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THE LAKATOSIAN METHODOLOGY IN TEACHING THE SURFACE AREA OF A CONE AND A STUDENT'S CONCEPTUAL TRANSITION

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ABSTRACT

Although many scholars support the Lakatosian method of mathematics education for enhancing students learning, the use of the method to foster students' transition from alternative to scientific conceptions in mathematics does not seem to have been a focus of any research. This study explored if the Lakatosian heuristic method of teaching could foster a students' transition from alternative to scientific conceptions of the surface area of a cone. The study used a qualitative, exploratory single case study design to undertake an in-depth study of a student in the 11th grade at a secondary school in Cyprus. Data was collected through lesson observation and analyzed using deductive content analysis. The study found that beginning from informal conjectures and proofs to more formal proofs the student was able to transit from alternative conception to the scientific conception of the surface area of a cone. The finding suggests that the Lakatosian method of mathematics can foster students' conceptual understanding of some mathematical concepts.

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1. INTRODUCTION

Students often have a misconception about the surface area of a cone. Some of them fail to see the relationship between the construction/deconstruction of a cone from a 2-D shape to a 3-D shape and vice versa. The misconceptions could be because Euclidean geometry, in general, is taught through the derivation of theorems from axioms in a deductive formalised logical approach. In contrast to the deductive approach, Lakatos' heuristic approach to mathematics education advocates the teaching and learning of mathematics through proofs and refutations. Lakatos' philosophy of mathematics tried to make a distinction between the Euclidean theories and the 'quasi-empiricists' theories. According to Lakatos (1978), "Euclidean heuristic separates the process of finding the truth and of proving it" (p.72). He claimed that the discovery of the truth has an element of guessing the necessary axioms or the appropriate statements (that have already been proved), from which we can start the process of deductive proof. The idea of "improving by proving" (Lakatos, 1976, p. 37) that is, demonstrating a conjecture via a series of gradual improvements/revisions of proofs and refutations never occurred in the Euclidean system. To support his opinion, Lakatos (1978) claimed that:

> *"The Greeks did not find a process of decision for their Geometry though they dreamt of one. However, they found a compromising solution: a heuristic procedure, which does not always produce the desired result, but which is still a heuristic rule, a standard pattern of the logic of discovery. This heuristic method was the method of analysis-synthesis". (p. 72)*

According to Lakatos (1978), a quasi-empiricist theory as a heuristic method is a method of analysis and synthesis, which he referred to as a rule. The analysis-synthesis rule consists of three parts: (a) make your conjecture, one after the other, assuming that it is true. If you reach a false conclusion, then your conjecture was false, (b) if you reach an indubitably true conclusion, your conjecture may have been true, and (c) in the case of (b), reverse the process, work backward, and try to deduce your original conjecture via the reverse route from the indubitable truth to the dubitable conjecture. If you succeed you have proved your conjecture (Lakatos, 1978). Lakatos called the first two parts (a and b) 'analysis' and the third part (c) 'synthesis'.

Lakatos strongly criticised the deductivist approach in mathematics. In the deductivist approach, definitions, axioms, and theorem statements are presented with no explanation about their development and are considered to be eternal, immutable truths (Pease et al., 2004). The deductivist approach "is constituted by a system of apodictically certain, a priori grounded Euclidean axioms, lemmas, and definitions from which theorems are deductively derived and thereby also secured as certainties" (Shaffer, 2015, p.1).

The Lakatos (1976) heuristic method is based on proofs and refutations as a method for the discovery of mathematical knowledge. Discovery is seen as a method of acquiring knowledge (Cellucci, 2020). To Lakatos, mathematics resembles natural science (Ravn & Skovsmose, 2019). His theory is that mathematics, "like the natural sciences, is fallible, not indubitable; it grows by the criticism and correction of theories, which are never entirely free of ambiguity or the possibility of error or oversight. Starting from a problem or a conjecture, there is a simultaneous search for proofs and counterexamples. New proofs explain old counterexamples, while new counterexamples undermine old proofs" (Dimitriou-Hadjichristou & Ogbonnaya, 2015, p. 186).

Lakatosian heuristic is characterized by thought (mental) experiment (quasi-experiment) (Morales-Carballo et al., 2018). Thought experiment "suggests a decomposition of the original conjecture into sub-conjectures or lemmas, thus embedding it in a possibly quite discrete body of knowledge" (Lakatos, 1976, p. 9). Thought experiment (deiknymi, 'δείκνυμι') was the most ancient pattern of mathematical proof. It prevailed in pre-Euclidean Greek mathematics according to Lakatos (1976). "Lakatos considers informal proof as just another name for thought-experiment" (Motterlini, 2002, p. 27).

In the heuristic methods, the teacher of mathematics, through proper activities and experiences, uses a combination of guided methods to help the students to become involved in 'the research processes' of finding the truth and proving it. The help/guidance is gradually minimized so that the student becomes autonomous, through continuous proof and refutation of hypotheses/conjectures. This process is adjusted by the teacher with suitable alternating questions arising from either criticism or refutation of hypotheses/conjectures, leading to the reconstruction of the initial hypotheses/conjectures in the light of new counterexamples, by the logic of proofs and refutations (Lakatos, 1976). Thus, the method is based on the quasi–

empirical system where the typical flow of the process is to bring “lies” back from the false “basic sentences” or “basic statements” (Popper, 1959, p. 78) in a down-up direction of the original hypothesis (Shaffer, 2015).

For students to transit from their alternative to the correct (scientific) conceptions, it is essential to provide alternative views that contradict their previous thinking. Students’ concepts that do not correspond with the consensus view of the scientific community are the students’ alternative conceptions (Weissová & Prokop, 2020). These alternative conceptions are not considered wrong, but rather regarded as models; perhaps in the same sense as used by scientists to simplify the complexity of a problem (Laburú & Niaz, 2002).

Studies have shown that conceptual change could be difficult to achieve (Bofferding, 2018; Lehtinen et al., 2020; Vamvakoussi, 2017). According to Chinn and Brewer (1993), students resist changes in their core beliefs (cf. ‘hard core’ as Lakatos (1970) puts it), more strongly than they resist change in the more peripheral aspects of a subject (Laburú & Niaz, 2002). For this reason, students look for an auxiliary hypothesis to defend their core beliefs. The new/alternative view must appear initially plausible to the students. Auxiliary hypotheses used by students to defend their core beliefs may provide clues and guidance for the construction of novel teaching strategies. This is based on the Lakatosian thesis that scientists do not abandon a theory based on contradictory evidence alone, and that ‘there is no falsification before the emergence of a better theory (Lakatos, 1970, p. 119). Hidden lemma (Lakatos, 1976) or “guilty lemma” was the one whose replacement leads to the most progressive problem-shift (Motterlini, 2002, p. 12). The modification of a theorem is stopped either when no more counterexamples can be found or when the theory has proved a conjecture (Pease et al., 2004, p. 13).

Euclidean geometry is taught deductively in a “proof scheme” (de Villiers, 2012, p. 1) in Cypriot secondary schools based on “accepted truths” Stylianides (2010, p. 41), which is in line with the rationalist epistemological tradition. This approach characterises the Euclidean theory (Shaffer, 2015), and it does not allow for the development of critical/creative thinking in mathematics. According to Sriraman and Mousoulides (2020), “Lakatos makes the point that this sort of Euclidean methodology is detrimental to the explanatory spirit of mathematics; it can also ignore the needs of students as they learn argumentation that constitutes a proof” (p. 2).

The Educational System in Cyprus is based on the principles of Encyclopedism which promoted teacher-centric methods. According to Persianis (1998) (cited in Karagiorgi & Symeou, 2006, p. 3), “the Greek Educational System, was influenced by the French System with its underlying epistemological tradition of encyclopaedism and its extensive centralization and uniformity”. The deductive method of teaching Euclidean geometry unlike a reliance on intuition and argumentation in mathematics promotes a lecturing method, which encourages the avoidance of in-depth discussion of the whys (European Evaluation Committee of the pre-service programme in Cyprus, 2009).

From the second author’s interaction with students as a curriculum specialist and supervisor of mathematics teaching and learning in Cypriot secondary schools, she found that geometry, in general, constitutes considerable difficulties for students. She observed that many students have a misconception about the surface area of a cone because they fail to see the relationship between the construction/deconstruction of a cone from a 2-D shape to a 3-D shape and vice versa. The objective of this study was to explore if the Lakatosian heuristic method of teaching could foster a students’ transition from alternative to scientific conceptions of surface area of a cone. The findings of this study will advance our knowledge of the Lakatosian heuristic method of teaching - proofs and refutations and how it promotes students’ conceptual learning of mathematical concepts.

2. METHOD

The study employed a qualitative exploratory single case study design (Merriam & Tisdell, 2016) This was to gain an in-depth insight into how the Lakatosian heuristic method can foster a student's transition from alternative to scientific conception of the surface area of a cone. A qualitative case study helps researchers to explore a phenomenon in a context (Rashid et al., 2019). This study is part of a larger study on the Lakatosian method of mathematics teaching in Cypriot secondary schools conducted in 2015. In this study, the Lakatosian method was used to teach the SAC to 11th grade students at a secondary school. Data was collected through lesson observation. The data for this study was from a recorded dialog between the teacher (the second author) and a student identified in this paper by the pseudonym Alexa.. Alexa was purposively and conveniently selected for the study because she was a low achiever in mathematics and lacked interest in learning mathematics. In addition, she showed misconception about the surface area of a cone and she consented that the recorded conversation she had with the teacher during the lesson be analysed in this study. Therefore, she was found suitable for the study.

In the teaching, the teacher posed the following questions to the class. It was requested that a cone-shaped tall hat be made for the junior school carnival show. Circle only one of the following shapes that is the proper one to be used for the model of the cone hat.

Data presentation involved a verbatim record of the transcribed conversation between the teacher and Alexa. Deductive content analysis (Bass & Semetko, 2021) employing Laburú and Niaz's (2002) model as the reference, was used to analyse the data to make sense of the student's transition from the alternative conception to the scientific conception of the SAC.

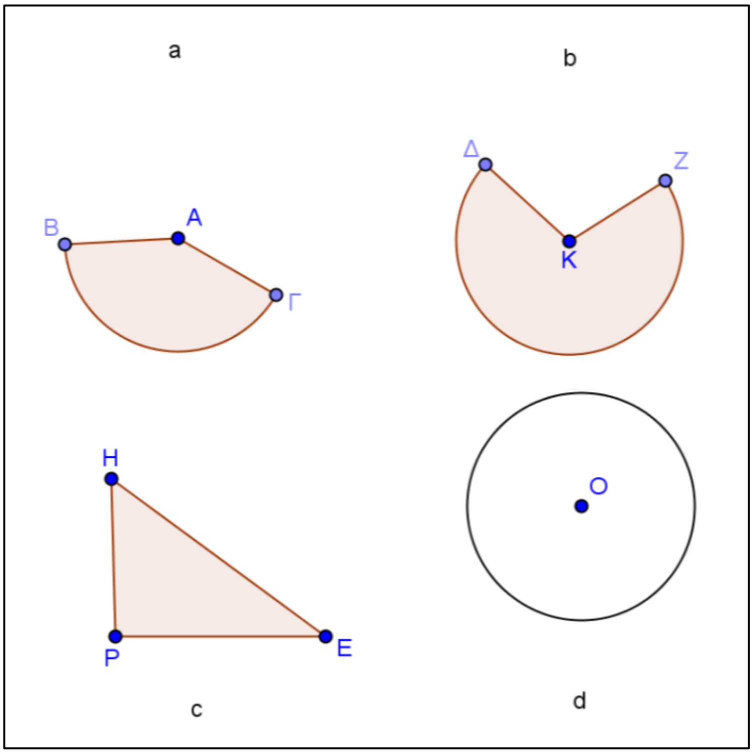

Figure 1. Task about the construction of the SAC

3. RESULTS AND DISCUSSION

3.1. Results

From the dialogue between the teacher and Alexa, we explored Alexa's transition from her alternative to scientific conceptions of a cone. The transition was analysed and presented in three phases according to Laburú and Niaz's (2002) model: from the Alternative Model (AM) to Transitory Model (TM) and finally to the Scientific Model (SM). Laburú and Niaz's (2002) model was based on the Lakatosian framework that one learns not by accepting or rejecting one single theory but by comparing one theory with another for theoretical, empirical, and heuristic progress (Lakatos, 1976).

Alternative Model (AM)

Teacher : *Looking at the question, which one of the four options do you think is the correct one?*
Alexa : *The circle. The circle because I can see only the base of the shape, I don't know how to see the shape in 3-D. I can see it only as its base because the cone has a circular base.*
Teacher : *How can it be a cone?*
Alexa : *I can hold it up from the point O, the centre of the circle, to be a cone*.

(The teacher was surprised. If Alexa had a rubber material, she could similarly think of how Lakatos (1976, p. 7) explained in his utopian class the development of a thin rubber cube in a flat network to prove the formula F+V=E+2).

Teacher : *Hm…What do you think about the triangle (Figure 1c)?*
Alexa : *I think that I see an 'empty space' from whichever angle I look at the cone there is a triangle. So, the triangle may be the answer not because this one forms a cone but because I can see it (triangle) inside a cone.*

(She refuted immediately her first answer about the circle).

Teacher : *A triangle inside a cone?*

Transitory Model (TM)

Alexa : *Yes! Like the traffic light cone in the road. It's a triangle above a circle.*
Teacher : *Please draw what you mean.*

(NB: she drew an isosceles triangle above a circle as an example of what she could "mentally see" (Hersh (1978). The shape was exactly the same as what Apollonius of Perga did and explained why students think like that which is be discussed later in this paper).

(NB: She was thinking silently by putting her hands together as a closed shape in 3-D and her hand in her attempt to 'see' the new shape in 2-D. By opening and closing her hands she realized how to construct or deconstruct a cone. She tried to convince herself, speaking aloud, that a circle and a triangle do not construct a cone. However, she was confused about what the true answer was. Her hands were reacted as a tool of a heuristic method).

Teacher : *How about the other two shapes?*
Alexa : *Shape a may be transformed into an 'empty space' (NB: she means a 'funnel' cone to be a hat) but no….it must have a base to stand on like a triangle*

(NB: She refuted herself again returning to a triangle as the correct answer. She still worked by her hands when she saw the sector as a possible answer. However, she refuted that. Finally, the teacher asked her to use a piece of paper to check if a right-angle triangle (Figure 1c) could form a cone hat. After many trials and errors, she folded the two edges H and E of a triangle. She was very surprised by the resulting extra paper which prevented the cone hat from standing on its base. She confirmed that a triangle could not construct a cone).

Teacher : *How about the shapes a and b? What's the difference between them?*
Alexa : *Yes! Shape b can also be a cone! And it can stand because I think it has a base to stand on!*
Teacher : *What do you mean?*
Alexa : *Its material (NB: area of a sector) is more than the shape a, b can stand on its base! If I connect point B with point Γ of shape a there will be a cone! The same applies to shape b if I connect point Δ and point Z there will be a cone too. I mean if I connect their radius!*
Teacher : *Excellent! Which one will be the tallest?*

(NB: The teacher believed that Alexa comprehended the concept believing that she could find the tallest cone. However, Alexa was wrong again. She thought that the radius of a sector in shape a seems to be greater than that in shape b).

Alexa : *Shape a is the tallest cone because it depends on its radius.*
Teacher : *Good. Which radius? (NB: unfortunately, Alexa means the sector radius which was the same not the base circle radius of a cone)*
Alexa : *It is AΓ or AB of the shape a, and in the shape b, it is any point of the circle to the centre O.*
Teacher : *These radii are equal as the sectors were cut from the same circle to form a cone hat!*

(NB: However, the teacher realized from her previous answer that Alexa was pseudo-conceptualizing (Vinner, 1997)).

Alexa refutes her thinking once more unable to find a new counterexample while she could not improve her proof. However, she was insisting to find the true answer.

Scientific Model (SM)

Alexa : *The shape b has more material*

(NB: She means that the sector/lateral area when it is folded forms a "funnel" cone with circle base) to form a base, which will be quite wide, and the cone can stand on its base. She demonstrates it by forming a cone using her hand as well as a piece of paper) so it won't be so tall compared to the shape a. However, because, for the shape a there is less material (i.e., sector/lateral area) to create a cone, it forces me to bring the two sides, i.e. the radii, closer together. This makes the cone (a) taller (NB: the paper hat works as a heuristic tool helping her to change her alternative conceptions about the correct shape used to construct a cone hat).

Teacher : *So?*
Alexa : *Shape b fulfills the criteria, but it is not tall.*
Teacher : *What do you mean?*
Alexa : *It is not as tall as a shape a.*
Teacher : *What bothers you about the shape (a)?*
Alexa : *The idea that I have in my mind is that shape a will not be able to stand as a shape b will.*
Teacher : *So, let's go back to the initial concern, of how it will stand.*
Alexa : *If I had a piece of paper (NB: means in her maths lessons) I could cut it to see the shape as it would actually be in reality. It would answer all of my questions. But I feel I would be cheating the process of learning because I feel I ought to be able to come up with the answer by looking at the question without having to construct it using a piece of paper (NB: she was influenced from the traditional teaching method where students cannot use the heuristic tools at all).*
Teacher : *Why do you think it is wrong to use a piece of paper?*

Alexa : *Because I feel that as a student, I am expected to have the knowledge and the ability to come up with the answer by just looking at the shape.*

Teacher : *Is it embarrassing to use a piece of paper to construct it?*

Alexa : *No, not embarrassing but it is like cheating–it is like taking a test and having an open book to look for the answers (NB: she was also influenced by the traditional teaching method where students have to learn by watching their teachers' lecturing concentrating only on the blackboard).*

3.2. Discussion

Alexa felt incapable (in her words 'stupid') due to her need to 'see' it experimentally to reason about the problem, and she thought that mathematical thinking was an abstract thinking capability that she did not possess. As the teacher and Alexa discussed the construction /deconstruction of a cone, Alexa was initially not sure of the correct answer though she guessed the correct answer twice she could not support it. Her pseudo-conceptualization (Vinner, 1997), when she said shape a is the tallest because it depends on its radius (Transitory model) could have been a hindrance to her understanding of the teacher had not asked her to explain the radius she meant. She was unable to see that the radius of a sector was equal to the lateral height of the cone. However, due to the 'improving by proving' process of the Lakatosian method she could see the hidden lemma that the tallest cone would be made from the sector the less sector material (the smaller surface area).

Though she was not able to prove the SAC, she changed her alternative conception to the scientific concept of a cone through the construct/deconstruct of a cone. By folding a sector paper, she discovered a "key" that helped her to see the relationship between the two shapes, such as the radius of a sector is the same as the lateral height of a cone.

In line with Laburú and Niaz's (2002) hypothesis, Alexa's conceptual transition progressed from Alternative Model (AM) to Transitory Model (TM), and Scientific Model (SM). The AM reflects Alexa's alternative conceptions of a cone. Alexa held strong core beliefs about the shapes in 2-D corresponding to a cone in 3-D, based on the reasoning that when she looks at a cone, she can see only the base of it or a triangle inside a cone. She believed that there is a space inside the cone, and she could not grasp how to transform a 2-D shape to a 3-D shape and vice versa, to construct/deconstruct a cone. This misconception could be because of how the students were taught the concept of a cone traditionally. By resisting change in her hard-core, Alexa attempted to invent an auxiliary hypothesis to support her belief. She had a "mental model" (Hersh, 1978) that a right-angle triangle creates a cone in 3-D when it is rotated about one of its vertical sides, in the alternative model (Laburú & Niaz, 2002) she could not imagine how the sector constructs a cone.

According to Laburú and Niaz (2002), scientists generally show resistance to changes in the hardcore of their research programmes by postulating auxiliary hypotheses. Guidance on what is to be done in the face of anomalies is provided by the positive heuristic of the programme which provides suggestions for developing the refutable parts of the research programme. In confronting anomalies, Alexa did her best to comprehend how the cone can stand on a plane: she moved to the TM. She came up with arguments, such as If I connect Β with Γ of shape a there will be a cone! The same applies to shape b if I connect Δ and Ζ there will be a cone. I mean their radius! to support her alternative hypothesis, just as the scientists built a protective belt to defend the hardcore of their research programs (Lakatos, 1970). In the face of anomalies, the positive heuristic was provided, enabling scientists to build models by ignoring "the actual counter-examples, the available data" (Lakatos, 1970, p. 135, original emphasis). The positive heuristic in this research programme is Alexa's understanding (by using experimental methods) that both shapes a and b are cones and can stand on a plane independently of their sector material, helping her to refine her theories into the SM. However, she found it difficult to understand that the height of the cone depends on its base radius. As a result, she was unable to understand the relations between the two spaces e.g., the tallest cone depends on the length of the arc of its sector and also the angle of this sector. The counterexample was posed to Alexa experimentally, helping her to refute her core belief when she realized that a triangle (Figure 1c) could not form a cone. She changed her initial conception (core belief) that the sector of a circle cannot be constructed into a cone, by posing the counterexample that an extra material (meant the segment) prevented it from standing on a base level. By cutting and folding a piece of paper she was led to the 'new knowledge' that if I connect point B and Γ of the shape a that will form a cone. She understood that a sector constructs an open cone, she realized that was its SAC in the scientific model. Even though, she was unable to prove the SAC's formula she gained important experiences during the thought experiment.

Lakatos (1970, p. 133) emphasized that "the hard-core of a program itself develops slowly by a long preliminary process of trial and error and does not emerge fully armed like Athena from the head of Zeus". In addition, according to Niaz (1998), "just as a hardcore of students' beliefs is constructed slowly any change perhaps will also follow a similar process" (p. 123). This line of thought explains why after she achieved the SM, insisted that the shape b fulfills the criterion (that the shape can be transformed into a cone), but it is not tall. She was resistant to changing her hard-core concept and insisted that the cone cannot stand on a plane, hence returning to the TM, reluctant to give up all the elements of the AM. Therefore, she built a protective belt to defend the hardcore of her 'research programme' by asserting that she would need a piece of paper to construct the cone. She claimed that she could not imagine a solid cone that was represented by one of the proposed shapes; hence she would try to construct it using a piece of paper. She still reflected on her AM by saying …the first thing that comes to my mind is the two-dimensional shapes… or If I were to draw a cone using a pencil, I would draw a triangle and a circle as a base. According to Laburú and Niaz (2002), the above outline of Alexa's thinking process confirms that at this stage (SM) "even if a student constructs it, it cannot be claimed that a change in the hard-core of students' understanding has been achieved" (p. 217).

3.3. Implications of the findings

It is clear in teaching and learning the concept of a cone that teachers must spend time with their students to construct the definition by their own words by using heuristic tools such as the cone hat to see the relationship between the solid cone's construction/deconstruction and the creation of its surfaces. According to the Lakatosian method they could use heuristic tools that can help students to resolve their misconceptions. In contrast, the visual representation of the rotation of the right-angle triangle about one of its vertical sides, on the whiteboard traditionally, prevents them from comprehending a cone in 3 dimensions.

As Wadhwa et al. (2020) observed, it is necessary for teachers to explore students' alternative conceptions because it helps teachers to understand the students' thinking and the subtle source of their conceptions. Students' core belief (cf. Lakatos, 1970, 'negative heuristic') is that the SAC in 2-D is a right-angle triangle. Another misconception is how they draw the cone in 3-D as a plane isosceles triangle above a circle. Teachers must pay attention to this in teaching. Appolonius of Perga explains this misunderstanding by constructing a cone from 3-D to 2-D, explaining to us why a cone in 2-D is constructed as a circle and a triangle above it. According to Flaumenhaft as cited in Densmore (2010), Apollonious asserted in his Proposition 3 that "If a cone is cut by a plane through the vertex, the section is a triangle" (p. 6). Apollonius was engaged in a study of lines obtained by the intersection of a cross-section of a cone, which is neither straight nor circular but closed.

The difficulty that Alexa encountered, and other students in general, in constructing or deconstructing a cone in 3-D may be due to the new kind of lines which is generated

> *by putting together the straight line and the circle to generate a conic surface–a surface that is neither flat nor spherical; and if we cut that curvy surface with a plane surface, we shall get (as the intersection of the surfaces) a kind of a curvy line (Densmore, 2010, p. xxviii).*

Apollonius as cited in Densmore (2010), examined this new kind of line concerning the cone surface area. In his first proposition, he showed that a conic surface is not wiggly in any direction: the straight lines drawn from the vertex of the conic surface to points on the surface are on that surface (Densmore, 2010, Proposition 1). Therefore, the locus of the points of a cone which are on the straight line drawn from the vertex to the base circle of a cone when they are rotated about their axis forms the SAC.

Apollonius also showed that a conic surface is like a circle in being everywhere curvy bulging outward (if you go from a point on it toward another point, but without going straight toward its vertex) [proposition 2]. Then he explained in his third proposition (Figure 2) that (by cutting through the vertex) one obtains a conic section that is straight-lined (being a triangle) and another circular conic section (by cutting to make an angle equal to one of the base angles: parallel [prop. 4] or sub contrariwise [prop. 5] (Densmore, 2010, p. xxix).

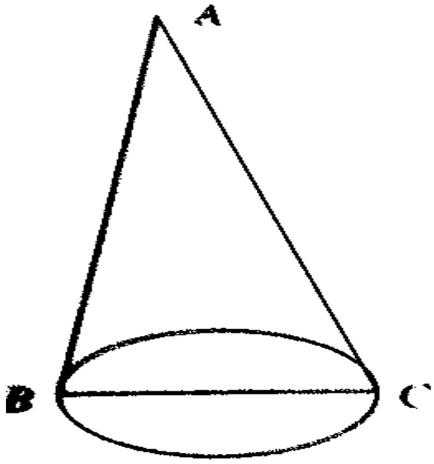

Figure 2. Apollonius Proposition 3 (Densmore, 2010, p. 7)

The new kind of 'mixed lines', as shown in Figure 2, is what exactly Alexa created in her mind to imagine a cone in a plane: the first thing that comes to my mind is the two-dimensional shapes my mind perceives when looking at a cone, just as looking at a cone in the street. For example, a circle (as a base) and a triangle (isosceles) above it. This misunderstanding (mixed-lines in 2-D) might have prevented her from realizing that a cone in 3-D is represented in 2-D, not as a cross-section of a cone by a new line together with a triangle and a circle, but as a sector of a circle. This is a result of a conceptual-embodiment (Tall, 2013) based on students' as well as other people's perception of the cone which can be overcome by using an experimental (down-up) method in teaching and learning, such as the Lakatosian method. It helps students not only to change their core beliefs but also to contribute to the understanding of the concept, and to develop problem-solving skills due to developing higher-order thinking as this study has demonstrated.

4. CONCLUSION

Students come to the classroom with knowledge or conceptions that they acquired from previous learnings and experiences. Sometimes the conceptions the students hold might be erroneous (misconceptions). In mathematics, many students have a misconception about the surface area of a cone. Some of them do not recognise the relationship between the construction/deconstruction of a cone from a 2-D shape to a 3-D shape and vice versa. This case study explored how the Lakatosian heuristic method of teaching could enhance students' transition from alternative conceptions to scientific concepts of a cone. The study showed that through the Lakatos proof and refutation approach the student was able to transit from her alternative conception to scientific concepts of the surface area of a cone. This study can be used as a reference for further studies on the effectiveness of the Lakatosian Methodology in teaching mathematics.

REFERENCES

Bass, L., & Semetko, H. A. (2021). Content Analysis. In J.-F. Morin, C. Olsson, & E. Ö. Atikcan (Eds.), *Research Methods in the Social Sciences: an AZ of Key Concepts* (Vol. 54, pp. 56-61). Oxford University Press.

Bofferding, L. (2018). Challenges of Promoting Conceptual Change with Instructional Contexts. In L. Bofferding & N. M. Wessman-Enzinger (Eds.), *Exploring the Integer Addition and Subtraction Landscape: Perspectives on Integer Thinking* (pp. 167-187). Springer International Publishing. https://doi.org/10.1007/978-3-319-90692-8_7

Cellucci, C. (2020). Reconnecting Logic with Discovery. *Topoi*, *39*(4), 869-880. https://doi.org/10.1007/s11245-017-9523-3

Chinn, C. A., & Brewer, W. F. (1993). The role of anomalous data in knowledge acquisition: A theoretical framework and implications for science instruction. *Review of Educational Research*, *63*(1), 1-49. https://doi.org/10.3102/00346543063001001

de Villiers, M. (2012). An illustration of the explanatory and discovery functions of proof. *Pythagoras*, *33*(3), 193. https://doi.org/10.4102/pythagoras.v33i3.193

Densmore, D. (2010). *Conics, books I-III*. Green Lion Press.

Dimitriou-Hadjichristou, C., & Ogbonnaya, U. I. (2015). The effect of using the lakatosian heuristic method to teach the surface area of a cone on students' achievement according to bloom's taxonomy levels. *African Journal of Research in Mathematics, Science and Technology Education*, *19*(2), 185-198. https://doi.org/10.1080/10288457.2015.1049013

European Evaluation Committee of the pre-service programme in Cyprus. (2009). *Report of Cyprus Pedagogical Institute*. Cyprus Pedagogical Institute (unpublished).

Hersh, R. (1978). Introducing imre lakatos. *The Mathematical Intelligencer*, *1*(3), 148-151. https://doi.org/10.1007/BF03023262

Karagiorgi, Y., & Symeou, L. (2006, 12-17 October 2006). Teacher professional development in focus: Reorganizing the existing in-service training scheme in Cyprus. In Conference Recreating Linkages between Theory and Praxis in Educational Leadership, Nicosia.

Laburú, C. E., & Niaz, M. (2002). A Lakatosian Framework to Analyze Situations of Cognitive Conflict and Controversy in Students' Understanding of Heat Energy and Temperature. *Journal of Science Education and Technology*, *11*(3), 211-219. https://doi.org/10.1023/A:1016064301034

Lakatos, I. (1970). Falsification and the methodology of scientific research programmes. In I. Lakatos & A. Musgrave (Eds.), *Criticism and the Growth of Knowledge*. Cambridge University Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge University Press.

Lakatos, I. (1978). *Philosophical Papers* (Vol. 54). Cambridge University Press.

Lehtinen, E., Gegenfurtner, A., Helle, L., & Säljö, R. (2020). Conceptual change in the development of visual expertise. *International Journal of Educational Research*, *100*, 101545. https://doi.org/10.1016/j.ijer.2020.101545

Merriam, S., & Tisdell, E. (2016). *Qualitative research: a guide to design and implementation*. Jossey-Bass.

Morales-Carballo, A., Locia-Espinoza, E., Ramirez-Barragan, M., Almira, J. M. S., & Mederos, O. B. (2018). The Theoretical didactic approach to the counterexample in mathematics. *International Journal of Research in Education Methodology*, *9*, 1510-1517. https://doi.org/10.24297/ijrem.v9i1.8013

Motterlini, M. (2002). Professor Lakatos Between the Hegelian Devil and the Popperian Deep Blue Sea. In G. Kampis, L. Kvasz, & M. Stöltzner (Eds.), *Appraising Lakatos: Mathematics, Methodology, and the Man* (pp. 23-52). Springer Netherlands. https://doi.org/10.1007/978-94-017-0769-5_3

Niaz, M. (1998). A lakatosian conceptual change teaching strategy based on student ability to build models with varying degrees of conceptual understanding of chemical equilibrium. *Science & Education*, *7*(2), 107-127. https://doi.org/10.1023/A:1008671632536

Pease, A., Colton, S., Smaill, A., & Lee, J. (2004). A model of Lakatos's philosophy of mathematics. In Proceedings of the European Computing and Philosophy Conference (ECAP 2004).

Persianis, P. (1998). 'Compensatory Legitimation' in Greek Educational Policy: An explanation for the abortive educational reforms in Greece in comparison with those in France. *Comparative Education*, *34*(1), 71-83. https://doi.org/10.1080/03050069828351

Popper, K. R. (1959). *The Logic of scientific discovery*. Basic Books, Inc.

Rashid, Y., Rashid, A., Warraich, M. A., Sabir, S. S., & Waseem, A. (2019). Case study method: A step-by-step guide for business researchers. *International journal of qualitative methods*, *18*, 1-13. https://doi.org/10.1177/1609406919862424

Ravn, O., & Skovsmose, O. (2019). Mathematics as Dialogue. In O. Ravn & O. Skovsmose (Eds.), *Connecting Humans to Equations : A Reinterpretation of the Philosophy of Mathematics* (pp. 107-119). Springer International Publishing. https://doi.org/10.1007/978-3-030-01337-0_8

Shaffer, M. (2015). Lakatos' quasi-empiricism in the philosophy of mathematics. *Polish Journal of Philosophy*, *9*(2), 71-80.

Sriraman, B., & Mousoulides, N. (2020). Quasi-empirical Reasoning (Lakatos). In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 703-705). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_131

Stylianides, G. (2010). Engaging secondary students in reasoning and proving. *Mathematics Teaching*, *219*, 39-44.

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Vamvakoussi, X. (2017). Using analogies to facilitate conceptual change in mathematics learning. *ZDM Mathematics Education*, *49*(4), 497-507. https://doi.org/10.1007/s11858-017-0857-5

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, *34*(2), 97-129. https://doi.org/10.1023/A:1002998529016

Wadhwa, M., Gahlawat, I. N., & Chhikara, A. (2020). Understanding students' alternative conceptions: A mirror to their thinking. *Integrated Journal of Social Sciences*, *7*(1), 9-13.

Weissová, M., & Prokop, P. (2020). Alternative conceptions of obesity and perception of obese people amongst children. *Journal of Biological Education*, *54*(5), 463-475. https://doi.org/10.1080/00219266.2019.1609549