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Cordiality of digraphs

Research Article

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Abstract: A $(0,1)$-labelling of a set is said to be *friendly* if approximately one half the elements of the set are labelled 0 and one half labelled 1. Let g be a labelling of the edge set of a graph that is induced by a labelling f of the vertex set. If both g and f are friendly then g is said to be a *cordial* labelling of the graph. We extend this concept to directed graphs and investigate the cordiality of sets of directed graphs. We investigate a specific type of cordiality on digraphs, a restriction of quasigroup-cordiality called $(2,3)$-cordiality. A directed graph is $(2,3)$-cordial if there is a friendly labelling f of the vertex set which induces a $(1,-1,0)$-labelling of the arc set g such that about one third of the arcs are labelled 1, about one third labelled -1 and about one third labelled 0. In particular we determine which tournaments are $(2,3)$-cordial, which orientations of the n-wheel are $(2,3)$-cordial, and which orientations of the $n-$fan are $(2,3)$-cordial.

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Keywords: Tournament, Wheel graph, Fan graph, $(2,3)$-cordial

1. Introduction

The study of cordial graphs began in 1987 with an article by I. Cahit [2]: "Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs". In 1991, Hovey [3] generalized this concept to $\mathcal{A}$-cordial graphs where $\mathcal{A}$ is an abelian group. A further generalization, one that included cordiality of directed graphs, appeared in 2012 with an article by Pechenik and Wise [4], where the $\mathcal{A}$ was allowed to be any quasi group, not necessarily Abelian. We modify this concept to $(A,\mathcal{A})$-cordial digraphs where A is a subset of the quasigroup $\mathcal{A}$.

Let $G=(V,E)$ be an undirected graph with vertex set V and edge set E. A $(0,1)$-labelling of the vertex set is a mapping $f: V \to \{0,1\}$ and is said to be *friendly* if approximately one half of the vertices are labelled 0 and the others labelled 1. An induced labelling of the edge set is a mapping $g: E \to \{0,1\}$ where for an edge $uv, g(uv) = \hat{g}(f(u), f(v))$ for some $\hat{g}: \{0,1\}\times\{0,1\} \to \{0,1\}$ and is said to be cordial

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if f is friendly and about one half the edges of G are labelled 0 and the others labelled 1. A graph, G, is called *cordial* if there exists a cordial induced labelling of the edge set of G. In this article, as in [1], we define a cordial labelling of directed graphs that is not merely a cordial labelling of the underlying undirected graph.

A specific type of $(A, \mathcal{A})$-cordial digraph is a $(2,3)$-cordial digraph defined by Beasley in [1]. Let $D = (V, A)$ be a directed graph with vertex set V and arc set A. Let $f : V \to \{0, 1\}$ be a friendly vertex labelling and let g be the induced labelling of the arc set, $g : A \to \{0, 1, -1\}$ where for an arc $\overrightarrow{uv}, g(\overrightarrow{uv}) = f(v) - f(u)$. The labellings f and g are $(2,3)$-*cordial* if f is friendly and about one third the arcs of D are labelled 1, one third are labelled -1 and one third labelled 0. A digraph, D, is called $(2,3)$-*cordial* if there exist $(2,3)$-cordial labellings f of the vertex set and g of the arc set of D.

Note that here and what follows, the term "about" when talking about fractions of a quantity we shall mean as close is possible in integral arithmetic, so about half of 9 is either 4 or 5, but not 3 or 6.

2. Preliminaries

Definition 2.1. *A* quasigroup *is a set $\mathcal{Q}$ with binary operation $\circ$ such that given any $a, b \in \mathcal{Q}$ there are $x, y \in \mathcal{Q}$ such that $a \circ x = b$ and $y \circ a = b$.*

Fact: *All two element quasigroups are Abelian.*

Proof. Suppose that $\mathcal{Q} = \{a, b\}$ is a quasi group with binary operation $\circ$. Then, there are $x, y \in \mathcal{Q}$ such that $a \circ x = a$ and $y \circ a = a$. If $x = y = b$ then $\mathcal{Q}$ is Abelian. Otherwise, we must have $a \circ a = a$. Similarly either $\mathcal{Q}$ is abelian or $b \circ b = b$.

Now, suppose that $a \circ a = a$ and $b \circ b = b$. Then there are $c, d \in \mathcal{Q}$ such that $a \circ d = b$ and $c \circ a = b$. Since $a \circ a = a$., we must have that both $c = b$ and $d = b$. That is $\mathcal{Q}$ is Abelian. ■

We now formalize the terms mentioned in the introduction. We let $\mathbb{Z}_k$ denote the set of integers $\{0, 1, \ldots, k\}$ with arithmetic modulo k. Further let $\mathbb{Z}_k^-$ denote the set $\mathbb{Z}_k$ with binary operation "$-$", subtraction modulo k. Clearly, for $k \geq 3$, $\mathbb{Z}_k^-$ is a nonabelian quasigroup.

Definition 2.2. *A $\mathbb{Z}_k$-labelling (or simply a k-labelling) of a finite set, $\mathcal{X}$, is a mapping $f : \mathcal{X} \to \mathbb{Z}_k$ and is said to be* <u>friendly</u> *if the labelling is evenly distributed over $\mathbb{Z}_k$, that is, given any $i, j \in \mathbb{Z}_k$, $-1 \leq |f^{-1}(i)| - |f^{-1}(j)| \leq 1$ where $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$.*

Definition 2.3. *Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, and let f be a friendly $(0,1)$-labelling of the vertex set V. Given this friendly vertex labelling f, an induced $(0,1)$-labelling of the edge set is a mapping $g : E \to \{0, 1\}$ where for an edge uv, $g(uv) = \hat{g}(f(u), f(v))$ for some $\hat{g} : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ and is said to be* <u>cordial</u> *if g is also friendly, that is about one half the edges of G are labelled 0 and the others are labelled 1, or $-1 \leq |g^{-1}(0)| - |g^{-1}(1)| \leq 1$. A graph, G, is called* <u>cordial</u> *if there exists a induced cordial labelling of the edge set of G.*

The induced labelling g in a cordial graph is usually $g(u, v) = \hat{g}(f(u), f(v)) = |f(v) - f(u)|$ [2], $g(u, v) = \hat{g}(f(u), f(v)) = f(v) + f(u)$ (in $\mathbb{Z}_2$) [3], or $g(u, v) = \hat{g}(f(u), f(v)) = f(v)f(u)$ (product cordiality) [5].

In [3], Hovey introduced $\mathcal{A}$-friendly labellings where $\mathcal{A}$ is an Abelian group. A labelling $f : V(G) \to \mathcal{A}$ is said to be <u>$\mathcal{A}$-*friendly*</u> if given any $a, b \in \mathcal{A}$, $-1 \leq |f^{-1}(b)| - |f^{-1}(a)| \leq 1$. If g is an induced edge labelling and f and g are both $\mathcal{A}$-friendly Then g is said to be an $\mathcal{A}$-cordial labelling and G is said to be $\mathcal{A}$-cordial. When $\mathcal{A} = \mathbb{Z}_k$ we say that G is k-cordial. We shall use this concept with digraphs.

Given an undirected graph or a digraph, G, let $\hat{G}$ denote the subgraph (or subdigraph) of G induced by its nonisolated vertices. So $\hat{G}$ never has an isolated vertex. The need for this will become apparent in Example 3.4.

In this article, we will be concerned mainly with digraphs. We let $\mathcal{D}_n$ denote the set of all simple directed graphs on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Note that the arc set of members of $\mathcal{D}_n$ may contain digons, a pair of arcs between two vertices each directed opposite from the other. We shall let $\mathcal{T}_n$ denote the set of all subdigraphs of a tournament digraph. So the members of $\mathcal{T}_n$ contain no digons. Let $D \in \mathcal{D}_n$, $D = (V, A)$ where A is the arc set of D. Then D has no loops, and no multiple arcs. An arc in D directed from vertex u to vertex v will be denoted $\overrightarrow{uv}, \overleftarrow{vu}$ or by the ordered pair (u, v). We also let $\mathcal{G}_n$ denote the set of all simple undirected graphs on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. So all members of $\mathcal{T}_n$ are orientations of graphs in $\mathcal{G}_n$.

In [4], Pechenik and Wise introduced quasigroup cordiality. When the quasigroup is nonabelian, this type of cordiality is quite suitable for studying labellings of directed graphs. In fact, if $\mathcal{Q}$ is a quasigroup with any binary operation $\circ$ with the property that for any $a, b \in \mathcal{Q}$ $a \circ b = b \circ a$ if and only if $a = b$, we have the best situation for directed graphs.

Now the set $\mathbb{Z}_k$ with binary operation $\circ$ where for $a, b \in \mathbb{Z}_k$, $a \circ b = (b - a) \bmod k$ is such a quasigroup.

In our investigations we make one further restriction: we will label our vertices with only a subset of $\mathcal{Q}$, not necessarily the whole set $\mathcal{Q}$:

Definition 2.4. *Let $\mathcal{Q}$ be a quasigroup with binary operation $\circ$ and let $\mathbb{Q}$ be a subset of $\mathcal{Q}$. Let $D = (V, A)$ be a directed graph with vertex set V and arc set A. Let $f : V \to \mathbb{Q}$ be a friendly $\mathbb{Q}$-labelling of V and let $g : A \to \mathcal{Q}$ be an induced arc labelling where for $\overrightarrow{uv} \in A$, $g(\overrightarrow{uv}) = \hat{g}(f(u), f(v))$ for some $\hat{g} : \mathbb{Q} \times \mathbb{Q} \to \mathcal{Q}$. The mapping g is said to be* $(\mathbb{Q}, \mathcal{Q})$-cordial *if g is also friendly, that is, given any $a, b \in \mathcal{Q}$, $-1 \leq |g^{-1}(a)| - |g^{-1}(b)| \leq 1$. A directed graph, D, is called* $(\mathbb{Q}, \mathcal{Q})$-cordial *if there exists a induced $(\mathbb{Q}, \mathcal{Q})$-cordial labelling of the arc set of D.*

We now shall restrict our attention to the smallest case of $(\mathbb{Q}, \mathcal{Q})$-cordiality that is appropriate for directed graphs, $(\mathbb{Z}_2, \mathbb{Z}_3^-)$-cordiality, that defined by Beasley in [1], $(2, 3)$-cordiality.

3. $(2, 3)$-orientable digraphs

Let $D = (V, A)$ be a directed graph with vertex set V and arc set A. Let $f : V \to \{0, 1\}$ be a friendly labelling of the vertices of D. As for undirected graphs, an induced labelling of the arc set is a mapping $g : A \to \mathcal{X}$ for some set $\mathcal{X}$ where for an arc $(u, v) = \overrightarrow{uv}, g(u, v) = \hat{g}(f(u), f(v))$ for some $\hat{g} : \{0, 1\} \times \{0, 1\} \to \mathcal{X}$. As we are dealing with directed graphs, it would be desirable for the induced labelling to distinguish between the label of the arc (u, v) and the label of the arc (v, u), otherwise, the labelling would be an induced labelling of the underlying undirected graph. If we let $\mathcal{X} = \{-1, 0, 1\}$ and $\hat{g}(f(u), f(v)) = f(v) - f(u)$ using real arithmetic, or arithmetic in $\mathbb{Z}_3$, we have an asymmetric labelling. In this case, if about one third of the arcs are labelled 0, about one third of the arcs are labelled 1 and about one third of the arcs are labelled -1 we say that the labelling is $(2, 3)$-cordial. Formally:

Definition 3.1. *Let $D \in \mathcal{T}_n$, $D = (V, A)$, be a digraph without isolated vertices. Let $f : V \to \{0, 1\}$ be a friendly labelling of the vertex set V of D. Let $g : A \to \{1, 0, -1\}$ be an induced labelling of the arcs of D such that for any $i, j \in \{1, 0, -1\}$, $-1 \leq |g^{-1}(i)| - |g^{-1}(j)| \leq 1$. Such a labelling is called a* $(2, 3)$-cordial *labelling.*

A digraph $D \in \mathcal{T}_n$ whose subgraph $\hat{G}$ can possess a $(2, 3)$-cordial labelling will be called a $(2, 3)$-cordial *digraph.*

An undirected graph G is said to be $(2, 3)$-orientable *if there exists an orientation of G that is $(2, 3)$-cordial.*

In [1] the concept of $(2, 3)$-cordial digraphs was introduced and paths and cycles were investigated. In [6] one can find further investigation of orientations of paths and trees as well as finding the maximum number of arcs possible in a $(2, 3)$-cordial digraph. In this article we continue this investigation, showing which tournaments, which orientations of the wheel graphs, and which orientations of the fan graphs are $(2,3)$-cordial.

Definition 3.2. *Let $D = (V, A)$ be a digraph with vertex labelling $f : V \to \{0, 1\}$ and with induced arc labelling $g : A \to \{0, 1, -1\}$. Define $\Lambda_{f,g} : \mathcal{D}_n \to \mathbb{N}^3$ by $\Lambda_{f,g}(D) = (\alpha, \beta, \gamma)$ where $\alpha = |g^{-1}(1)|, \beta = |g^{-1}(-1)|$, and $\gamma = |g^{-1}(0)|$.*

Let $D \in \mathcal{T}_n$ and let D^R be the digraph such that every arc of D is reversed, so that $\overrightarrow{uv}$ is an arc in D^R if and only if $\overrightarrow{vu}$ is an arc in D. Let f be a $(0,1)$-labelling of the vertices of D and let $g(\overrightarrow{uv}) = f(v) - f(u)$ so that g is a $(1, -1, 0)$-labelling of the arcs of D. Let $\overline{f}$ be the complementary $(0, 1)$-labelling of the vertices of D, so that $\overline{f}(v) = 0$ if and only if $f(v) = 1$. Let $\overline{g}$ be the corresponding induced arc labelling of D, $\overline{g}(\overrightarrow{uv}) = \overline{f}(v) - \overline{f}(u)$.

Lemma 3.3. *Let $D \in \mathcal{T}_n$ with vertex labelling f and induced arc labelling g. Let $\Lambda_{f,g}(D) = (\alpha, \beta, \gamma)$. Then*

1. $\Lambda_{f,g}(D^R) = (\beta, \alpha, \gamma)$.
2. $\Lambda_{\overline{f},\overline{g}}(D) = (\beta, \alpha, \gamma)$, *and*
3. $\Lambda_{\overline{f},\overline{g}}(D^R) = \Lambda_{f,g}(D)$.

Proof. If an arc is labelled $1, -1, 0$ respectively then reversing the labelling of the incident vertices gives a labelling of $-1, 1, 0$ respectively. If an arc $\overrightarrow{uv}$ is labelled $1, -1, 0$ respectively, then $\overrightarrow{vu}$ would be labelled $-1, 1, 0$ respectively. ■

Example 3.4. *Now, consider a graph, III_n in $\mathcal{G}_n$ consisting of three parallel edges and n-6 isolated vertices. Is III_n $(2,3)$-orientable? If $n = 6$, the answer is no, since any friendly labelling of the six vertices would have either no arcs labelled 0 or two arcs labelled 0. In either case, the orientation would never be $(2,3)$-cordial. That is III_6 is not $(2,3)$-orientable, however with additional vertices like III_7 the graph is $(2,3)$-orientable.*

Thus, for our investigation here, we will use the convention that a graph, G, is $(2,3)$-orientable/$(2,3)$-cordial if and only if the subgraph of G induced by its nonisolated vertices, $\hat{G}$, is $(2,3)$-orientable/$(2,3)$-cordial.

3.1. $(2,3)$-orientations of a complete graph-tournaments

It is an easy exercise to show that every 3-tournament is $(2,3)$-cordial and that two of the four non isomorphic 4-tournaments are $(2,3)$-cordial. See Figures 2 and 3. Note that the 4-tournaments that are not $(2,3)$-cordial may require more than a cursory glance to verify that they are not $(2,3)$-cordial.

Lemma 3.5. *Every 5-tournament is $(2,3)$-cordial.*

Proof. Let $T \in \mathcal{D}_5$ be a tournament. Then there are two vertices, without loss of generality, v_1 and v_2, whose total out degree is four. (And hence the total in-degree of v_1 and v_2 is also four.) Let f be the vertex labelling and let $f(v_1) = f(v_2) = 1$ and $f(v_3) = f(v_4) = f(v_5) = 0$. Let g be the arc labelling $g(\overrightarrow{v_iv_j}) = f(v_j) - f(v_i)$. Then, the arc between v_1 and v_2 is labelled 0, as are the three arcs between v_3, v_4 and v_5. Thus there are four arcs labelled 0. The three arcs from v_1 or v_2 to vertices v_3, v_4 or v_5 are labelled 1 and the three arcs from v_3, v_4 or v_5 to vertices v_1 or v_2 are labelled -1. In Figure 1 is an example of the labelling described above. Thus $\Lambda_{f,g}(T) = (3, 3, 4)$. That is T is $(2,3)$-cordial. ■

Lemma 3.6. *If $n \geq 6$ and $T \in \mathcal{D}_n$ is a tournament on n vertices then T is not $(2,3)$-cordial.*

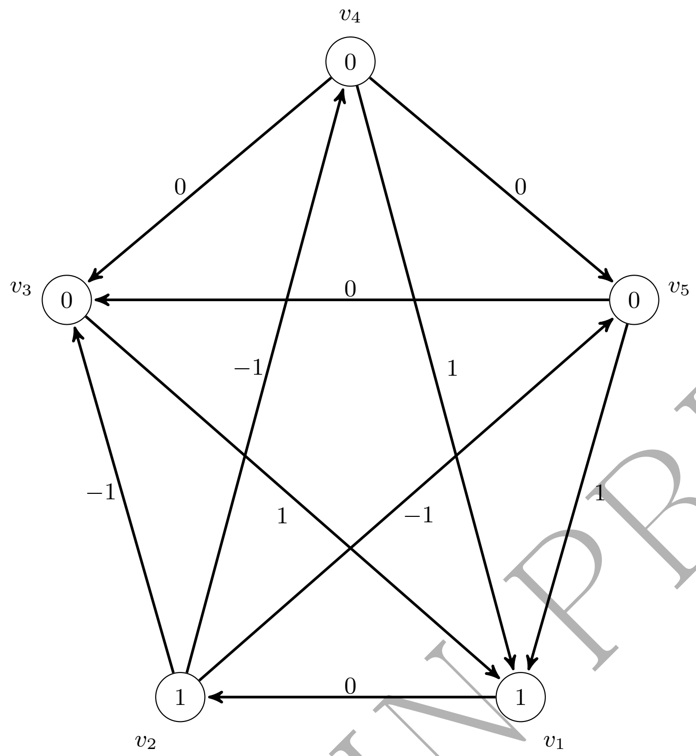

Figure 1. **A $(2,3)$-cordial labelling of a 5-tournament**

Proof. We divide the proof into two cases:

Case 1. ***n* is even.** Let $n = 2k$. We shall show that there must be more arcs labelled 0 than is allowed in any $(2,3)$-cordial digraph with $\frac{n(n-1)}{2}$ arcs. For any vertex labelled 0, there are $k-1$ other vertices also labelled 0 so that there are $k-1$ arcs labelled 0 that either begin or terminate at that vertex. Also there are k such vertices so there are $k(k-1)/2$ arcs between pairs of vertices each labelled 0. (Note, since each arc is adjacent to two vertices we have divided the total number by 2 to get the number of distinct arcs labelled 0.) There are also $k(k-1)/2$ arcs between pairs of vertices each labelled 1. Thus we must have $k(k-1)$ arcs labelled 0.

Now, there must be at most one third the number of arcs labelled 0, so we must have $3k(k-1) \leq \frac{n(n-1)}{2} + 2 = \frac{4k^2-2k+4}{2}$. That is, we must have $k^2 - 2k - 2 \leq 0$. But that only happens if $k \leq 2$. So if $k \geq 3$ or $n \geq 6$, T is not $(2,3)$-cordial.

Case 2. ***n* is odd.** Let $n = 2k+1$. Without loss of generality, we may assume that there are k vertices labelled 0 and $k+1$ vertices labelled 1. Thus there are $\frac{1}{2}k(k-1)$ arcs labelled 0 that connect two vertices labelled 0 and $\frac{1}{2}(k+1)k$ arcs labelled 0 that connect two vertices labelled 1. Thus there are at least k^2 arcs labelled 0. To be $(2,3)$-cordial we must have that $3k^2 \leq \frac{n(n-1)}{2} + 2$, or $k^2 - k - 2 \leq 0$. That happens only if $k \leq 2$. But, since n is odd, $n \geq 7$ so $k \geq 3$. Thus, T is not $(2,3)$-cordial. ∎

Lemma 3.7. *The tournaments $T_{4,3}$ and $T_{4,4}$ of Figure 3 are not $(2,3)$-cordial.*

Proof. Since $T_{4,4}$ is the reversal of $T_{4,3}$, by Lemma 3.3 we only need show that $T_{4,3}$ is not $(2,3)$-cordial. Further, by Lemma 3.3 we may assume that the upper left vertex of $T_{4,3}$ in Figure 3 is labelled 0. Since any permutation of the other three vertices results in an isomorphic graph we may assume that the upper

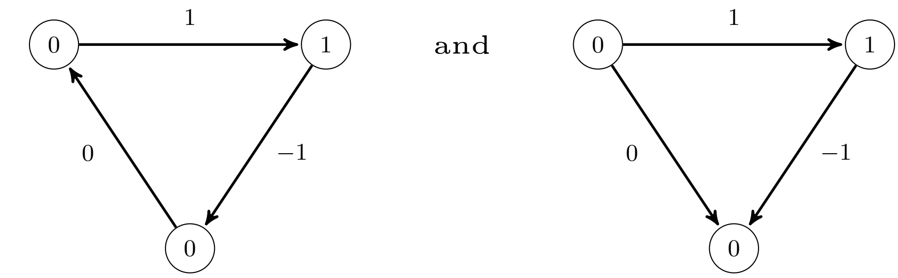

Figure 2. $(2,3)$**-cordial labellings of 3- tournaments**

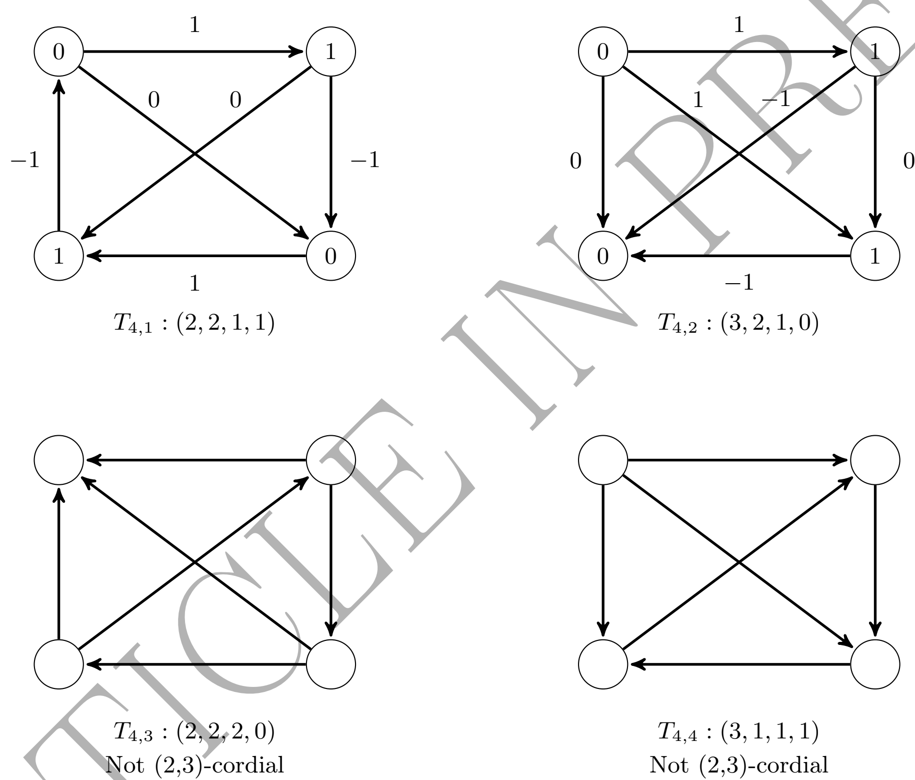

Figure 3. $(2,3)$**-cordial labellings of two 4-tournaments and two non (2,3) cordial 4-tournaments with their out-degree sequences.**

right vertex is labelled 0 and the bottom two are labelled 1. This results in one arc labelled 1, three arcs labelled -1 and two arcs labelled 0. Thus $T_{4,3}$ is not $(2,3)$-cordial. ■

Theorem 3.8. *Let T be an n-tournament. Then T is $(2,3)$-cordial if and only if $n \leq 5$ and T is not isomorphic to $T_{4,3}$ or $T_{4,4}$.*

Proof. Lemmas 3.7, 3.5 and 3.6 together with Figures 2 and 3 establish the theorem. ■

We end this section with a couple of observations we label as corollaries:

Corollary 3.9. *The property of being (or not being) $(2,3)$-cordial is not closed under vertex deletion.*

Proof. Every tournament on k vertices is a vertex deletion of a tournament on $k+1$ vertices. Thus, $T_{4,3}$, which is not $(2,3)$-cordial, is a vertex deletion of a tournament on 5 vertices, which is $(2,3)$-cordial, and this tournament is a vertex deletion of a tournament on 6 vertices, which is not $(2,3)$-cordial. ■

Corollary 3.10. *The property of being (or not being) $(2,3)$-cordial is not closed under arc contraction.*

Proof. As in the above corollary, every tournament on k vertices is an arc contraction of a tournament on $k+1$ vertices. Thus, $T_{4,3}$, which is not $(2,3)$-cordial, is an arc contraction of a tournament on 5 vertices, which is $(2,3)$-cordial, and this tournament is an arc contraction of a tournament on 6 vertices, which is not $(2,3)$-cordial. ■

3.2. $(2,3)$-orientations of wheel graphs

A wheel graph on n vertices consists of an $(n-1)$-star together with edges joining the non central vertices in a cycle. A 6-wheel is shown in Figure 4. Since we are not concerned with digraphs that contain digons, we shall assume that $n \geq 4$ in this section.

An orientation of the wheel graph with the central vertex being a source/sink is called an *out/in-wheel*. If the outer cycle of the wheel is oriented in a directed cycle the wheel is called a *cycle-wheel*. If the n-wheel is oriented such that it is both an out-wheel and a cycle-wheel it is called an *n-cycle-out-wheel*. See Figure 5.

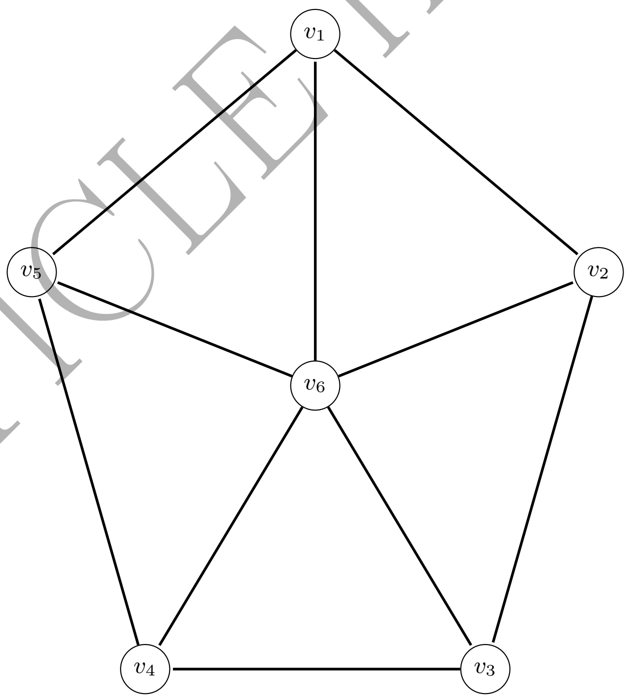

Figure 4. A 6-wheel graph.

Definition 3.11. *Let $W_n = (V, A)$ be an n-wheel digraph with central vertex v_n and with vertex labelling $f : V \to \{0,1\}$. Let g be the induced arc labelling $g : A \to \{0,1,-1\}$ where $g(\overrightarrow{uv}) = f(v) - f(u)$. Let*

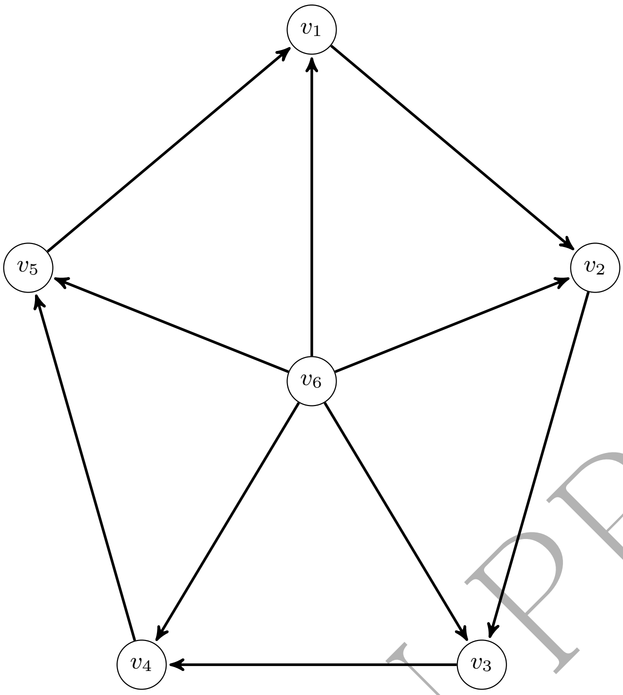

Figure 5. A 6-cycle-out-wheel graph.

S be the set of arcs incident with the central vertex and let T be the set of arcs not incident with the central vertex. Define $\Lambda^S_{f,g}$ to be the real triple $\Lambda^S_{f,g}(D) = (\alpha_S, \beta_S, \gamma_S)$ where $\alpha_S = |g^{-1}(1) \cap S|, \beta_S = |g^{-1}(-1) \cap S|$, and $\gamma_S = |g^{-1}(0) \cap S|$ and $\Lambda^T_{f,g}$ to be the real triple $\Lambda^T_{f,g}(D) = (\alpha_T, \beta_T, \gamma_T)$ where $\alpha_T = |g^{-1}(1) \cap T|, \beta_T = |g^{-1}(-1) \cap T|$, and $\gamma_T = |g^{-1}(0) \cap T|$. Since $S \cup T = A$, the set of all arcs of D, $\alpha_S + \alpha_T = \alpha, \beta_S + \beta_T = \beta$, and $\gamma_S + \gamma_T = \gamma$, where $\Lambda_{f,g}(D) = (\alpha, \beta, \gamma)$.

Theorem 3.12. *Let $\overrightarrow{W_n}$ be an n-cycle-out-wheel graph with central vertex v_n. Then $\overrightarrow{W_n}$ is not $(2,3)$-cordial.*

Proof. We proceed with two cases, the case that n is even then the case that n is odd. Let $\overrightarrow{W_n} = (V, A)$ and let $f : V \to \{0, 1\}$ be a vertex labelling and $g : A \to \{0, 1, -1\}$ be the induced arc labelling, $g(\overrightarrow{uv} = f(v) - f(u)$. Suppose that f and g is a $(2,3)$-cordial labelling.

Case 1: $n = 2k$. Without loss of generality, we may assume that $f(v_n) = 0$. Thus, $\alpha_S = k, \beta_S = 0$ and $\gamma_S = k - 1$. Further, since the orientation is cyclic, $\alpha_T = \beta_T$. Since $\alpha - 1 \leq \beta \leq \alpha + 1$ we have $\alpha_T + k - 1 = \alpha_T + \alpha_S - 1 = \alpha - 1 \leq \beta = \beta_T + \beta_S = \beta_T = \alpha_T$. Thus $k - 1 \leq 0$ or $k \leq 1$, a contradiction since $n \geq 4$.

Case 2: $n = 2k + 1$. Without loss of generality we may assume that $f(v_n) = 0$. Since either k or $k + 1$ of the non central vertices must be labelled 0, we have two possibilities: $\alpha_S = k$ or $\alpha_S = k + 1$.

Subcase 1. $\alpha_S = k$ Here we have $\gamma_S = k$ and $\beta_S = 0$. Further, $\alpha_T = \beta_T$ since $\overrightarrow{W}$ is cyclic. Since the labelling is $(2,3)$-cordial, $\alpha - 1 \leq \beta \leq \alpha + 1$. Thus $k + \alpha_T - 1 = \alpha_S = \alpha_T - 1 = \alpha - 1 \leq \beta = \beta_S + \beta_T = \alpha_T$. That is $k - 1 \leq 0$ or $k \leq 1$, a contradiction since $n \geq 4$.

Subcase 2. $\alpha_S = k + 1$ Here we have $\gamma_S = k - 1$ and $\beta_S = 0$. Further, $\alpha_T = \beta_T$ since $\overrightarrow{W}$ is cyclic. Since the labelling is $(2,3)$-cordial, $\alpha - 1 \leq \beta \leq \alpha + 1$. Thus $k + \alpha_T = k + 1 + \alpha_T - 1 = \alpha_S + \alpha_T - 1 = \alpha - 1 \leq \beta = \beta_S + \beta_T = \alpha_T$. That is $k \leq 0$, a contradiction since $n \geq 4$.

In all cases we have arrived at a contradiction thus we must have that $\overrightarrow{W}$ is not $(2,3)$-cordial. ■

Lemma 3.13. *Let C be an undirected cycle with a $(0,1)$-vertex labelling. Then, there is an even number of edges in C whose incident vertices are labelled differently.*

Proof. We may assume that the vertex v_1 is labelled 0. Going around the cycle, the labelling goes from 0 to 1 then back again to zero. This two step change must happen a fixed number of times then return to vertex v_1. Thus there are an equal number of changes in labelling from 0 to 1 and from 1 to 0. That is, the total number of changes is an even number. ■

Theorem 3.14. *Let W_n be the undirected wheel graph on n vertices. Then, W_n is not $(2,3)$-orientable if and only if $n = 2k$ for some integer k, 4 does not divide n, and $2n - 2 = 3z$ for some integer z.*

Proof. Let $\overrightarrow{W_n}$ be an orientation of the wheel graph on n vertices with central vertex v_n. Let A_H be the set of arcs incident with v_n, and let A_R be the arcs not incident with v_n. Then $A = A_H \cup A_R$. Let f be a friendly vertex labelling and g the induced arc labelling of $\overrightarrow{W_n}$. Define , $\lambda_{f,H}(x) = |g^{-1}(x) \cap A_H|$, and $\lambda_{f,R}(x) = |g^{-1}(x) \cap A_R|$, Define $\lambda_f(x) = \lambda_{f,H}(x) + \lambda_{f,R}(x)$, that is $\lambda_f(x) = |g^{-1}(x)|$.

We begin by showing for $n = 2k$ for some integer k, $k = 2\ell + 1$ for some integer ℓ, and $2n - 2 = 3z$ for some integer z that W_n is not $(2,3)$-orientable. In this case, We may assume that $f(v_n) = 0$ and $\lambda_{f,H}(1) + \lambda_{f,H}(-1) = k$, an odd integer. By Lemma 3.13 the number of arcs that are not incident with v_n and labelled either 1 or -1 is even. Thus $\lambda_f(1) + \lambda_f(-1) = (\lambda_{f,H}(1) + \lambda_{f,H}(-1)) + (\lambda_{f,R}(1) + \lambda_{f,R}(-1))$ is the sum of an even integer and an odd integer, so that $\lambda_f(1) + \lambda_f(-1)$ is an odd integer. But since the total number of arcs is $2n - 2 = 3z$, if $\overrightarrow{W_n}$ is $(2,3)$-cordial, we must have $\lambda_f(1) + \lambda_f(-1) = 2z$, an even integer, a contradiction. Thus, in this case W_n is not $(2,3)$-orientable.

We now show that for all other cases W_n is $(2,3)$-orientable. We divide the proof into three cases, those being whether the total number of edges in W_n is a multiple of three, one more than a multiple of three, or two more than a multiple of three.

Case 1. $2n - 2 = 3z$ for some integer z.

Subcase 1.1. $n = 2k$ and $k = 2\ell$. In this case, let f be the labelling such that the labelling of the cycle has $2(z - \frac{k}{2})$ edges incident with vertices labelled differently. Orient all arcs not incident with v_n clockwise around the cycle. Orient half the arcs incident with v_n that are labelled 1 away from v_n, and half toward v_n. In this case, $\lambda_{f,H}(0) = k - 1$, and $\lambda_{f,H}(1) = \lambda_{f,H}(-1) = \frac{k}{2} = \ell$. Further, $\lambda_{f,R}(1) = \lambda_{f,R}(-1) = z - \frac{k}{2}$. Thus, $\lambda_f(1) = \lambda_f(-1) = \lambda_{f,H}(1) + \lambda_{f,R}(1) = \ell + z - \frac{k}{2} = z$. Thus, we must also have $\lambda_f(0) = z$, and that W_n is $(2,3)$-orientable.

Subcase 1.2. $n = 2k + 1$. In this case proceed as in Subcase 1.1, labelling the vertices not incident with v_n with an even number of 1's (either k or $k+1$). Let ℓ be half of this even number. Then, we produce a $(2,3)$-cordial orientation of W_n the same as in Subcase 1.1.

Case 2. $2n - 2 = 3z + 1$ for some integer z.

Subcase 2.1. $n = 2k$, $k = 2\ell$. In this case, let f be the labelling such that the labelling of the cycle has $2(z - \frac{k}{2})$ edges incident with vertices labelled differently. Orient all arcs not incident with v_n clockwise around the cycle. Orient half the arcs incident with v_n that are labelled 1 away from v_n, and half toward v_n. In this case, $\lambda_{f,H}(0) = k - 1, \lambda_{f,H}(1) = \lambda_{f,H}(-1) = \frac{k}{2} = \ell$. Further, $\lambda_{f,R}(1) = \lambda_{f,R}(-1) = z - \frac{k}{2}$. Thus, $\lambda_f(1) = \lambda_f(-1) = \lambda_{f,H}(1) + \lambda_{f,R}(1) = \ell + z - \frac{k}{2} = z$. Thus, we must also have $\lambda_f(0) = z + 1$, and that W_n is $(2,3)$-orientable.

Subcase 2.2. $n = 2k$, $k = 2\ell + 1$ In this case, let f be the labelling such that the labelling of the cycle has $2(z - \frac{k-1}{2})$ edges incident with vertices labelled differently. Orient ℓ of the arcs incident with v_n that are labelled 1 away from v_n, and $\ell + 1$ of those arcs toward v_n. In this case, $\lambda_{f,H}(0) = k - 1, \lambda_{f,H}(1) = \ell$ and $\lambda_{f,H}(-1) = \ell + 1$. Further, $\lambda_{f,R}(1) = \lambda_{f,R}(-1) = z - \frac{k}{2} = z = \ell$. Thus, $\lambda_f(1) = \lambda_{f,H}(1) + \lambda_{f,R}(1) = \ell + z - \ell = z$, and $\lambda_f(-1) = \lambda_{f,H}(-1) + \lambda_{f,R}(-1) = \ell + 1 + z - \ell = z + 1$. Thus, we must also have $\lambda_f(0) = 2n - 2 - (z) - (z + 1) = z$, and thus W_n is $(2,3)$-orientable.

Subcase 2.3. $n = 2k+1$. In this case proceed as in Subcase 2.1 labelling the vertices not incident with v_n with an even number of 1's (either k or $k+1$ depending upon whether k is even or odd). Let ℓ be half of this even number. Then, we produce a $(2,3)$-cordial orientation of W_n the same as in Subcase 2.1.

Case 3. $2n-2 = 3z+2$ for some integer z.

Subcase 3.1. $n = 2k$, $k = 2\ell$. In this case, let f be the labelling such that the labelling of the cycle has $2(z-\frac{k}{2})$ edges incident with vertices labelled differently. Orient all arcs not incident with v_n clockwise around the cycle. Orient half the arcs incident with v_n that are labelled 1 away from v_n, and half toward v_n. In this case, $\lambda_{f,H}(0) = k-1, \lambda_{f,H}(1) = \lambda_{f,H}(-1) = \frac{k}{2} = \ell$. Further, $\lambda_{f,R}(1) = \lambda_{f,R}(-1) = z - \frac{k}{2}$. Thus, $\lambda_f(1) = \lambda_f(-1) = \lambda_{f,H}(1) + \lambda_{f,R}(1) = \ell + z - \frac{k}{2} = z$. Thus, we must also have $\lambda_f(0) = z+1$, and that W_n is $(2,3)$-orientable.

Subcase 3.2. $n = 2k$, $k = 2\ell+1$ In this case, let f be the labelling such that the labelling of the cycle has $2(z-\frac{k-1}{2})$ edges incident with vertices labelled differently. Orient ℓ of the arcs incident with v_n that are labelled 1 away from v_n, and $\ell+1$ toward v_n. In this case, $\lambda_{f,H}(0) = k-1, \lambda_{f,H}(1) = \ell$ and $\lambda_{f,H}(-1) = \ell+1$. Further, $\lambda_{f,R}(1) = \lambda_{f,R}(-1) = z - \frac{k}{2} = z = \ell$. Thus, $\lambda_f(1) = \lambda_{f,H}(1) + \lambda_{f,R}(1) = \ell + z - \ell = z$, and $\lambda_f(-1) = \lambda_{f,H}(-1) + \lambda_{f,R}(-1) = \ell + 1 + z - \ell = z+1$. Thus, we must also have $\lambda_f(0) = z$, and that W_n is $(2,3)$-orientable.

Subcase 3.3. $n = 2k+1$. In this case proceed as in Subcase 3.1 labelling the vertices not incident with v_n with an even number of 1's (either k or $k+1$ depending upon whether k is even or odd. Let ℓ be half of this even number. Then, we produce a $(2,3)$-cordial orientation of W_n the same as in Subcase 3.1.

We have now established the theorem. ■

3.3. $(2,3)$-orientations of fan graphs

A fan graph is isomorphic to a wheel graph with one edge of the cycle deleted. Thus, by deleting one properly chosen arc from the cycle of a $(2,3)$-oriented n-wheel graph we obtain an orientation of the n-fan graph that is $(2,3)$-cordial. Note that if there are at least as many arcs labelled x ($x = 1, -1$ or 0) as any other labelling, the properly chosen arc would be in the set of arcs labelled x. Thus there is only one case to consider, the case where $2n-2 = 3z, n = 2k$ and $k = 2\ell+1$ for some z, k, and ℓ.

Theorem 3.15. *Let $n \geq 5$ and let F_n be the n-fan graph with central vertex v_1, that is the edges not on the cycle are all incident to v_1. Let $\overrightarrow{F}$ be a cyclic-out orientation of F_n. Then $\overrightarrow{F}$ is not $(2,3)$-cordial.*

Proof. As for wheel graphs, the number of arcs labelled 1 on the cycle is equal to the number of arcs labelled -1 and there are at least two arcs labelled 1 on the interior of the cycle. Thus, the number of arcs labelled 1 in $\overrightarrow{F}$ is at least two more that the arcs labelled -1 in $\overrightarrow{F}$. That is $\overrightarrow{F}$ is not $(2,3)$-cordial. ■

Theorem 3.16. *Let F_n be the fan graph on n vertices, $2n-3 = 3z+2$, $n = 2k$ and $k = 2\ell+1$ for some integers k, ℓ, and z. Then there is an orientation of F_n that is $(2,3)$-cordial.*

Proof. Let $\alpha = z-\ell+1$, and define $f : V \to \{0,1\}$ by $f(v_{2i-1}) = 0$, $i = 1,\ldots,\alpha$, $f(v_{2i}) = 1, i = 1,\ldots,\alpha$, $f(v_{2\alpha+i}) = 1, i = 1,\ldots,k-\alpha$, and $f(v_{k+\alpha+i}) = 0, i = 1,\ldots,k-\alpha$, Note that $(k-\alpha)+(k+\alpha) = 2k = n$, so all vertices are labelled. Orient the cycle clockwise, so that the oriented cycle is $\overrightarrow{v_1v_2}$, $\overrightarrow{v_2v_3}$, $\ldots$, $\overrightarrow{v_{n-1}v_n}$, $\overrightarrow{v_nv_1}$. See Figure 6 where the vertex labellings are outside the cycle.

Now, orient ℓ of the inner arcs from v_1 to arcs labelled 1 (except for v_2 which is not an inner arc) away from v_1 and the remaining ℓ such arcs inward so that we get $\overrightarrow{F_n} = (V, A)$. Let $g : A \to \{0, 1, -1\}$ be the induced labelling, $g(\overrightarrow{uv}) = f(v) - f(u)$. Then there are α arcs labelled 1 on the cycle, α arcs labelled -1 on the cycle, ℓ of the inner arcs are labelled -1 and ℓ of the inner arcs are labelled 1. Thus, in all of $\overrightarrow{F_n}$ there are $\alpha+\ell = z+1$ arcs labelled -1, $\alpha+\ell = z+1$ arcs labelled 1 and (hence) z arcs labelled 0. That is, this orientation of F_n is $(2,3)$-cordial. ■

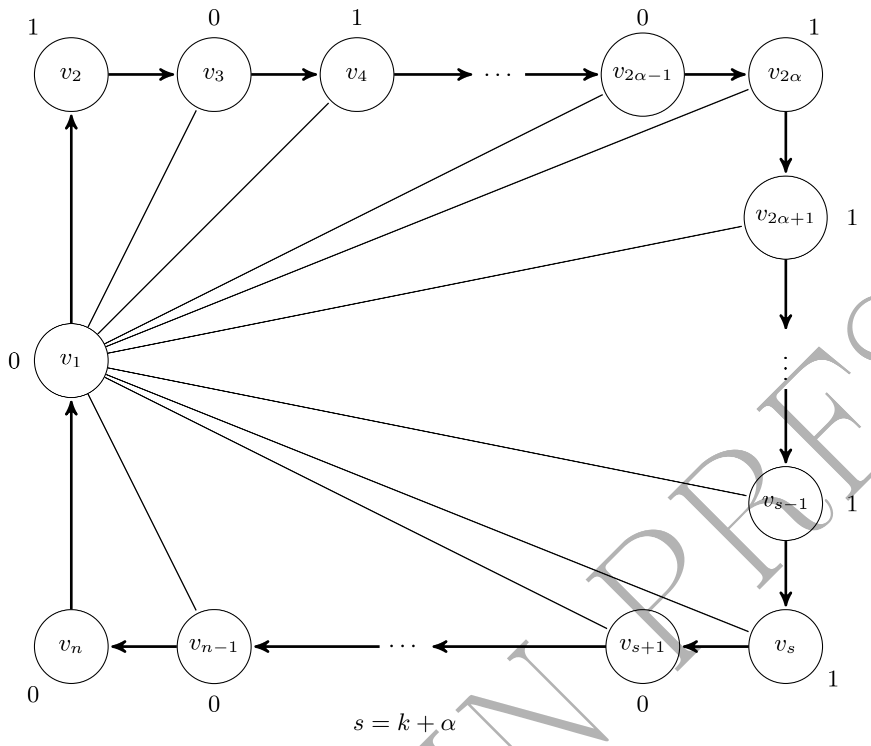

Figure 6. F_n **with oriented cycle arcs.**

3.4. Extremes of $(2,3)$-cordiality

As seen in section 3, complete graphs are not $(2,3)$-orientable if $n \geq 6$. So the question arises: How large can a $(2,3)$-orientable graph be (how many edges)? Or: How large can a $(2,3)$-cordial digraph be? That question was fully answered by M. A. Santana in [6] For completeness we shall include the proofs of his results.

Theorem 3.17. *[6, Theorem 3.1] Every simple directed graph is $(2,3)$-cordial if and only if there exists a friendly vertex labelling such that about $\frac{1}{3}$ of the edges are connected by vertices of the same label.*

Proof. Let G be a graph such that there exists a friendly labelling on G such that about $\frac{1}{3}$ of the edges are connected by vertices of the same label. This would mean about $\frac{2}{3}$ of the edges are connected by vertices of different labels, and therefore arcs may be assigned such that G is cordial. Now let H be a graph such that there does not exist a friendly labelling on H such that about $\frac{1}{3}$ of the edges are connected by vertices of the same label then there will be no way H can be cordial since only then could about one third of the edges be labelled 0. ∎

Santana's application of Theorem 3.17 is

Theorem 3.18. *[6, Theorem 4.2] Given a directed graph $G = (V, E)$ with vertex set V and $n = |V|$ with $n \geq 6$, and edge set E, the maximum size of E such that G is cordial for any given n is*

$$|E|_{max} = \binom{n}{2} - Z + \left\lceil \frac{1}{2}\left(\binom{n}{2} - Z\right) \right\rceil Z = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}. \tag{1}$$

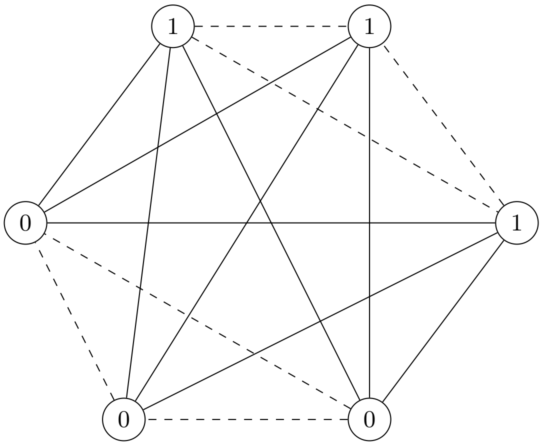

Figure 7. A complete graph. Dashed lines represent edges labelled zero regardless of arc orientation

From section 3 we have that for any tournament with $n \leq 5$ there exists a cordial labelling, save for the case when $n = 4$ thus we begin with a complete graph with $n \geq 6$. Recall that the number of edges on a complete graph is $\binom{n}{2}$. Due to our cordial labelling the number of edges with an induced labelling of 0 will be our Z. This is because it will be the number of edges connected by two vertices of the same label, as shown in Figure 7. If n is even that will mean that $Z = 2\binom{\frac{n}{2}}{2}$, i.e., it will be the number of edges on two complete graphs each on $\frac{n}{2}$ vertices represented by the labellings of ones and zeros. The floor and ceiling function in (1) simply account for the odd case.
For every tournament with $n \geq 6$ vertices, $Z > \frac{1}{3}\binom{n}{2}$. Therefore some of the arcs labelled zero will need to be removed to get a cordial graph. How many arcs need to be removed is going to be equal to how much greater Z is than the number of half the number of arcs not labelled zero. By the definition of a directed cordial graph we know that Z can be larger than α or β and we can still have a cordial graph, hence the ceiling function. ■

As mentioned in the introduction, the smallest non $(2, 3)$-cordial digraph is an orientation of Ш_n, three parallel arcs. A question may be asked: What is the minimum number of arcs in a non $(2, 3)$-cordial digraph that has no isolated vertices?

4. Conclusions

In this article, we have shown that the only tournaments that are $(2, 3)$-cordial are when $n \leq 5$ and then not for two 4-tournaments. Except for one case when n is even, the n-wheel graph has an orientation that is $(2, 3)$-cordial and that at least one orientation of any wheel graph is not $(2, 3)$-cordial. Further, we show that every fan graph has a $(2, 3)$-cordial orientation, and there is always an orientation of the n-fan that is not $(2, 3)$-cordial.

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