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On unit group of finite semisimple group algebras of non-metabelian groups of order 108

Research Article

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Abstract: In this paper, we characterize the unit groups of semisimple group algebras $\mathbb{F}_qG$ of non-metabelian groups of order 108, where F_q is a field with $q = p^k$ elements for some prime $p > 3$ and positive integer k. Upto isomorphism, there are 45 groups of order 108 but only 4 of them are non-metabelian. We consider all the non-metabelian groups of order 108 and find the Wedderburn decomposition of their semisimple group algebras. And as a by-product obtain the unit groups.

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1. Introduction

Let $\mathbb{F}_q$ denote a finite field with $q = p^k$ elements for odd prime $p > 3$, G be a finite group and $\mathbb{F}_qG$ be the group algebra. The study of the unit groups of group algebras is a classical problem and has applications in cryptography [4] as well as in coding theory [5] etc. For the exploration of Lie properties of group algebras and isomorphism problems, units are very useful see, e.g. [1]. We refer to [11] for elementary definitions and results about the group algebras and [2, 14] for the abelian group algebras and their units. Recall that a group G is metabelian if there is a normal subgroup N of G such that both N and G/N are abelian. The unit groups of the finite semisimple group algebras of metabelian groups have been well studied.

In this paper, we are concerned about the unit groups of the group algebras of non-metabelian groups. Let us first mention the available literature in this direction. From [13], we know that all the groups up to order 23 are metabelian. The only non-metabelian groups of order 24 are S_4 and $SL(2, 3)$ and the unit group of their group algebras have been discussed in [7, 9]. Further, from [13], we also

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deduce that there are non-metabelian groups of order of $48, 54, 60, 72, 108$ etc. The unit groups of the group algebras of non-metabelian groups up to order 72 have been discussed in [10, 12]. The unit group of the semisimple group algebra of the non-metabelian group $SL(2, 5)$ has been discussed in [15].

The main motive of this paper is to characterize the unit groups of $\mathbb{F}_qG$, where G represents a non-metabelian group of order 108. It can be verified that, upto isomorphism, there are 45 groups of order 108 and only 4 of them are non-metabelian. We deduce the Wedderburn decomposition of group algebras of all the 4 non-metabelian groups and then characterize the respective unit groups. The rest of the paper is organized in following manner: We recall all the basic definitions and results to be needed in our work in Section 2. Our main results on the characterization of the unit groups are presented in third section and the last section includes some discussion.

2. Preliminaries

Let e denote the exponent of G, ζ a primitive e^{th} root of unity. On the lines of [3], we define

$$I_{\mathbb{F}} = \{n \mid \zeta \mapsto \zeta^n \text{ is an automorphism of } \mathbb{F}(\zeta) \text{ over } \mathbb{F}\},$$

where $\mathbb{F}$ is an arbitrary finite field. Since, the Galois group $\mathrm{Gal}\big(\mathbb{F}(\zeta), \mathbb{F}\big)$ is a cyclic group, for any $\tau \in \mathrm{Gal}\big(\mathbb{F}(\zeta), \mathbb{F}\big)$, there exists a positive integer s which is invertible modulo e such that $\tau(\zeta) = \zeta^s$. In other words, $I_{\mathbb{F}}$ is a subgroup of the multiplicative group $\mathbb{Z}_e^*$ (group of integers which are invertible with respect to multiplication modulo e). For any p-regular element $g \in G$, i.e. an element whose order is not divisible by p, let the sum of all conjugates of g be denoted by γ_g, and the cyclotomic $\mathbb{F}$-class of γ_g be denoted by $S(\gamma_g) = \{\gamma_{g^n} \mid n \in I_{\mathbb{F}}\}$. The cardinality of $S(\gamma_g)$ and the number of cyclotomic $\mathbb{F}$-classes will be incorporated later on for the characterization of the unit groups.

Next, we recall two important results from [3]. First one relates the number of cyclotomic $\mathbb{F}$-classes with the number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$. Here $J(\mathbb{F}G)$ denotes the Jacobson radical of $\mathbb{F}G$. Second one is about the cardinality of any cyclotomic $\mathbb{F}$-class in G.

Theorem 2.1. *The number of simple components of $\mathbb{F}G/J(\mathbb{F}G)$ and the number of cyclotomic $\mathbb{F}$-classes in G are equal.*

Theorem 2.2. *Let ζ be defined as above and j be the number of cyclotomic $\mathbb{F}$-classes in G. If $K_i, 1 \leq i \leq j$, are the simple components of center of $\mathbb{F}G/J(\mathbb{F}G)$ and $S_i, 1 \leq i \leq j$, are the cyclotomic $\mathbb{F}$-classes in G, then $|S_i| = [K_i : \mathbb{F}]$ for each i after suitable ordering of the indices if required.*

To determine the structure of unit group $U(\mathbb{F}G)$, we need to determine the Wedderburn decomposition of the group algebra $\mathbb{F}G$. In other words, we want to determine the simple components of $\mathbb{F}G$. Based on the existing literature, we can always claim that $\mathbb{F}$ is one of the simple component of decomposition of $\mathbb{F}G/J(\mathbb{F}G)$. The simple proof is given here for the completeness.

Lemma 2.3. *Let A_1 and A_2 denote the finite dimensional algebras over $\mathbb{F}$. Further, let A_2 be semisimple and g be an onto homomorphism between A_1 and A_2, then we must have $A_1/J(A_1) \cong A_3 + A_2$, where A_3 is some semisimple $\mathbb{F}$-algebra.*

Proof. From [6], we have $J(A_1) \subseteq \mathrm{Ker}(g)$. This means there exists $\mathbb{F}$-algebra homomorphism g_1 from $A_1/J(A_1)$ to A_2 which is also onto. In other words, we have

$$g_1 : A_1/J(A_1) \longmapsto A_2 \text{ defined by } g_1(a + J(A_1)) = g(a), \qquad a \in A_1.$$

As $A_1/J(A_1)$ is semisimple, there exists an ideal I of $A_1/J(A_1)$ such that

$$A_1/J(A_1) = \ker(g_1) \oplus I.$$

Our claim is that $I \cong A_2$. For this to prove, note that any element $a \in A_1/J(A_1)$ can be uniquely written as $a = a_1 + a_2$ where $a_1 \in \ker(g_1), a_2 \in I$. So, define

$$g_2 : A_1/J(A_1) \mapsto \ker(g_1) \oplus A_2 \text{ by } g_2(a) = (a_1, g_1(a_2)).$$

Since, $\ker(g_1)$ is a semisimple algebra over $\mathbb{F}$ and A_2 is an isomorphic $\mathbb{F}$-algebra, claim and the result holds. $\square$

Above lemma concludes that $\mathbb{F}$ is one of the simple components of $\mathbb{F}G$ provided $J(\mathbb{F}G) = 0$. Now, we recall a result which characterize the set $I_{\mathbb{F}}$ defined in the beginning of this section. For proof, see Theorem 2.21 in [8].

Theorem 2.4. *Let $\mathbb{F}$ be a finite field with prime power order q. If e is such that $\gcd(e, q) = 1$, ζ is the primitive e^{th} root of unity and $|q|$ is the order of q modulo e, then modulo e, we have $I_{\mathbb{F}} = \{1, q, q^2, \ldots, q^{|q|-1}\}$.*

Next result is Proposition 3.6.11 from [11] and is useful for the determination of commutative simple components of the group algebra $\mathbb{F}_qG$.

Theorem 2.5. *If RG is a semisimple group algebra, then $RG \cong R\big(G/G'\big) \oplus \Delta(G, G')$, where G' is the commutator subgroup of G, $R\big(G/G'\big)$ is the sum of all commutative simple components of RG, and $\Delta(G, G')$ is the sum of all others.*

We end this section by recalling a Proposition 3.6.7 from [11] which is a generalized version of the last result.

Theorem 2.6. *Let RG be a semisimple group algebra and H be a normal subgroup of G. Then $RG \cong R\big(G/H\big) \oplus \Delta(G, H)$, where $\Delta(G, H)$ is a left ideal of RG generated by the set $\{h - 1 : h \in H\}$.*

3. Unit group of $\mathbb{F}_qG$ where G is a non-metabelian group of order 108

The main objective of this section is to characterize the unit group of $\mathbb{F}_qG$ where G is a non-metabelian group of order 108. Upto isomorphism, there are 4 non-metabelian groups of order 108 namely: (1) $G_1 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4$. (2) $G_2 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4$.

(3) $G_3 = ((C_3 \times C_3) \rtimes C_3) \rtimes (C_2 \times C_2)$. (4) $G_4 = C_2 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$.

Here G_1 and G_2 are two non-isomorphic groups formed by the semi-direct product of $(C_3 \times C_3) \rtimes C_3$ and C_4 which will be clear once we discuss the presentation of these groups. We consider each of these 4 groups one by one and discuss the unit groups of their respective group algebras along with the Wedderburn decompositions in subsequent subsections. Throughout this paper, we use the notation $[x, y] = x^{-1}y^{-1}xy$.

3.1. The group $G_1 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4$

Group G_1 has the following presentation:

$$G_1 = \langle x, y, z, w, t \mid x^2y^{-1},\ [y, x],\ [z, x]z^{-1},\ [w, x]w^{-1},\ [t, x],\ y^2,\ [z, y],\ [w, y],$$

$$[t, y],\ z^3,\ [w, z]t^{-1},\ [t, z],\ w^3,\ [t, w],\ t^3\rangle.$$

Also G_1 has 20 conjugacy classes as shown in the table below.

r	e	x	y	z	w	t	xy	xt	yz	yw	yt	zw	t^2	xyt	xt^2	yzw	yt^2	z^2w	xyt^2	yz^2w
s	1	9	1	6	6	1	9	9	6	6	1	6	1	9	9	6	1	6	9	6
o	1	4	2	3	3	3	4	12	6	6	6	3	3	12	12	6	6	3	12	6

where r, s and o denote the representative of conjugacy class, size and order of the representative of the conjugacy class, respectively. From the above discussion, it is clear that exponent of G_1 is 12. Also $G_1' \cong (C_3 \times C_3) \rtimes C_3$ with $G_1/G_1' \cong C_4$. Next, we discuss the unit group of the group algebra $\mathbb{F}_q G_1$ when $p > 3$.

Theorem 3.1. *The unit group of $\mathbb{F}_q G_1$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

1. *for any p and k even or $p^k \equiv 1 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_q G_1) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^8.$$
2. *for $p^k \equiv 5 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_q G_1) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_{q^2})^4.$$
3. *for $p^k \equiv 7 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_q G_1) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^4 \oplus GL_3(\mathbb{F}_{q^2})^2.$$
4. *for $p^k \equiv 11 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_q G_1) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_{q^2})^4.$$

Proof. Since $\mathbb{F}_q G_1$ is semisimple, using Lemma 2.3 we get
$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r), \text{ for some } t \in \mathbb{Z}. \tag{1}$$
First assume that k is even which means for any prime $p > 3$, we have
$$p^k \equiv 1 \bmod 3 \text{ and } p^k \equiv 1 \bmod 4.$$
Using Chinese remainder theorem, we get $p^k \equiv 1 \bmod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_1$ as $I_{\mathbb{F}} = \{1\}$. Hence, (1), Theorems 2.1 and 2.2 imply that
$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{19} M_{n_r}(\mathbb{F}_q). \tag{2}$$
Incorporating Theorem 2.5 with $G_1' \cong (C_3 \times C_3) \rtimes C_3$ in (2) to obtain
$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus_{r=1}^{16} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{16} n_r^2. \tag{3}$$
Above equation gives the only possibility $(2^8, 3^8)$ for the values of n_r's where a^b means $(a, a, \cdots, b \text{ times})$ and therefore, (3) implies that
$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^8. \tag{4}$$

Now we consider that k is odd. We shall discuss this possibility in the following four cases:
Case 1: $p^k \equiv 1 \bmod 3$ and $p^k \equiv 1 \bmod 4$. In this case, Wedderburn decomposition is given by (4).
Case 2. $p^k \equiv 5 \bmod 12$. In this case, we have
$$S(\gamma_t) = \{\gamma_t, \gamma_{t^2}\}, \ S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xt^2}\}, \ S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yt^2}\}, \ S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xyt^2}\},$$

and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Therefore, (1) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus_{r=12}^{15} M_{n_r}(\mathbb{F}_{q^2}).$$

Since $G_1/G_1' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q^4$. This with above and Theorem 2.5 yields

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{8} n_r^2 + 2\sum_{r=9}^{12} n_r^2. \tag{5}$$

Further, consider the normal subgroup $H_1 = \langle t \rangle$ of G_1 having order 3 with $K_1 = G_1/H_1 \cong (C_3 \times C_3) \rtimes C_4$. The quotient group K_1 has 12 conjugacy classes as shown in the table below. Here elements of K_1 are cosets, for instance, $x \in K_1$ is xH_1 but we keep the same notation.

r	e	x	y	z	w	xy	yz	yw	zw	yzw	z^2w	yz^2w
s	1	9	1	2	2	9	2	2	2	2	2	2
o	1	4	2	3	3	4	6	6	3	6	3	6

It can be verified that for all the representatives g of K_1, $|S(\gamma_g)| = 1$. Therefore, from Theorems 2.1 and 2.2, we have

$$\mathbb{F}_q K_1 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{t_r}(\mathbb{F}_q),\ t_r \in \mathbb{Z}.$$

Observe that $K_1/K_1' \cong C_4$. So, Theorem 2.5 implies that

$$\mathbb{F}_q K_1 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{t_r}(\mathbb{F}_q), \text{ with } 32 = \sum_{r=1}^{8} t_r^2,\ t_r \geq 2.$$

This gives us the only choice (2^8) for values of t_r's and therefore, Theorem 2.5 and (5) yields

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 36 = \sum_{r=1}^{4} n_r^2.$$

Above leaves us with the only choice (3^4) for values of n_r's which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_{q^2})^4.$$

Case 3. $p^k \equiv 7 \bmod 12$. In this case, we have

$$S(\gamma_x) = \{\gamma_x, \gamma_{xy}\},\ S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt}\},\ S(\gamma_{xt^2}) = \{\gamma_{xt^2}, \gamma_{xyt^2}\}, \text{ and } S(\gamma_g) = \{\gamma_g\}$$

for the remaining representatives g of conjugacy classes. Therefore (1) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q) \oplus_{r=14}^{16} M_{n_r}(\mathbb{F}_{q^2}).$$

Since $G_1/G_1' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$. This with above and Theorem 2.5 yields

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{12} M_{n_r}(\mathbb{F}_q) \oplus_{r=13}^{14} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{12} n_r^2 + 2\sum_{r=13}^{14} n_r^2. \tag{6}$$

Again consider the normal subgroup H_1 of G_1. In this case, it can be verified that $|S(\gamma_g)| = 1$ for all the representatives g of K_1 except x and xy for which $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$. Therefore, employing Theorems 2.1 and 2.2 to obtain

$$\mathbb{F}_q K_1 \cong \mathbb{F}_q \oplus_{r=1}^{9} M_{t_r}(\mathbb{F}_q) \oplus M_{t_{10}}(\mathbb{F}_{q^2}),\ t_r \in \mathbb{Z}.$$

Since $K_1/K_1' \cong C_4$, above and Theorem 2.5 imply that

$$\mathbb{F}_q K_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{8} M_{t_r}(\mathbb{F}_q),\ \text{with } 32 = \sum_{r=1}^{8} t_r^2,\ t_r \geq 2.$$

This gives us the only choice (2^8) for values of t_r's. Hence, Theorem 2.6 and (6) imply that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2}),\ \text{with } 72 = \sum_{r=1}^{4} n_r^2 + 2\sum_{r=5}^{6} n_r^2.$$

Above leaves us with the only choice (3^6) for values of n_r's which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_{q^2})^2.$$

Case 4. $p^k \equiv 11 \bmod 12$. In this case, we have

$$S(\gamma_t) = \{\gamma_t, \gamma_{t^2}\},\ S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyt^2}\},\ S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yt^2}\},\ S(\gamma_{xyt}) = \{\gamma_{xyt}, \gamma_{xt^2}\},$$

$$S(\gamma_x) = \{\gamma_x, \gamma_{xy}\},\ \text{and } S(\gamma_g) = \{\gamma_g\}$$

for the remaining representatives g of conjugacy classes. Therefore, (1) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q \oplus_{r=1}^{9} M_{n_r}(\mathbb{F}_q) \oplus_{r=10}^{14} M_{n_r}(\mathbb{F}_{q^2}).$$

Since $G_1/G_1' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$. This with above and Theorem 2.5 yields

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{8} n_r^2 + 2\sum_{r=9}^{12} n_r^2. \tag{7}$$

In this case, again we have $|S(\gamma_g)| = 1$ for all representatives g of K_1 except x and xy which means the Wedderburn decomposition of $\mathbb{F}_q K_1$ is same as obtained in case 3, i.e.

$$\mathbb{F}_q K_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8.$$

Now employ Theorem 2.6 and (7) to obtain

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_{q^2}),\ \text{with } 36 = \sum_{r=1}^{4} n_r^2.$$

This leaves us with the only choice (3^4) for values of n_r's which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_1 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_{q^2})^4.$$

□

3.2. The group $G_2 = ((C_3 \times C_3) \rtimes C_3) \rtimes C_4$

Group G_2 has the following presentation:

$$G_2 = \langle x, y, z, w, t \mid x^2y^{-1},\ [y,x],\ [z,x]w^{-2},\ [w,x]t^{-1}w^{-1}z^{-2},\ [t,x],\ y^2,\ [z,y]z^{-1},$$

$$[w,y]t^{-1}w^{-1},\ [t,y],\ z^3,\ [w,z]t^{-1},\ [t,z],\ w^3,\ [t,w],\ t^3\rangle.$$

Further, G_2 has 14 conjugacy classes as shown in the table below.

r	e	x	y	z	w	t	xy	xz	xt	yw	yt	t^2	xyz	$xyzw$
s	1	9	9	12	12	1	9	9	9	9	9	1	9	9
o	1	4	2	3	3	3	4	12	12	6	6	3	12	12

From above discussion, clearly the exponent of G_2 is 12. Also $G_2' \cong (C_3 \times C_3) \rtimes C_3$ and $G_2/G_2' \cong C_4$. Next, we discuss the unit group of $\mathbb{F}_qG_2$ when $p > 3$.

Theorem 3.2. *The unit group of $\mathbb{F}_qG_2$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

1. *for any p and k even or $p^k \equiv 1 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_qG_2) \cong (\mathbb{F}_q^*)^4 \oplus GL_3(\mathbb{F}_q)^8 \oplus GL_4(\mathbb{F}_q)^2.$$
2. *for $p^k \equiv 5 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_qG_2) \cong (\mathbb{F}_q^*)^4 \oplus GL_4(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2})^4.$$
3. *for $p^k \equiv 7 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_qG_2) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_3(\mathbb{F}_q)^4 \oplus GL_4(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2})^2.$$
4. *for $p^k \equiv 11 \bmod 12$ with k odd, we have*
$$U(\mathbb{F}_qG_2) \cong (\mathbb{F}_q^*)^2 \oplus \mathbb{F}_{q^2}^* \oplus GL_4(\mathbb{F}_q)^2 \oplus GL_3(\mathbb{F}_{q^2})^4.$$

Proof. Since $\mathbb{F}_qG_2$ is semisimple, we have

$$\mathbb{F}_qG_2 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r), \text{ for some } t \in \mathbb{Z}. \tag{8}$$

First assume that k is even which means for any prime $p > 3$, $p^k \equiv 1 \bmod 12$. This means $|S(\gamma_g)| = 1$ for each $g \in G_2$. Hence, (8), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_qG_2 \cong \mathbb{F}_q \oplus_{r=1}^{13} M_{n_r}(\mathbb{F}_q).$$

Using Theorem 2.5 with $G_2' \cong (C_3 \times C_3) \rtimes C_3$ to obtain

$$\mathbb{F}_qG_2 \cong \mathbb{F}_q^4 \oplus_{r=1}^{10} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{10} n_r^2. \tag{9}$$

Above equation gives us four possibilities $(2^8, 6^2)$, $(2^5, 3^2, 4, 5^2)$, $(2^4, 3^4, 4, 6)$ and $(3^8, 4^2)$ for the values of n_r's. Further, consider the normal subgroup $H_2 = \langle t \rangle$ of G_2 having order 3 with $K_2 = G_2/H_2 \cong (C_3 \times C_3) \rtimes C_4$. It can be verified that K_2 has 6 conjugacy classes as shown in the table below.

r	e	x	y	z	w	xy
s	1	9	9	4	4	9
o	1	4	2	3	3	4

Also, for all the representatives g of K_2, $|S(\gamma_g)| = 1$ which means by Theorems 2.1 and 2.2, we have

$$\mathbb{F}_q K_2 \cong \mathbb{F}_q \oplus_{r=1}^{5} M_{t_r}(\mathbb{F}_q),\ t_r \in \mathbb{Z}.$$

Observe that $K_2/K_2' \cong C_4$. This with above and Theorem 2.4 imply that

$$\mathbb{F}_q K_2 \cong \mathbb{F}_q^4 \oplus_{r=1}^{2} M_{t_r}(\mathbb{F}_q),\ \text{with } 32 = \sum_{r=1}^{2} t_r^2,\ t_r \geq 2.$$

This gives us the only choice (4^2) for values of t_r's. Therefore, Theorem 2.6 and (9) imply that $(3^8, 4^2)$ is the correct choice for values of n_r's and therefore, we have

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus M_3(\mathbb{F}_q)^8 \oplus M_4(\mathbb{F}_q)^2. \tag{10}$$

Now we consider that k is odd. We shall discuss this possibility in the following four cases:
Case 1: $p^k \equiv 1 \bmod 12$. In this case, Wedderburn decomposition is given by (10).
Case 2. $p^k \equiv 5 \bmod 12$. In this case, we have

$$S(\gamma_t) = \{\gamma_t, \gamma_{t^2}\},\ S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xt}\},\ S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yt}\},\ S(\gamma_{xyz}) = \{\gamma_{xyz}, \gamma_{xyzw}\},$$

and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representatives g of conjugacy classes. Therefore, (8) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus_{r=1}^{5} M_{n_r}(\mathbb{F}_q) \oplus_{r=6}^{9} M_{n_r}(\mathbb{F}_{q^2}).$$

Since $G_2/G_2' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q^4$. This with above and Theorem 2.5 yields

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus_{r=3}^{6} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{2} n_r^2 + 2\sum_{r=3}^{6} n_r^2. \tag{11}$$

Further, again consider the normal subgroup $H_2 = \langle t \rangle$ of G_2. It can be verified that for all the representatives g of K_2, $|S(\gamma_g)| = 1$ which means (as earlier)

$$\mathbb{F}_q K_2 \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2.$$

This with Theorem 2.6 and (11) imply that

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 36 = \sum_{r=1}^{4} n_r^2.$$

Above leaves us with the only choice (3^4) for values of n_r's which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^4 \oplus M_4(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2})^4.$$

Case 3. $p^k \equiv 7 \bmod 12$. In this case, we have

$$S(\gamma_x) = \{\gamma_x, \gamma_{xy}\},\ S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xyzw}\},\ S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyz}\},\ \text{and } S(\gamma_g) = \{\gamma_g\}$$

for the remaining representatives g of conjugacy classes. Therefore, (8) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q) \oplus_{r=8}^{10} M_{n_r}(\mathbb{F}_{q^2}).$$

Since $G_2/G_2' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$. This with Theorem 2.5 yields

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{6} M_{n_r}(\mathbb{F}_q) \oplus_{r=7}^{8} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{6} n_r^2 + 2\sum_{r=7}^{8} n_r^2. \tag{12}$$

Further, it can be verified that $|S(\gamma_g)| = 1$ for all the representatives g of K_2 except x and xy. For these, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xy}\}$ which means by Theorems 2.1 and 2.2, we have

$$\mathbb{F}_q K_2 \cong \mathbb{F}_q \oplus_{r=1}^{3} M_{t_r}(\mathbb{F}_q) \oplus M_{t_4}(\mathbb{F}_{q^2}),\ t_r \in \mathbb{Z}.$$

Incorporating $K_2/K_2' \cong C_4$ with Theorem 2.5 to obtain

$$\mathbb{F}_q K_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{2} M_{t_r}(\mathbb{F}_q),\ \text{with } 32 = \sum_{r=1}^{2} t_r^2,\ t_r \geq 2.$$

This gives us only choice (4^2) for values of t_r's. Therefore, Theorem 2.6 and (12) yields

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_q) \oplus_{r=5}^{6} M_{n_r}(\mathbb{F}_{q^2}),\ \text{with } 72 = \sum_{r=1}^{4} n_r^2 + 2\sum_{r=5}^{6} n_r^2.$$

Above leaves us with the only choice (3^6) for values of n_r' which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_q)^4 \oplus M_3(\mathbb{F}_{q^2})^2.$$

Case 4. $p^k \equiv 11 \bmod 12$. In this case, we have

$$S(\gamma_t) = \{\gamma_t, \gamma_{t^2}\},\ S(\gamma_x) = \{\gamma_x, \gamma_{xy}\},\ S(\gamma_{xz}) = \{\gamma_{xz}, \gamma_{xyz}\},\ S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xyzw}\},$$

$$S(\gamma_{yw}) = \{\gamma_{yw}, \gamma_{yt}\},\ \text{and } S(\gamma_g) = \{\gamma_g\}$$

for the remaining representatives g of conjugacy classes. Therefore, (8) and Theorems 2.1, 2.2 imply that

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q \oplus_{r=1}^{3} M_{n_r}(\mathbb{F}_q) \oplus_{r=4}^{8} M_{n_r}(\mathbb{F}_{q^2}).$$

Since $G_2/G_2' \cong C_4$, we have $\mathbb{F}_q C_4 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}$. This with above and Theorem 2.5 yields

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus_{r=1}^{2} M_{n_r}(\mathbb{F}_q) \oplus_{r=3}^{6} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{2} n_r^2 + 2\sum_{r=3}^{6} n_r^2. \tag{13}$$

Further, it can be verified that $|S(\gamma_g)| = 1$ for all the representatives g of K_2 except x and xy which means (as in case 3),

$$\mathbb{F}_q K_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2.$$

Now employ Theorem 2.6 and (13) to obtain

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_{q^2}),\ \text{with } 36 = \sum_{r=1}^{4} n_r^2.$$

Above leaves us with the only choice (3^4) for values of n_r' which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_2 \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2} \oplus M_4(\mathbb{F}_q)^2 \oplus M_3(\mathbb{F}_{q^2})^4.$$

□

3.3. The group $G_3 = ((C_3 \times C_3) \rtimes C_3) \rtimes (C_2 \times C_2)$

Group G_3 has the following presentation:

$$G_3 = \langle x, y, z, w, t \mid x^2,\ [y, x],\ [z, x],\ [w, x]w^{-1},\ [t, x]t^{-1},\ y^2,\ [z, y]z^{-1},\ [w, y],\ [t, y]t^{-1},$$

$$z^3,\ [w, z]t^{-1},\ [t, z],\ w^3,\ [t, w],\ t^3\rangle.$$

Further, G_3 has 11 conjugacy classes as shown in the table below.

rep	1	x	y	z	w	t	xy	xz	yw	zw	xyt
size of class	1	9	9	6	6	2	9	18	18	12	18
order of rep	1	2	2	3	3	3	2	6	6	3	6

From above discussion, clearly the exponent of G_3 is 6. Also $G_3' \cong (C_3 \times C_3) \rtimes C_3$ with $G_3/G_3' \cong C_2 \times C_2$. Next, we discuss the unit group of $\mathbb{F}_q G_3$ when $p > 3$.

Theorem 3.3. *The unit group of $\mathbb{F}_q G_3$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

$$U(\mathbb{F}_q G_3) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^4 \oplus GL_4(\mathbb{F}_q) \oplus GL_6(\mathbb{F}_q)^2.$$

Proof. Since $\mathbb{F}_q G_3$ is semisimple, we have

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r), \text{ for some } t \in \mathbb{Z}. \tag{14}$$

First assume that k is even which means for any prime $p > 3$, $p^k \equiv 1 \bmod 6$. This means $|S(\gamma_g)| = 1$ for each $g \in G_3$. Hence, (14), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q \oplus_{r=1}^{10} M_{n_r}(\mathbb{F}_q).$$

Incorporating Theorem 2.5 with $G_3' \cong (C_3 \times C_3) \rtimes C_3$ in above to obtain

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \oplus_{r=1}^{7} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{7} n_r^2. \tag{15}$$

Above equation gives us four possibilities $(2^4, 4, 6^2)$, $(2^3, 3^2, 5, 7)$, $(2, 3^2, 4^2, 5^2)$ and $(3^4, 4^2, 6)$ for the values of n_r's. Further, consider the normal subgroup $H_3 = \langle t \rangle$ of G_3 having order 3 with $K_3 = G_3/H_3 \cong S_3 \times S_3$. It can be verified that K_3 has 9 conjugacy classes as shown in the table below.

r	e	x	y	z	w	xy	xz	yw	zw
s	1	3	3	2	2	9	6	6	4
o	1	2	2	3	3	2	6	6	3

Further, for all the representatives g of K_3, $|S(\gamma_g)| = 1$ which means by Theorems 2.1 and 2.2, we have

$$\mathbb{F}_q K_3 \cong \mathbb{F}_q \oplus_{r=1}^{8} M_{t_r}(\mathbb{F}_q),\ t_r \in \mathbb{Z}.$$

Observe that $K_3/K_3' \cong C_2 \times C_2$. So, above and Theorem 2.5 imply that

$$\mathbb{F}_q K_3 \cong \mathbb{F}_q^4 \oplus_{r=1}^{5} M_{t_r}(\mathbb{F}_q), \text{ with } 32 = \sum_{r=1}^{5} t_r^2,\ t_r \geq 2.$$

This gives us the only choice $(2^4, 4)$ for values of t_r's. Therefore, from Theorem 2.6 and (15), we conclude that $(2^4, 4, 6^2)$ is the correct choice for n_r's which means

$$\mathbb{F}_q G_3 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^4 \oplus M_4(\mathbb{F}_q) \oplus M_6(\mathbb{F}_q)^2. \tag{16}$$

Now we consider that k is odd. We shall discuss this possibility in the following two cases:
Case 1: $p^k \equiv 1 \bmod 6$. In this case, Wedderburn decomposition is given by (16).
Case 2. $p^k \equiv 5 \bmod 6$. In this case, we have $S(\gamma_g) = \{\gamma_g\}$ for all the representatives g of conjugacy classes. Therefore, Wedderburn decomposition is again given by (16). ☐

3.4. The group $G_4 = C_2 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2$

Group G_4 has the following presentation:

$$G_4 = \langle x, y, z, w, t \mid x^2,\ [y, x],\ [z, x]z^{-1},\ [w, x]w^{-1},\ [t, x],\ y^2,\ [z, y],\ [w, y],\ [t, y]t^{-1},\ z^3,$$
$$[w, z]t^{-1},\ [t, z],\ w^3,\ [t, w],\ t^3\rangle.$$

Further, G_4 has 20 conjugacy classes as shown in the table below.

r	e	x	y	z	w	t	xy	xt	yz	yw	yt	zw	t^2	xyt	xt^2	yzw	yt^2	z^2w	xyt^2	yz^2w
s	1	9	1	6	6	1	9	9	6	6	1	6	1	9	9	6	1	6	9	6
o	1	2	2	3	3	3	2	6	6	6	6	3	3	6	6	6	6	3	6	6

From above discussion, clearly the exponent of G_4 is 6. Also $G_4' \cong (C_3 \times C_3) \rtimes C_3$ with $G_4/G_4' \cong C_2 \times C_2$. Next, we discuss the unit group of $\mathbb{F}_q G_4$ when $p > 3$.

Theorem 3.4. *The unit group of $\mathbb{F}_q G_4$, for $q = p^k$, $p > 3$ where $\mathbb{F}_q$ is a finite field having $q = p^k$ elements is as follows:*

1. *for any p and k even or $p^k \equiv 1 \bmod 6$ with k odd, we have*
$$U(\mathbb{F}_q G_4) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_q)^8.$$

2. *for $p^k \equiv 5 \bmod 6$ with k odd, we have*
$$U(\mathbb{F}_q G_4) \cong (\mathbb{F}_q^*)^4 \oplus GL_2(\mathbb{F}_q)^8 \oplus GL_3(\mathbb{F}_{q^2})^4.$$

Proof. Since $\mathbb{F}_q G_4$ is semisimple, we have

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \oplus_{r=1}^{t-1} M_{n_r}(\mathbb{F}_r), \text{ for some } t \in \mathbb{Z}. \tag{17}$$

First assume that k is even which means for any prime $p > 3$, $p^k \equiv 1 \bmod 6$. This means $|S(\gamma_g)| = 1$ for each $g \in G_4$. Hence, (17), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \oplus_{r=1}^{19} M_{n_r}(\mathbb{F}_q).$$

Using Theorem 2.5 with $G_4' \cong (C_3 \times C_3) \rtimes C_3$ in above to obtain

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^4 \oplus_{r=1}^{16} M_{n_r}(\mathbb{F}_q), \text{ where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{16} n_r^2. \tag{18}$$

Above equation gives us the only possibility $(2^8, 3^8)$ for values of n_r's which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_q)^8. \tag{19}$$

Now we consider that k is odd. We shall discuss this possibility in the following two cases:
Case 1: $p^k \equiv 1 \mod 6$. In this case, Wedderburn decomposition is given by (19).
Case 2. $p^k \equiv 5 \mod 6$. In this case, we have

$$S(\gamma_t) = \{\gamma_t, \gamma_{t^2}\},\ S(\gamma_{xt}) = \{\gamma_{xt}, \gamma_{xt^2}\},\ S(\gamma_{yt}) = \{\gamma_{yt}, \gamma_{yt^2}\},\ S(\gamma_{xyt}) = \{\gamma_{xyt},\ \gamma_{xyt^2}\},$$

and $S(\gamma_g) = \{\gamma_g\}$ for all the remaining representatives g of conjugacy classes. Hence, (17), Theorems 2.1 and 2.2 imply that

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{n_r}(\mathbb{F}_q) \oplus_{r=12}^{15} M_{n_r}(\mathbb{F}_{q^2}).$$

Above with Theorem 2.5 yields

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{n_r}(\mathbb{F}_q) \oplus_{r=9}^{12} M_{n_r}(\mathbb{F}_{q^2}),\ \text{where } n_r \geq 2 \text{ with } 104 = \sum_{r=1}^{8} n_r^2 + 2\sum_{r=9}^{12} n_r^2. \tag{20}$$

Further, consider the normal subgroup $H_4 = \langle t \rangle$ of G_4 having order 3 with $K_4 = G_4/H_4 \cong C_2 \times ((C_3 \times C_3) \rtimes C_2)$. It can be verified that K_4 has 12 conjugacy classes as shown in the table below.

r	e	x	y	z	w	xy	yz	yw	zw	yzw	z^2w	yz^2w
s	1	9	1	2	2	9	2	2	2	2	2	2
o	1	2	2	3	3	2	6	6	3	6	3	6

It can be seen that for all the representatives g of K_4, $|S(\gamma_g)| = 1$ which means by Theorems 2.1 and 2.2, we have

$$\mathbb{F}_q K_4 \cong \mathbb{F}_q \oplus_{r=1}^{11} M_{t_r}(\mathbb{F}_q),\ t_r \in \mathbb{Z}.$$

Observe that $K_4/K_4' \cong C_2 \times C_2$. This with above and Theorem 2.5 imply that

$$\mathbb{F}_q K_4 \cong \mathbb{F}_q^4 \oplus_{r=1}^{8} M_{t_r}(\mathbb{F}_q),\ \text{with } 32 = \sum_{r=1}^{8} t_r^2,\ t_r \geq 2.$$

This gives us the only choice (2^8) for values of t_r's. Therefore, Theorem 2.6 and (20) yields

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus_{r=1}^{4} M_{n_r}(\mathbb{F}_{q^2}),\ n_r \geq 2 \text{ with } 36 = \sum_{r=1}^{4} n_r^2.$$

Above leaves us with the only choice (3^4) for values of n_r's which means the required Wedderburn decomposition is

$$\mathbb{F}_q G_4 \cong \mathbb{F}_q^4 \oplus M_2(\mathbb{F}_q)^8 \oplus M_3(\mathbb{F}_{q^2})^4.$$

□

4. Discussion

We have characterized the unit groups of semisimple group algebras of 4 non-metabelian groups having order 108 and the results are verified using GAP. Clearly, the complexity in the calculation of Wedderburn decomposition upsurges with the increase in order of the group and we need to look into the Wedderburn decompositions of the quotient groups. The technique used for obtaining the Wedderburn decomposition works well provided the group has non-trivial normal subgroups of small order.

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