P, and on the strength of the contradiction we are entitled to conclude that the hypothesis, not-Q, is false, or simply that Q or that Q is true. Or take the argument that from not-Q and if P then Q we can infer that therefore not-P. This can be restructured as a deductively valid *reductio* inference by supposing as a hypothesis for *reductio* that P. It follows from P and if P then Q that Q, which contradicts the assumption that not-Q, and on the strength of the contradiction once again we are validly entitled to conclude the negation of the hypothesis, thereby inferring that not-P, just as in the original *modus tollendo tollens* inference. Finally, to reconstruct hypothetical syllogism as a *reductio*, assume that it is not the case that if P then R, for the inference if P then Q, if Q then R, therefore, if P then R. From not-(if P then R) it follows that P and not-R, from P and if P then Q it follows that Q, and from Q and if Q then R it follows that R. Thus, we derive the contradiction R and not-R, on the strength of which we are validly entitled to deduce the negation of the *reductio* hypothesis, not-not-(if P then R), or simply if P then R.

The point is that none of these standard forms of argument is overtly *reductio* in form because none involves the explicit deduction of a syntactical contradiction. We can nevertheless convert any such argument into a *reductio* by introducing the negation of the conclusion into an updated assumption set, and making use, just as we would in more forthrightly *reductio* arguments, of other common inference rules, in order to produce an outright contradiction, from which the original conclusion of the deductively valid inference follows as a deductively valid consequence. More generally, we can see in a classical bivalent truth functional propositional logic that if from the assumptions P,…,Q in any argument it deductively validly follows that R, then we can convert the argument P,…,Q ⊢ R to a deductively valid

reductio style inference with a modified assumption set including the hypothesis not-R, whereby ¬R, P,…,Q ⊢ R. The reason is that (classically) if P,…,Q ⊢ R, then ¬R, P,…,Q ⊢ S, ¬S (or S ∧ ¬S), for any proposition S; and from S, ¬S (or S ∧ ¬S) it deductively validly follows that R.

Given that any deductively valid inference can be converted into a deductively valid *reductio* counterpart inference, what remains of interest in *reductio* reasoning? Are we saying anything special when we say that a certain conclusion follows *reductio ad absurdum* that we could not say by appealing to another form of deductively valid logical inference? The right answer to this pertinent question has two parts, one of them obvious and the second perhaps somewhat less so. We note in the first place that just because any deductively valid inference can be morphed into a deductively valid *reductio* inference, it does not yet follow that any deductively valid *reductio* inference can similarly be redesigned as an equivalent deductively valid non-*reductio* inference. Many standard forms of argument, as we have seen, including but certainly not limited to *modus ponendo ponens*, *modus tollendo tollens*, and hypothetical syllogism, among numerous others, unlike *reductio*, are fully justified by truth table analysis. There is therefore a very clearcut formal difference between at least some *reductio* arguments and standard inferences, such that we are saying something significant when we say that a particular inference holds *reductio ad absurdum* and that a particular conclusion is derivable *reductio ad absurdum*.

More interestingly, perhaps, especially from the standpoint of argumentation theory as it applies to mathematical reasoning, the mere fact, whenever it is a fact, that a *reductio* inference is reducible to another non-*reductio* type of inference, does nothing to change the relative epistemic value of *reductio* reasoning as distinct from other standard modes of inference. Reasoning by *reductio*, a matter of art, and hence with a style of its own, as we have now argued, calls upon the reasoner to pursue specific tasks. The reasoner must look for and advance a particular *reductio* hypothesis, consider its truth value and implications against a background of other assumptions, work toward an explicit contradiction, and then arrive at the negation of the hypothesis as a conclusion. In this process, in the course of meeting the formal requirements of a *reductio* in contrast with another mode of proof, as we have dramatically seen in the serendipitous discovery by *reductio* methods of non-Euclidean geometries in the history of mathematics, logic is led to reveal truths that might otherwise have never or at least not so easily been disclosed. It is precisely in the activity of seeking a distinctively *reductio*-style proof, more or less experimentally,   to   determine   if   an   appropriate   hypothesis

supporting a contradiction can be identified, that the kinds of mathematical intuitions Schopenhauer regards as standing in opposition to proof by contradiction *reductio ad absurdum,* can often most efficiently be gained.[6]

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