J. Algebra Comb. Discrete Appl.
5(2) • 85–99

Received: 19 August 2017
Accepted: 26 Ferbruary 2018

Journal of Algebra Combinatorics Discrete Structures and Applications

Some results on the comaximal ideal graph of a commutative ring

Research Article

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Abstract: The rings considered in this article are commutative with identity which admit at least two maximal ideals. Let R be a ring such that R admits at least two maximal ideals. Recall from Ye and Wu (J. Algebra Appl. 11(6): 1250114, 2012) that the comaximal ideal graph of R, denoted by $\mathscr{C}(R)$ is an undirected simple graph whose vertex set is the set of all proper ideals I of R such that $I \not\subseteq J(R)$, where $J(R)$ is the Jacobson radical of R and distinct vertices I_1, I_2 are joined by an edge in $\mathscr{C}(R)$ if and only if $I_1 + I_2 = R$. In Section 2 of this article, we classify rings R such that $\mathscr{C}(R)$ is planar. In Section 3 of this article, we classify rings R such that $\mathscr{C}(R)$ is a split graph. In Section 4 of this article, we classify rings R such that $\mathscr{C}(R)$ is complemented and moreover, we determine the S-vertices of $\mathscr{C}(R)$.

2010 MSC: 13A15, 05C25

Keywords: Comaximal ideal graph, Special principal ideal ring, Planar graph, Split graph, Complement of a vertex in a graph

1. Introduction

The rings considered in this article are commutative with identity which admit at least two maximal ideals. Let R be a ring. We denote the set of all maximal ideals of R by $Max(R)$. We denote the Jacobson radical of R by $J(R)$. We denote the cardinality of a set A using the notation $|A|$. Motivated by the research work done on *comaximal graphs of rings* in [9, 12, 13, 15, 16] and on the *annihilating-ideal graphs of rings* in [5, 6], M. Ye and T. Wu in [18] introduced a graph structure on a ring R, whose vertex set is the set of all proper ideals I of R such that $I \not\subseteq J(R)$ and distinct vertices I_1 and I_2 are joined by an edge if and only if $I_1 + I_2 = R$. M. Ye and T. Wu called the graph introduced by them in [18] as the *comaximal ideal graph* of R and denoted it using the notation $\mathscr{C}(R)$ and investigated the influence of certain graph parameters of $\mathscr{C}(R)$ on the ring structure of R. Let R be a ring such that $|Max(R)| \geq 2$.

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http://dx.doi.org/10.13069/jacodesmath.423751 ISSN 2148-838X

The aim of this article is to classify rings R such that $\mathscr{C}(R)$ is planar; $\mathscr{C}(R)$ is a split graph; and $\mathscr{C}(R)$ is complemented.

It is useful to recall the following definitions from commutative ring theory. A ring R is said to be *quasilocal* (respectively, *semiquasilocal*) if R has a unique maximal ideal (respectively, R has only a finite number of maximal ideals). A Noetherian quasilocal (respectively, semiquasilocal) ring is referred to as a *local ring* (respectively, a *semilocal ring*). Recall that a principal ideal ring is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal $\mathfrak{m}$. It is clear that $\mathfrak{m}$ is nilpotent. If R is a SPIR with $\mathfrak{m}$ as its only prime ideal, then we denote it by saying that $(R, \mathfrak{m})$ is a SPIR. Suppose that a ring T is quasilocal with $\mathfrak{m}$ as its unique maximal ideal such that $\mathfrak{m} \neq (0)$ and nilpotent. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. If $\mathfrak{m}$ is principal, then it follows from $(iii) \Rightarrow (i)$ of [3, Proposition 8.8] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ is the set of all nonzero proper ideals of T. Hence, $(T, \mathfrak{m})$ is a SPIR.

We next recall the following definitions and results from graph theory. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Recall from [4, Definition 1.2.2] that a *clique* of G is a complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$ is defined as the largest integer $n \geq 1$ such that G contains a clique on n vertices [4, Definition, page 185]. We set $\omega(G) = \infty$ if G contains a clique on n vertices for all $n \geq 1$.

Let $n \in \mathbb{N}$. A complete graph on n vertices is denoted by K_n. A graph $G = (V, E)$ is said to be *bipartite* if V can be partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. Let $m, n \in \mathbb{N}$. Let $G = (V, E)$ be a complete bipartite graph with vertex partition V_1 and V_2. If $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$ [4, Definition 1.1.12].

Let $G = (V, E)$ be a graph. Recall from [4, Definition 8.1.1] that G is said to be *planar* if G can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G. Recall that two adjacent edges are said to be in *series* if their common end vertex is of degree two [7, page 9]. Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by insertion of vertices of degree two or by the merger of edges in series [7, page 100]. It is useful to note from [7, page 93] that the graph K_5 is referred to as *Kuratowski's first graph* and the graph $K_{3,3}$ is referred to as *Kuratowski's second graph*. The celebrated theorem of Kuratowski states that a graph G is planar if and only if G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [7, Theorem 5.9].

Let $G = (V, E)$ be a graph. It is convenient to name the following conditions satisfied by G so that it can be used throughout Section 2 of this article.

(i) We say that G *satisfies* (Ku_1) if G does not contain K_5 as a subgraph (that is, equivalently, if $\omega(G) \leq 4$).

(ii) We say that G *satisfies* (Ku_1^*) if G satisfies (Ku_1) and moreover, G does not contain any subgraph homeomorphic to K_5.

(iii) We say that G *satisfies* (Ku_2) if G does not contain $K_{3,3}$ as a subgraph.

(iv) We say that G *satisfies* (Ku_2^*) if G satisfies (Ku_2) and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

Suppose that a graph $G = (V, E)$ is planar. It follows from Kuratowski's theorem [7, Theorem 5.9] that G satisfies both (Ku_1^*) and (Ku_2^*). Hence, G satisfies both (Ku_1) and (Ku_2). It is interesting to note that a graph G can be nonplanar even if it satisfies both (Ku_1) and (Ku_2). For an example of this type, refer [7, Figure 5.9(a), page 101] and the graph G given in this example does not satisfy (Ku_2^*). It is not hard to construct an example of a graph G such that G satisfies (Ku_1) but G does not satisfy (Ku_1^*).

Let R be a ring such that $|Max(R)| \geq 2$. In Section 2 of this article, we try to classify rings R such that $\mathscr{C}(R)$ is planar. It is proved in [18, Theorem 3.1] that $\omega(\mathscr{C}(R)) = |Max(R)|$. Hence, $\mathscr{C}(R)$ satisfies (Ku_1) if and only if $|Max(R)| \leq 4$. In Section 2 of this article, we first focus on classifying rings R such that $\mathscr{C}(R)$ satisfies (Ku_2). It is shown in Lemma 2.1 that if $\mathscr{C}(R)$ satisfies (Ku_2) , then $|Max(R)| \leq 3$.

Let R be a ring such that $|Max(R)| = 2$. It is proved in Proposition 2.7 that $\mathscr{C}(R)$ is planar if and only if $\mathscr{C}(R)$ satisfies (Ku_2) if and only if $R \cong R_1 \times R_2$ as rings, where $(R_i, \mathfrak{m}_i)$ is a quasilocal ring for each $i \in \{1,2\}$ and for at least one $i \in \{1,2\}$, $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$. Let R be a ring with $|Max(R)| = 3$. It is shown in Proposition 2.13 that $\mathscr{C}(R)$ satisfies (Ku_2) if and only if $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1,2,3\}$. It is proved in Theorem 2.18 that $\mathscr{C}(R)$ is planar if and only if $R \cong R_1 \times R_2 \times R_3$ as rings, where R_i is a field for at least two values of $i \in \{1,2,3\}$ and if $i \in \{1,2,3\}$ is such that R_i is not a field, then $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$.

Let $G = (V,E)$ be a graph. Recall that G is a *split graph* if V is the disjoint union of two nonempty subsets K and S such that the subgraph of G induced on K is complete and S is an independent set of G. Let R be a commutative ring with identity. In [16] P. K. Sharma and S.M. Bhatwadekar introduced and investigated a graph associated with R, whose vertex set is the set of all elements of R and distinct vertices x, y are joined by an edge if and only if $Rx + Ry = R$. The graph studied in [16] is named as the *comaximal graph* of R in [12]. In [9], M.I. Jinnah and S.C. Mathew classified rings R such that the comaximal graph of R is a split graph. Let R be a ring such that $|Max(R)| \geq 2$. In Section 3 of this article, we try to classify rings R such that $\mathscr{C}(R)$ is a split graph. It is proved in Lemma 3.2 that if $\mathscr{C}(R)$ is a split graph, then $|Max(R)| \leq 3$. Let R be a ring such that $|Max(R)| = 3$. It is shown in Theorem 3.3 that $\mathscr{C}(R)$ is a split graph if and only if $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1,2,3\}$. Let R be a ring such that $|Max(R)| = 2$. It is proved in Theorem 3.5 that $\mathscr{C}(R)$ is a split graph if and only if $R \cong F \times S$ as rings, where F is a field and S is a quasilocal ring.

Let $G = (V,E)$ be a graph. Recall from [2, 11] that two distinct vertices u, v of G are said to be *orthogonal*, written $u \perp v$ if u and v are adjacent in G and there is no vertex of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not an edge of any triangle in G. Let $u \in V$. A vertex v of G is said to be a *complement* of u if $u \perp v$ [2]. Moreover, we recall from [2] that G is *complemented* if each vertex of G admits a complement in G. Furthermore, G is said to be *uniquely complemented* if G is complemented and whenever the vertices u, v, w of G such that $u \perp v$ and $u \perp w$, then a vertex x of G is adjacent to v in G if and only if x is adjacent to w in G. Let R be a ring such that R is not an integral domain. Recall from [1] that the *zero-divisor graph* of R denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)\backslash\{0\}$ (here $Z(R)$ denotes the set of all zero-divisors of R) and distinct vertices x, y are joined by an edge if and only if $xy = 0$. The authors of [2] determined in Section 3 of [2] rings R such that $\Gamma(R)$ is complemented or uniquely complemented. For a ring R, we denote the set of all units of R by $U(R)$ and we denote the set of all nonunits of R by $NU(R)$. The Krull dimension of a ring R is simply denoted by $dim R$. In [15, Proposition 3.11] it is proved that the subgraph of the comaximal graph of R induced on $NU(R)\backslash J(R)$ is complemented if and only if $dim(\frac{R}{J(R)}) = 0$. Section 4 of this article is devoted to find a classification of rings R such that $\mathscr{C}(R)$ is complemented. Let R be a ring such that $|Max(R)| \geq 2$. It is verified in Remark 4.1(ii) that if $\mathscr{C}(R)$ is complemented, then it is uniquely complemented. It is shown in Theorem 4.7 that $\mathscr{C}(R)$ is complemented if and only if R is semiquasilocal. Moreover, in Section 4, a discussion on the S-vertices of $\mathscr{C}(R)$ is included. Let $G = (V,E)$ be a graph. Recall from [14, Definition 2.9] a vertex a of G is said to be a *Smarandache vertex* or simply a S-vertex if there exist distinct vertices x, y, and b of G such that $a - x, a - b$, and $b - y$ are edges of G but there is no edge joining x and y in G. In [14], A.M. Rahimi investigated the S-vertices of the zero-divisor graph of a commutative ring and the zero-divisor graph of a ring with respect to an ideal. For a ring R with $|Max(R)| \geq 2$, it is noted in Remark 4.8 that if $|MaxR)| = 2$, then no vertex of $\mathscr{C}(R)$ is a S-vertex. Let R be a ring such that $|Max(R)| \geq 3$. It is shown in Proposition 4.9 that a vertex I of $\mathscr{C}(R)$ is a S-vertex if and only if I is not contained in at least two distinct maximal ideals of R.

Let A, B be sets. If A is a subset of B and $A \neq B$, then we denote it symbolically using the notation $A \subset B$. Let G be a graph. We denote the vertex set of G by $V(G)$. Let R be a ring. For a proper ideal I of R, as in [15], we denote $\{\mathfrak{m} \in Max(R) | \mathfrak{m} \supseteq I\}$ by $M(I)$.

2. Some preliminary results and on the planarity of $\mathscr{C}(R)$

As is already mentioned in the introduction, the rings considered in this article are commutative with identity which admit at least two maximal ideals.

Lemma 2.1. *Let R be a ring. If $\mathscr{C}(R)$ satisfies (Ku_2), then $|Max(R)| \leq 3$.*

Proof. Suppose that $|Max(R)| \geq 4$. Let $\{\mathfrak{m}_i | i \in \{1,2,3,4\}\} \subseteq Max(R)$. Let $V_1 = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_1\mathfrak{m}_2\}$ and let $V_2 = \{\mathfrak{m}_3, \mathfrak{m}_4, \mathfrak{m}_3\mathfrak{m}_4\}$. Observe that $V_1 \cup V_2 \subseteq V(\mathscr{C}(R))$, $V_1 \cap V_2 = \emptyset$, and the subgraph of $\mathscr{C}(R)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. Hence, $\mathscr{C}(R)$ does not satisfy (Ku_2). This is in contradiction to the hypothesis that $\mathscr{C}(R)$ satisfies (Ku_2). Therefore, $|Max(R)| \leq 3$. $\square$

Lemma 2.2. *Let R be a ring such that $|Max(R)| \geq 2$. If $\mathscr{C}(R)$ satisfies (Ku_2), then there exist nonzero rings R_1 and R_2 such that $R \cong R_1 \times R_2$ as rings.*

Proof. Assume that $\mathscr{C}(R)$ satisfies (Ku_2). We assert that R admits a nontrivial idempotent. Suppose that R does not have any nontrivial idempotent. By hypothesis, $|Max(R)| \geq 2$. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be distinct maximal ideals of R. Observe that $\mathfrak{m}_1 + \mathfrak{m}_2 = R$. Hence, there exist $a \in \mathfrak{m}_1$ and $b \in \mathfrak{m}_2$ such that $a + b = 1$. Therefore, $Ra + Rb = R$. It is clear that for all $i, j \in \mathbb{N}$, $Ra^i + Rb^j = R$. Since we are assuming that R has no nontrivial idempotent, we obtain that $Ra^i \neq Ra^j$ and $Rb^i \neq Rb^j$ for all distinct $i, j \in \mathbb{N}$. Let $V_1 = \{Ra, Ra^2, Ra^3\}$ and let $V_2 = \{Rb, Rb^2, Rb^3\}$. Note that $V_1 \cup V_2 \subseteq V(\mathscr{C}(R))$ and $V_1 \cap V_2 = \emptyset$. For all $i, j \in \mathbb{N}$, $Ra^i + Ra^j \subseteq \mathfrak{m}_1$ and $Rb^i + Rb^j \subseteq \mathfrak{m}_2$. Hence, no two members of V_i are adjacent in $\mathscr{C}(R)$ for each $i \in \{1,2\}$. It is clear from the above discussion that the subgraph of $\mathscr{C}(R)$ induced on $V_1 \cup V_2$ is $K_{3,3}$. This is a contradiction. Therefore, R admits at least one nontrivial idempotent. Let e be a nontrivial idempotent of R. Observe that the mapping $f : R \to Re \times R(1-e)$ defined by $f(r) = (re, r(1-e))$ is an isomorphism of rings. Let us denote the ring Re by R_1 and $R(1-e)$ by R_2. It is clear that R_1 and R_2 are nonzero rings and $R \cong R_1 \times R_2$ as rings. $\square$

Remark 2.3. *Let R be a ring such that $|Max(R)| = 2$. If $\mathscr{C}(R)$ satisfies (Ku_2), then we know from Lemma 2.2 that there exist nonzero rings R_1, R_2 such that $R \cong R_1 \times R_2$ as rings. As $|Max(R)| = 2$, it follows that R_i is quasilocal for each $i \in \{1,2\}$. We assume that $R = R_1 \times R_2$ where $(R_i, \mathfrak{m}_i)$ is a quasilocal ring for each $i \in \{1,2\}$ and try to classify such rings R in order that $\mathscr{C}(R)$ satisfies (Ku_2).*

Lemma 2.4. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. Suppose that R_i has at least two nonzero proper ideals for each $i \in \{1,2\}$. Then $\mathscr{C}(R)$ does not satisfy (Ku_2).*

Proof. We are assuming that R_i has at least two nonzero proper ideals for each $i \in \{1,2\}$. Let I_1, I_2 be distinct nonzero proper ideals of R_1 and let J_1, J_2 be distinct nonzero proper ideals of R_2. Let $V_1 = \{I_1 \times R_2, I_2 \times R_2, (0) \times R_2\}$ and let $V_2 = \{R_1 \times J_1, R_1 \times J_2, R_1 \times (0)\}$. Observe that $V_1 \cup V_2 \subseteq V(\mathscr{C}(R))$ and $V_1 \cap V_2 = \emptyset$. As $(I_i \times R_2) + (R_1 \times J_k) = R_1 \times R_2$ for all $i, k \in \{1,2,3\}$ (where we set I_3 is the zero ideal of R_1 and J_3 = zero ideal of R_2), it follows that the subgraph of $\mathscr{C}(R)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. Therefore, $\mathscr{C}(R)$ does not satisfy (Ku_2). $\square$

Let I be an ideal of a ring R. Then the *annihilator of I in R*, denoted by $Ann_R I$ is defined as $Ann_R I = \{r \in R | Ir = (0)\}$.

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a local ring which is not a field. The following statements are equivalent:*

(i) R has only one nonzero proper ideal.

(ii) $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^2 = (0)$.

Proof. $(i) \Rightarrow (ii)$ We are assuming that R has only one nonzero proper ideal. Hence, $\mathfrak{m}$ is the only nonzero proper ideal of R. Let $x \in \mathfrak{m}, x \neq 0$. Then $\mathfrak{m} = Rx$. Note that $Ann_R\mathfrak{m}$ is a nonzero proper ideal of R and so, $Ann_R\mathfrak{m} = \mathfrak{m}$. Hence, $\mathfrak{m}^2 = (0)$. Therefore, $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^2 = (0)$.

$(ii) \Rightarrow (i)$ As R is not a field, $\mathfrak{m} \neq (0)$. Thus if $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^2 = (0)$, then it is clear that $\mathfrak{m}$ is the only nonzero proper ideal of R. □

Lemma 2.6. *Let $R = R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a quasilocal ring for each $i \in \{1, 2\}$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ satisfies (Ku_2).

(ii) For at least one $i \in \{1, 2\}$, $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathscr{C}(R)$ satisfies (Ku_2). We know from Lemma 2.4 that for at least one $i \in \{1, 2\}$, R_i has at most one nonzero proper ideal. Hence, for that i, either $\mathfrak{m}_i = (0)$ or in the case $\mathfrak{m}_i \neq (0)$, we obtain from Lemma 2.5 that $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$. Therefore, there exists at least one $i \in \{1, 2\}$ such that $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$.

$(ii) \Rightarrow (i)$ Without loss of generality, we can assume that $(R_1, \mathfrak{m}_1)$ is a SPIR with $\mathfrak{m}_1^2 = (0)$. Observe that $\{\mathfrak{M}_1 = \mathfrak{m}_1 \times R_2, \mathfrak{M}_2 = R_1 \times \mathfrak{m}_2\}$ is the set of all maximal ideals of R. Since $|Max(R)| = 2$, it follows from $(3) \Rightarrow (1)$ of [18, Theorem 4.5] that $\mathscr{C}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2, where V_i is the set of all proper ideals A of R such that $M(A) = \{\mathfrak{M}_i\}$ for each $i \in \{1, 2\}$. Note that if $A \in V_1$, then $A = I \times R_2$ for some ideal I of R_1 such that $I \subseteq \mathfrak{m}_1$. Since there are at most two proper ideals of R_1, we obtain that $|V_1| \leq 2$. It is now clear that $\mathscr{C}(R)$ satisfies (Ku_2). □

Proposition 2.7. *Let R be a ring such that $|Max(R)| = 2$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ is planar.

(ii) $\mathscr{C}(R)$ satisfies both (Ku_1^*) and (Ku_2^*).

(iii) $\mathscr{C}(R)$ satisfies (Ku_2).

(iv) $R \cong R_1 \times R_2$ as rings, where $(R_i, \mathfrak{m}_i)$ is a quasilocal ring for each $i \in \{1, 2\}$ and for at least one $i \in \{1, 2\}$, $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [7, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ This follows from Remark 2.3 and Lemma 2.6.

$(iv) \Rightarrow (i)$ Let us denote the ring $R_1 \times R_2$ by T. Without loss of generality, we can assume that $(R_1, \mathfrak{m}_1)$ is a SPIR with $\mathfrak{m}_1^2 = (0)$. Let V_1, V_2 be as in the proof of $(ii) \Rightarrow (i)$ of Lemma 2.6 and it is already noted there that $|V_1| \leq 2$ and $\mathscr{C}(T)$ is a complete bipartite graph with vertex partition V_1 and V_2. It is now clear that $\mathscr{C}(T)$ is planar. Since $R \cong T$ as rings, we get that $\mathscr{C}(R)$ is planar. □

Let R be a ring such that $|Max(R)| = 3$. We next try to classify such rings R in order that $\mathscr{C}(R)$ satisfies (Ku_2).

Lemma 2.8. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. If R_1 admits at least two maximal ideals and if $\mathscr{C}(R_1)$ does not satisfy (Ku_2), then $\mathscr{C}(R)$ does not satisfy (Ku_2).*

Proof. We are assuming that $\mathscr{C}(R_1)$ does not satisfy (Ku_2). Then there exist subsets $A = \{I_1, I_2, I_3\}$ and $B = \{J_1, J_2, J_3\}$ of $V(\mathscr{C}(R_1))$ such that $A \cap B = \emptyset$ and $I_i + J_k = R_1$ for all $i, k \in \{1, 2, 3\}$. Let $V_1 = \{I_1 \times R_2, I_2 \times R_2, I_3 \times R_2\}$ and let $V_2 = \{J_1 \times R_2, J_2 \times R_2, J_3 \times R_2\}$. Observe that $V_1 \cup V_2 \subseteq V(\mathscr{C}(R))$, $V_1 \cap V_2 = \emptyset$, and as $(I_i \times R_2) + (J_k \times R_2) = R_1 \times R_2 = R$ for all $i, k \in \{1, 2, 3\}$, it follows that the subgraph of $\mathscr{C}(R)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This proves that $\mathscr{C}(R)$ does not satisfy (Ku_2). □

Lemma 2.9. *Let R be a ring such that $|Max(R)| = 3$. If $\mathscr{C}(R)$ satisfies (Ku_2), then $R \cong R_1 \times R_2 \times R_3$ as rings, where R_i is a quasilocal ring for each $i \in \{1, 2, 3\}$.*

Proof. Assume that $\mathscr{C}(R)$ satisfies (Ku_2). As $|Max(R)| = 3$, it follows from Lemma 2.2 that there exist nonzero rings T_1 and T_2 such that $R \cong T_1 \times T_2$ as rings. Since R has exactly three maximal ideals, it follows that either T_1 or T_2 is not quasilocal. Without loss of generality, we can assume that T_1 is not quasilocal. Hence, the number of maximal ideals of T_1 is exactly two. Let us denote the ring $T_1 \times T_2$ by T. Since $R \cong T$ as rings, we obtain that $\mathscr{C}(T)$ satisfies (Ku_2). Now, it follows from Lemma 2.8 that $\mathscr{C}(T_1)$ satisfies (Ku_2). Hence, we obtain from Lemma 2.2 that there exist nonzero rings T_{11} and T_{12} such that $T_1 \cong T_{11} \times T_{12}$ as rings. Therefore, $R \cong T_{11} \times T_{12} \times T_2$ as rings. Hence, on renaming the rings T_{11}, T_{12}, and T_2, we obtain that there exist rings R_1, R_2, and R_3 such that $R \cong R_1 \times R_2 \times R_3$ as rings. Since $|Max(R)| = 3$, it is clear that R_i is quasilocal for each $i \in \{1, 2, 3\}$. □

Lemma 2.10. *Let R be a ring such that $|Max(R)| = 3$. If $\mathscr{C}(R)$ satisfies (Ku_2), then $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2, 3\}$.*

Proof. Assume that $\mathscr{C}(R)$ satisfies (Ku_2). We know from Lemma 2.9 that $R \cong R_1 \times R_2 \times R_3$ as rings, where R_i is a quasilocal ring for each $i \in \{1, 2, 3\}$. Let $\mathfrak{m}_i$ denote the unique maximal ideal of R_i for each $i \in \{1, 2, 3\}$. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. Since $R \cong T$ as rings, we obtain that $\mathscr{C}(T)$ satisfies (Ku_2). Let $i \in \{1, 2, 3\}$. It follows from Lemma 2.4 that R_i has at most one nonzero proper ideal. Hence, either $\mathfrak{m}_i = (0)$ in which case, R_i is a field or $\mathfrak{m}_i \neq (0)$ is the only nonzero proper ideal of R_i in which case, we obtain from Lemma 2.5 that $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$. This proves that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2, 3\}$. □

Lemma 2.11. *Let R be a ring such that $|Max(R)| = 3$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ denote the set of all maximal ideals of R. Let $i \in \{1, 2, 3\}$. Let us denote the set of all proper ideals I of R such that $M(I) = \{\mathfrak{m}_i\}$ by W_i. If $|W_i| \leq 2$ for each $i \in \{1, 2, 3\}$, then $\mathscr{C}(R)$ satisfies (Ku_2).*

Proof. Suppose that $\mathscr{C}(R)$ does not satisfy (Ku_2). Then there exist subsets $V_1 = \{I_1, I_2, I_3\}$ and $V_2 = \{J_1, J_2, J_3\}$ of $V(\mathscr{C}(R))$ such that $V_1 \cap V_2 = \emptyset$ and $I_i + J_k = R$ for all $i, k \in \{1, 2, 3\}$. After renaming the maximal ideals of R (if necessary), we can assume without loss of generality that $I_1 \subseteq \mathfrak{m}_1$. Since $I_1 + J_k = R$ for each $k \in \{1, 2, 3\}$, it follows that $J_k \not\subseteq \mathfrak{m}_1$ for each $k \in \{1, 2, 3\}$. By hypothesis, $|W_2| \leq 2$ and $|W_3| \leq 2$. Therefore, we obtain that $W_2 \cap V_2 \neq \emptyset$ and $W_3 \cap V_2 \neq \emptyset$. This implies that $J_1 J_2 J_3 \subseteq \mathfrak{m}_2\mathfrak{m}_3$. It follows from $I_i + J_k = R$ for all $i, k \in \{1, 2, 3\}$ that $I_i + J_1 J_2 J_3 = R$ for each $i \in \{1, 2, 3\}$ and so, $I_i + \mathfrak{m}_2\mathfrak{m}_3 = R$. Hence, we get that $I_i \in W_1$ for each $i \in \{1, 2, 3\}$. This implies that $|W_1| \geq 3$. This is in contradiction to the assumption that $|W_1| \leq 2$. Therefore, $\mathscr{C}(R)$ satisfies (Ku_2). □

Let R be a ring such that $|Max(R)| = 3$. Let $i \in \{1, 2, 3\}$ and let $\mathfrak{m}_i, W_i$ be as in the statement of Lemma 2.11. In Proposition 2.12, we classify such rings R in order that $|W_i| \leq 2$ for each $i \in \{1, 2, 3\}$.

Proposition 2.12. *Let R be a ring such that $|Max(R)| = 3$. Let $\{\mathfrak{m}_i | i \in \{1, 2, 3\}\}$ denote the set of all maximal ideals of R. Let $i \in \{1, 2, 3\}$. Let us denote the set of all proper ideals I of R such that $M(I) = \{\mathfrak{m}_i\}$ by W_i. Then the following statements are equivalent:*

(i) $|W_i| \leq 2$ for each $i \in \{1, 2, 3\}$.

(ii) $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{n}_i)$ is a SPIR with $\mathfrak{n}_i^2 = (0)$ for each $i \in \{1, 2, 3\}$.

Proof. $(i) \Rightarrow (ii)$ Let $i \in \{1, 2, 3\}$. We claim that $\mathfrak{m}_i$ is principal. First, we verify that $\mathfrak{m}_1$ is principal. Suppose that $\mathfrak{m}_1$ is not principal. Observe that $\mathfrak{m}_1 \not\subseteq \mathfrak{m}_2 \cup \mathfrak{m}_3$. Let $a_1 \in \mathfrak{m}_1 \backslash (\mathfrak{m}_2 \cup \mathfrak{m}_3)$. As $\mathfrak{m}_1 \neq Ra_1$ by assumption, it follows from [10, Theorem 81] that there exists $a_2 \in \mathfrak{m}_1 \backslash (Ra_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3)$. Note that $\mathfrak{m}_1 \neq Ra_2$ and it is clear from the choice of the elements a_1, a_2 that $Ra_1 \neq Ra_2$ and $\{Ra_1, Ra_2, \mathfrak{m}_1\} \subseteq W_1$. This is in contradiction to the assumption that $|W_1| \leq 2$. Therefore, $\mathfrak{m}_1$ is principal. Similarly, it can be shown that $\mathfrak{m}_2$ and $\mathfrak{m}_3$ are principal. Let $i \in \{1, 2, 3\}$. Observe that $\mathfrak{m}_i^2 = \mathfrak{m}_i^3$. Suppose that $\mathfrak{m}_i^2 \neq \mathfrak{m}_i^3$. Then $\{\mathfrak{m}_i, \mathfrak{m}_i^2, \mathfrak{m}_i^3\} \subseteq W_i$. This is impossible, since $|W_i| \leq 2$. Therefore, $\mathfrak{m}_i^2 = \mathfrak{m}_i^3$. Since $\mathfrak{m}_i + \mathfrak{m}_j = R$ for all distinct $i, j \in \{1, 2, 3\}$, it follows from [3, Proposition 1.10(i)] that $J(R) = \cap_{i=1}^3 \mathfrak{m}_i = \prod_{i=1}^3 \mathfrak{m}_i$. Hence, $(J(R))^2 = \prod_{i=1}^3 \mathfrak{m}_i^2 = \prod_{i=1}^3 \mathfrak{m}_i^3 = (J(R))^3$. Now, as $J(R) = \prod_{i=1}^3 \mathfrak{m}_i$ is principal, there exists $a \in J(R)$ such that $J(R) = Ra$. From $(J(R))^2 = (J(R))^3$, we obtain that $Ra^2 = Ra^3$. Hence, $a^2 = ra^3$ for some $r \in R$. Since $1 - ra$ is a unit in R, we obtain that $a^2 = 0$ and so, $(J(R))^2 = (0)$. Since

$\mathfrak{m}_i^2 + \mathfrak{m}_j^2 = R$ for all distinct $i, j \in \{1,2,3\}$ and $\cap_{i=1}^3 \mathfrak{m}_i^2 = \prod_{i=1}^3 \mathfrak{m}_i^2 = (0)$, we obtain from the Chinese remainder theorem [3, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1^2} \times \frac{R}{\mathfrak{m}_2^2} \times \frac{R}{\mathfrak{m}_3^2}$ given by $f(r) = (r + \mathfrak{m}_1^2, r + \mathfrak{m}_2^2, r + \mathfrak{m}_3^2)$ is an isomorphism of rings. Let $i \in \{1,2,3\}$. Let us denote the ring $\frac{R}{\mathfrak{m}_i^2}$ by R_i. Let us denote $\frac{\mathfrak{m}_i}{\mathfrak{m}_i^2}$ by $\mathfrak{n}_i$. Since $\mathfrak{m}_i$ is a principal ideal of R, we obtain that $\mathfrak{n}_i$ is a principal ideal of R_i and it is clear that $\mathfrak{n}_i^2 = (0 + \mathfrak{m}_i^2)$. This shows that $(R_i, \mathfrak{n}_i)$ is a SPIR with $\mathfrak{n}_i^2$ is the zero ideal of R_i for each $i \in \{1,2,3\}$ and $R \cong R_1 \times R_2 \times R_3$ as rings.

$(ii) \Rightarrow (i)$ Assume that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{n}_i)$ is a SPIR with $\mathfrak{n}_i^2 = (0)$ for each $i \in \{1,2,3\}$. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. Observe that T is semilocal with $\{\mathfrak{N}_1 = \mathfrak{n}_1 \times R_2 \times R_3, \mathfrak{N}_2 = R_1 \times \mathfrak{n}_2 \times R_3, \mathfrak{N}_3 = R_1 \times R_2 \times \mathfrak{n}_3\}$ as its set of all maximal ideals. Let us denote the set of all proper ideals A of T such that $M(A) = \{\mathfrak{N}_i\}$ by U_i for each $i \in \{1,2,3\}$. Since R_i has at most one nonzero proper ideal for each $i \in \{1,2,3\}$, it follows that $|U_i| \leq 2$. From $R \cong T$ as rings, we obtain that $|W_i| \leq 2$ for each $i \in \{1,2,3\}$. □

Proposition 2.13. *Let R be a ring such that $|Max(R)| = 3$. The following statements are equivalent:*

- *(i) $\mathscr{C}(R)$ satisfies (Ku_2).*
- *(ii) $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1,2,3\}$.*

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathscr{C}(R)$ satisfies (Ku_2). We know from Lemma 2.10 that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1,2,3\}$.

$(ii) \Rightarrow (i)$ Assume that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1,2,3\}$. Let $\{\mathfrak{M}_i | i \in \{1,2,3\}\}$ denote the set of all maximal ideals of R. Let $i \in \{1,2,3\}$ and let us denote the set of all proper ideals I of R such that $M(I) = \{\mathfrak{M}_i\}$ by W_i. We know from $(ii) \Rightarrow (i)$ of Proposition 2.12 that $|W_i| \leq 2$. Hence, we obtain from Lemma 2.11 that $\mathscr{C}(R)$ satisfies (Ku_2). □

Let R be a ring such that $|Max(R)| = 3$. We try to classify such rings R in order that $\mathscr{C}(R)$ is planar. If $\mathscr{C}(R)$ is planar, then we know from Kuratowski's theorem [7, Theorem 5.9] that $\mathscr{C}(R)$ satisfies (Ku_2). Hence, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.13 that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1,2,3\}$.

Lemma 2.14. *Let $R = R_1 \times R_2 \times R_3$, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i \neq (0)$ but $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1,2,3\}$. Then $\mathscr{C}(R)$ does not satisfy (Ku_2^*).*

Proof. The proof of this lemma closely follows the proof given in [17, Lemma 3.13]. Note that $|Max(R)| = 3$ and $\{\mathfrak{M}_1 = \mathfrak{m}_1 \times R_2 \times R_3, \mathfrak{M}_2 = R_1 \times \mathfrak{m}_2 \times R_3, \mathfrak{M}_3 = R_1 \times R_2 \times \mathfrak{m}_3\}$ is the set of all maximal ideals of R. Let us denote the subgraph of $\mathscr{C}(R)$ induced on $W = \{v_1 = \mathfrak{M}_1, v_2 = \mathfrak{M}_1^2, v_3 = \mathfrak{M}_3, v_4 = \mathfrak{M}_2, v_5 = \mathfrak{M}_2^2, v_6 = \mathfrak{M}_3^2, v_7 = \mathfrak{M}_1 \cap \mathfrak{M}_2\}$ by H. Observe that in H, the edges $\mathfrak{M}_3 - \mathfrak{M}_1 \cap \mathfrak{M}_2$ and $\mathfrak{M}_1 \cap \mathfrak{M}_2 - \mathfrak{M}_3^2$ are in series and moreover, in H, v_i is adjacent to v_4 and v_5 for each $i \in \{1,2,3\}$. Furthermore in H, v_1 and v_2 are adjacent to v_6. Therefore, on merging the edges $v_3 - v_7$ and $v_7 - v_6$, we obtain a graph H_1 which contains $K_{3,3}$ as a subgraph. Hence, H contains a subgraph which is homeomorphic to $K_{3,3}$. This shows that $\mathscr{C}(R)$ does not satisfy (Ku_2^*). □

Lemma 2.15. *Let $R = F \times R_2 \times R_3$, where F is a field and $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i \neq (0)$ but $\mathfrak{m}_i^2 = (0)$ for each $i \in \{2,3\}$. Then $\mathscr{C}(R)$ does not satisfy (Ku_1^*).*

Proof. The proof of this lemma closely follows the proof given in [17, Lemma 3.14]. Observe that $|Max(R)| = 3$ and $\{\mathfrak{M}_1 = (0) \times R_2 \times R_3, \mathfrak{M}_2 = F \times \mathfrak{m}_2 \times R_3, \mathfrak{M}_3 = F \times R_2 \times \mathfrak{m}_3\}$ is the set of all maximal ideals of R. Let us denote the subgraph of $\mathscr{C}(R)$ induced on $W = \{v_1 = \mathfrak{M}_2, v_2 = \mathfrak{M}_1\mathfrak{M}_3, v_3 = \mathfrak{M}_2^2, v_4 = \mathfrak{M}_3, v_5 = \mathfrak{M}_1\mathfrak{M}_2, v_6 = \mathfrak{M}_3^2, v_7 = \mathfrak{M}_1\}$ by H. Note that in H, the edges $e_1 : v_1 - v_2$, $e_2 : v_2 - v_3$ are edges in series and the edges $e_3 : v_4 - v_5$, $e_4 : v_5 - v_6$ are edges in series. Observe that in H, v_1 is adjacent to all the elements of W except v_3 and v_5; v_3 is adjacent to all the elements of W except v_1 and v_5; v_4 is adjacent to all the elements of W except v_2 and v_6; v_6 is adjacent to all the elements of W except v_2

and v_4; v_7 is adjacent to all the elements of W except v_2 and v_5. Let H_1 be the graph obtained from H on merging the edges e_1 and e_2 and on merging the edges e_3 and e_4. It is clear that H_1 is a complete graph on five vertices. This proves that $\mathscr{C}(R)$ contains a subgraph H such that H is homeomorphic to K_5. Therefore, $\mathscr{C}(R)$ does not satisfy (Ku_1^*). □

Lemma 2.16. *Let $R = F_1 \times F_2 \times R_3$, where F_1 and F_2 are fields and $(R_3, \mathfrak{m}_3)$ is a SPIR with $\mathfrak{m}_3 \neq (0)$ but $\mathfrak{m}_3^2 = (0)$. Then $\mathscr{C}(R)$ is planar.*

Proof. Observe that $|Max(R)| = 3$ and $\{\mathfrak{M}_1 = (0) \times R_2 \times R_3, \mathfrak{M}_2 = F_1 \times (0) \times R_3, \mathfrak{M}_3 = F_1 \times F_2 \times \mathfrak{m}_3\}$ is the set of all maximal ideals of R. Observe that $V(\mathscr{C}(R))$ equals $\{v_1 = \mathfrak{M}_1, v_2 = \mathfrak{M}_2, v_3 = \mathfrak{M}_3, v_4 = \mathfrak{M}_1\mathfrak{M}_2, v_5 = \mathfrak{M}_2\mathfrak{M}_3, v_6 = \mathfrak{M}_1\mathfrak{M}_3, v_7 = \mathfrak{M}_3^2, v_8 = \mathfrak{M}_1\mathfrak{M}_3^2, v_9 = \mathfrak{M}_2\mathfrak{M}_3^2\}$. It is not hard to verify that $\mathscr{C}(R)$ is the union of the cycle $\Gamma : v_1 - v_3 - v_2 - v_7 - v_1$, the edges $e_1 : v_3 - v_4, e_2 : v_4 - v_7, e_3 : v_1 - v_2$, and the pendant edges $e_4 : v_1 - v_5, e_5 : v_1 - v_9, e_6 : v_2 - v_6$, and $e_7 : v_2 - v_8$. Note that Γ can be represented by means of a rectangle. The edges e_1, e_2 are edges in series and their common end vertex v_4 can be plotted inside the rectangle representing Γ and the edges e_1, e_2 can be drawn inside this rectangle. The edges $e_i (i \in \{3, 4, 5, 6, 7\})$ are such that one of their end vertices $\in \{v_1, v_2\}$ and they can be drawn outside the rectangle representing Γ in such a way that there are no crossing over of the edges. This proves that $\mathscr{C}(R)$ is planar. □

Lemma 2.17. *Let $R = F_1 \times F_2 \times F_3$, where F_i is a field for each $i \in \{1, 2, 3\}$. Then $\mathscr{C}(R)$ is planar.*

Proof. Note that $|Max(R)| = 3$ and $\{\mathfrak{M}_1 = (0) \times F_2 \times F_3, \mathfrak{M}_2 = F_1 \times (0) \times F_3, \mathfrak{M}_3 = F_1 \times F_2 \times (0)\}$ is the set of all maximal ideals of R. Observe that $V(\mathscr{C}(R))$ equals $\{v_1 = \mathfrak{M}_1, v_2 = \mathfrak{M}_2, v_3 = \mathfrak{M}_3, v_4 = \mathfrak{M}_1\mathfrak{M}_2, v_5 = \mathfrak{M}_2\mathfrak{M}_3, v_6 = \mathfrak{M}_1\mathfrak{M}_3\}$. It is clear that $\mathscr{C}(R)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_1$ and the pendant edges $e_1 : v_1 - v_5, e_2 : v_2 - v_6$, and $e_3 : v_3 - v_4$. The cycle Γ can be represented by means of a triangle and the one of the end vertex of e_i is v_i for each $i \in \{1, 2, 3\}$ and the edges e_1, e_2, e_3 can be drawn outside the triangle representing Γ in such a way that there are no crossing over of the edges. This proves that $\mathscr{C}(R)$ is planar. □

Theorem 2.18. *Let R be a ring such that $|Max(R)| = 3$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ is planar.

(ii) $\mathscr{C}(R)$ satisfies both (Ku_1^*) and (Ku_2^*).

(iii) $R \cong R_1 \times R_2 \times R_3$ as rings, where R_i is a field for at least two values of $i \in \{1, 2, 3\}$ and if $i \in \{1, 2, 3\}$ is such that R_i is not a field, then $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [7, Theorem 5.9].

$(ii) \Rightarrow (iii)$ Since $\mathscr{C}(R)$ satisfies (Ku_2^*), we get that $\mathscr{C}(R)$ satisfies (Ku_2). Therefore, we obtain from Proposition 2.13 that $R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2, 3\}$. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. Since $R \cong T$ as rings, we obtain that $\mathscr{C}(T)$ satisfies (Ku_1^*) and (Ku_2^*). As $\mathscr{C}(T)$ satisfies (Ku_2^*), it follows from Lemma 2.14 that R_i is a field for at least one value of $i \in \{1, 2, 3\}$. Suppose that R_i is a field for exactly one value of $i \in \{1, 2, 3\}$. Without loss of generality, we can assume that R_1 is a field and R_2, R_3 are not fields. In such a case, we obtain from Lemma 2.15 that $\mathscr{C}(T)$ does not satisfy (Ku_1^*). This is a contradiction. Therefore, R_i is a field for at least two values of $i \in \{1, 2, 3\}$. This proves that $R \cong R_1 \times R_2 \times R_3$ as rings, where R_i is a field for at least two values of $i \in \{1, 2, 3\}$ and if $i \in \{1, 2, 3\}$ is such that R_i is not a field, then $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$.

$(iii) \Rightarrow (i)$ Suppose that $R \cong R_1 \times R_2 \times R_3$ as rings, where R_i is a field for at least two values of $i \in \{1, 2, 3\}$ and if $i \in \{1, 2, 3\}$ is such that R_i is not a field, then $(R_i, \mathfrak{m}_i)$ is a SPIR with $\mathfrak{m}_i^2 = (0)$. Let us denote the ring $R_1 \times R_2 \times R_3$ by T. Note that either R_i is a field for each $i \in \{1, 2, 3\}$, in which case, we obtain from Lemma 2.17 that $\mathscr{C}(T)$ is planar or there are exactly two values of $i \in \{1, 2, 3\}$ such that R_i is a field and in such a case, we obtain from Lemma 2.16 that $\mathscr{C}(T)$ is planar. Since $R \cong T$ as rings, we get that $\mathscr{C}(R)$ is planar. □

3. When is $\mathscr{C}(R)$ a split graph?

Let R be a ring such that $|Max(R)| \geq 2$. The aim of this section is to classify rings R such that $\mathscr{C}(R)$ is a split graph. Throughout this section, we assume that K is a nonempty subset of $V(\mathscr{C}(R))$ such that the subgraph of $\mathscr{C}(R)$ induced on K is complete and S is a nonempty subset of $V(\mathscr{C}(R))$ such that S is an independent set of $\mathscr{C}(R)$.

Lemma 3.1. *Let R be a ring such that $\mathscr{C}(R)$ is a split graph with $V(\mathscr{C}(R)) = K \cup S$ and $K \cap S = \emptyset$. If $|Max(R)| \geq 3$, then $Max(R) = K$.*

Proof. First, we claim that $Max(R)) \subseteq K$. Suppose that $Max(R) \not\subseteq K$. Then there exists $\mathfrak{m} \in Max(R)$ such that $\mathfrak{m} \notin K$. Hence, $\mathfrak{m} \in S$. We are assuming that $|Max(R)| \geq 3$. Therefore, there exist distinct maximal ideals $\mathfrak{m}', \mathfrak{m}''$ of R such that $\mathfrak{m}' \neq \mathfrak{m}$ and $\mathfrak{m}'' \neq \mathfrak{m}$. Since $\mathfrak{m}+\mathfrak{m}' = \mathfrak{m}+\mathfrak{m}'' = \mathfrak{m}+\mathfrak{m}'\mathfrak{m}'' = R$, we get that $\mathfrak{m}', \mathfrak{m}''$, and $\mathfrak{m}'\mathfrak{m}''$ are adjacent to $\mathfrak{m}$ in $\mathscr{C}(R)$. As $\mathfrak{m} \in S$, we obtain that $\mathfrak{m}', \mathfrak{m}'', \mathfrak{m}'\mathfrak{m}'' \in K$. Hence, $\mathfrak{m}'$ and $\mathfrak{m}'\mathfrak{m}''$ must be adjacent in $\mathscr{C}(R)$. This is impossible, since $\mathfrak{m}' + \mathfrak{m}'\mathfrak{m}'' = \mathfrak{m}' \neq R$. Therefore, $Max(R) \subseteq K$. We next verify that $K \subseteq Max(R)$. Let $I \in K$. Then I is a proper ideal of R and so, there exists a maximal ideal $\mathfrak{m}$ of R such that $I \subseteq \mathfrak{m}$. We assert that $I = \mathfrak{m}$. Suppose that $I \neq \mathfrak{m}$. Then I and $\mathfrak{m}$ are adjacent in $\mathscr{C}(R)$. This is impossible, since $I + \mathfrak{m} = \mathfrak{m} \neq R$. Hence, $I = \mathfrak{m}$ and this proves that $K \subseteq Max(R)$ and so, $K = Max(R)$. $\square$

Lemma 3.2. *Let R be a ring. If $\mathscr{C}(R)$ is a split graph, then $|Max(R)| \leq 3$.*

Proof. Suppose that $|Max(R)| \geq 4$. Now, $V(\mathscr{C}(R)) = K \cup S$ with $K \cap S = \emptyset$. Since $\mathscr{C}(R)$ is a split graph by assumption, we obtain from Lemma 3.1 that $Max(R) = K$. Let $\{\mathfrak{m}_i | i \in \{1,2,3,4\}\} \subseteq Max(R)$. Note that for all distinct $i, j \in \{1,2,3,4\}$, $\mathfrak{m}_i\mathfrak{m}_j \notin Max(R) = K$ and so, $\mathfrak{m}_i\mathfrak{m}_j \in S$. Hence, both $\mathfrak{m}_1\mathfrak{m}_2$ and $\mathfrak{m}_3\mathfrak{m}_4$ must be in S. Therefore, $\mathfrak{m}_1\mathfrak{m}_2$ cannot be adjacent to $\mathfrak{m}_3\mathfrak{m}_4$ in $\mathscr{C}(R)$. This is impossible, since $\mathfrak{m}_1\mathfrak{m}_2 + \mathfrak{m}_3\mathfrak{m}_4 = R$. Therefore, $|Max(R)| \leq 3$. $\square$

Let R be a ring such that $|Max(R)| = 3$. In Theorem 3.3, we classify such rings R in order that $\mathscr{C}(R)$ is a split graph.

Theorem 3.3. *Let R be a ring such that $|Max(R)| = 3$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ is a split graph.

(ii) $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1,2,3\}$.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathscr{C}(R)$ is a split graph. Then $V(\mathscr{C}(R)) = K \cup S$ with $K \cap S = \emptyset$. As $|Max(R)| = 3$, we obtain from Lemma 3.1 that $Max(R) = K$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ denote the set of all maximal ideals of R. Let $i \in \{1,2,3\}$ and let us denote the set of all proper ideals I of R such that $M(I) = \{\mathfrak{m}_i\}$ by W_i. We assert that $W_i = \{\mathfrak{m}_i\}$ for each $i \in \{1,2,3\}$. First, we show that $W_1 = \{\mathfrak{m}_1\}$. It is clear that $\mathfrak{m}_1 \in W_1$. Let $I \in W_1$ be such that $I \neq \mathfrak{m}_1$. As $K = Max(R)$, it follows that I must be in S. Note that $\mathfrak{m}_2\mathfrak{m}_3 \in S$. It is clear that $I \neq \mathfrak{m}_2\mathfrak{m}_3$. Now, $I, \mathfrak{m}_2\mathfrak{m}_3 \in S$ and S is an independent set of $\mathscr{C}(R)$, we get that I and $\mathfrak{m}_2\mathfrak{m}_3$ cannot be adjacent in $\mathscr{C}(R)$. However, $I + \mathfrak{m}_2\mathfrak{m}_3 = R$. This is a contradiction and so, $I = \mathfrak{m}_1$. This shows that $W_1 = \{\mathfrak{m}_1\}$. Similarly, it can be shown that $W_2 = \{\mathfrak{m}_2\}$ and $W_3 = \{\mathfrak{m}_3\}$. We next show that $\mathfrak{m}_i$ is principal for each $i \in \{1,2,3\}$. Note that $\mathfrak{m}_1 \not\subseteq \mathfrak{m}_2 \cup \mathfrak{m}_3$. Hence, there exists $x_1 \in \mathfrak{m}_1 \backslash (\mathfrak{m}_2 \cup \mathfrak{m}_3)$. Observe that $Rx_1 \in W_1 = \{\mathfrak{m}_1\}$ and so, $\mathfrak{m}_1 = Rx_1$. Similarly, using the facts that $W_2 = \{\mathfrak{m}_2\}$ and $W_3 = \{\mathfrak{m}_3\}$, it can be proved that $\mathfrak{m}_2 = Rx_2$ for any $x_2 \in \mathfrak{m}_2 \backslash (\mathfrak{m}_1 \cup \mathfrak{m}_3)$ and $\mathfrak{m}_3 = Rx_3$ for any $x_3 \in \mathfrak{m}_3 \backslash (\mathfrak{m}_1 \cup \mathfrak{m}_2)$. It is clear that $J(R) = \cap_{i=1}^{3} \mathfrak{m}_i = \prod_{i=1}^{3} \mathfrak{m}_i = Rx_1x_2x_3$. Let $i \in \{1,2,3\}$. Note that $\mathfrak{m}_i^2 \in W_i = \{\mathfrak{m}_i\}$ and so, $\mathfrak{m}_i = \mathfrak{m}_i^2$. This implies that $\prod_{i=1}^{3} \mathfrak{m}_i = \prod_{i=1}^{3} \mathfrak{m}_i^2$. Hence, $Rx_1x_2x_3 = Rx_1^2x_2^2x_3^2$. Therefore, $x_1x_2x_3 = rx_1^2x_2^2x_3^2$ for some $r \in R$. As $x_1x_2x_3 \in J(R)$, we obtain that $1 - rx_1x_2x_3$ is a unit in R and so, $x_1x_2x_3 = 0$. Since $\mathfrak{m}_i + \mathfrak{m}_j = R$ for all distinct $i, j \in \{1,2,3\}$ and $J(R) = (0)$, we obtain from the Chinese remainder theorem [3, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \frac{R}{\mathfrak{m}_3}$ given by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2, r + \mathfrak{m}_3)$ is an isomorphism of rings. Let us denote the field $\frac{R}{\mathfrak{m}_i}$ by F_i for each $i \in \{1,2,3\}$. Therefore, $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1,2,3\}$.

$(ii) \Rightarrow (i)$ We are assuming that $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1,2,3\}$. Let us denote the ring $F_1 \times F_2 \times F_3$ by T. Note that $V(\mathscr{C}(T)) = \{\mathfrak{m}_1 = (0) \times F_2 \times F_3, \mathfrak{m}_2 = F_1 \times (0) \times F_3, \mathfrak{m}_3 = F_1 \times F_2 \times (0), \mathfrak{m}_1\mathfrak{m}_2 = (0) \times (0) \times F_3, \mathfrak{m}_2\mathfrak{m}_3 = F_1 \times (0) \times (0), \mathfrak{m}_1\mathfrak{m}_3 = (0) \times F_2 \times (0)\}$. Let $K = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ and let $S = \{\mathfrak{m}_1\mathfrak{m}_2, \mathfrak{m}_2\mathfrak{m}_3, \mathfrak{m}_1\mathfrak{m}_3\}$. It is clear that $V(\mathscr{C}(T)) = K \cup S$, $K \cap S = \emptyset$, the subgraph of $\mathscr{C}(T)$ induced on K is complete and S is an independent set of $\mathscr{C}(T)$. Therefore, $\mathscr{C}(T)$ is a split graph. As $R \cong T$ as rings, we obtain that $\mathscr{C}(R)$ is a split graph. $\square$

Let R be a ring such that $|Max(R)| = 2$. We next try to classify such rings in order that $\mathscr{C}(R)$ is a split graph. Lemma 3.4 is well-known. We include a proof of Lemma 3.4 for the sake of completeness.

Lemma 3.4. *Let $G = (V, E)$ be a complete bipartite graph. The following statements are equivalent:*

(i) *G is a split graph.*

(ii) *G is a star graph.*

Proof. $(i) \Rightarrow (ii)$ Assume that G is a split graph. Hence, there exist nonempty subsets K, S of V such that $V = K \cup S$, $K \cap S = \emptyset$, the subgraph of G induced on K is complete, and S is an independent set of G. By hypothesis, G is a complete bipartite graph. Let G be complete bipartite with vertex partition V_1 and V_2. We claim that $S \cap V_i = \emptyset$ for some $i \in \{1,2\}$. Suppose that $S \cap V_i \neq \emptyset$ for each $i \in \{1,2\}$. Let $s_i \in S \cap V_i$ for each $i \in \{1,2\}$. Then s_1 and s_2 are adjacent in G. This is impossible, since S is an independent set of G. Therefore, either $S \cap V_1 = \emptyset$ or $S \cap V_2 = \emptyset$. Without loss of generality, we can assume that $S \cap V_2 = \emptyset$. Hence, $S = S \cap V = S \cap (V_1 \cup V_2) = (S \cap V_1) \cup (S \cap V_2) = S \cap V_1$ and so, $S \subseteq V_1$. It follows from $V = V_1 \cup V_2 = S \cup K$ and $S \subseteq V_1$ that $V_2 \subseteq K$. Since no two distinct elements of V_2 are adjacent in G, whereas any two distinct vertices of K are adjacent in G, it follows that $|V_2| = 1$. This shows that G is a star graph.

$(ii) \Rightarrow (i)$ Suppose that G is a star graph. Hence, G is a complete bipartite graph with vertex partition V_1 and V_2 such that $|V_i| = 1$ for at least one $i \in \{1,2\}$. Without loss of generality, we can assume that $|V_1| = 1$. With $K = V_1$ and $S = V_2$, it is clear that $V = K \cup S, K \cap S = \emptyset$, the subgraph of G induced on K is complete, and S is an independent of G. Therefore, G is a split graph. $\square$

Theorem 3.5. *Let R be a ring such that $|Max(R)| = 2$. The following statements are equivalent:*

(i) *$\mathscr{C}(R)$ is a split graph.*

(ii) *$R \cong F \times S$ as rings, where F is a field and S is a quasilocal ring.*

Proof. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R.

$(i) \Rightarrow (ii)$ Assume that $\mathscr{C}(R)$ is a split graph. As $|Max(R)| = 2$, we know from $(3) \Rightarrow (1)$ of [18, Theorem 4.5] that $\mathscr{C}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2, where for each $i \in \{1,2\}$, V_i is the set of all proper ideals I of R such that $M(I) = \{\mathfrak{m}_i\}$. As we are assuming that $\mathscr{C}(R)$ is a split graph, we obtain from Lemma 3.4 that $\mathscr{C}(R)$ is a star graph. Hence, there exists a vertex I of $\mathscr{C}(R)$ such that I is adjacent to each vertex J of $\mathscr{C}(R)$ with $J \neq I$. We can assume without loss of generality that $I \in V_1$. In such a case, we obtain that $V_1 = \{I\}$. It is clear that $\mathfrak{m}_1 \in V_1$ and so, $I = \mathfrak{m}_1$. Let $a \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. As $Ra \in V_1$, we get that $\mathfrak{m}_1 = Ra$. Observe that $a^2 \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. Hence, $Ra^2 \in V_1$ and so, $Ra = Ra^2 = \mathfrak{m}_1$. Now, there exists $r \in R$ such that $a = ra^2$. This implies that $e = ra$ is a nontrivial idempotent element of R and moreover, $\mathfrak{m}_1 = Ra = Re$. Note that the mapping $f : R \to R(1-e) \times Re$ defined by $f(x) = (x(1-e), xe)$ is an isomorphism of rings. Hence, $f(\mathfrak{m}_1) = (0) \times Re$ is a maximal ideal of $R(1-e) \times Re$. Therefore, $R(1-e)$ is a field. Since $|Max(R)| = 2$, it follows that the ring Re is quasilocal. Thus with $F = R(1-e)$ and $S = Re$, we obtain that F is a field and S is a quasilocal ring and $R \cong F \times S$ as rings.

$(ii) \Rightarrow (i)$ Assume that $R \cong F \times S$ as rings, where F is a field and S is a quasilocal ring. Let us denote the ring $F \times S$ by T. Let $V_1 = \{(0) \times S\}$ and $V_2 = \{F \times I | I$ is a proper ideal of $S\}$. Note that $\mathscr{C}(T)$ is a complete bipartite graph with vertex partition V_1 and V_2 and as $|V_1| = 1$, it follows that $\mathscr{C}(T)$ is a star graph. Hence, we obtain from $(ii) \Rightarrow (i)$ of Lemma 3.4 that $\mathscr{C}(T)$ is a split graph. Since $R \cong T$ as rings, we get that $\mathscr{C}(R)$ is a split graph. □

4. Some more results on $\mathscr{C}(R)$

Let R be a ring such that $|Max(R)| \geq 2$. The aim of this section is to classify rings R such that $\mathscr{C}(R)$ is complemented and to determine the S-vertices of $\mathscr{C}(R)$.

Let R be a ring. Let X be the set of all prime ideals of R. Recall from [3, Exercise 15, page 12] that for a subset E of R, the set of all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \supseteq E$ is denoted by $V(E)$. We know from [3, Exercise 15, page 12] that the collection $\{V(E)|E \subseteq R\}$ satisfies the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. The topological space X is called the *prime spectrum* of R and is denoted by $Spec(R)$. Let $E \subseteq R$. We know from [3, Exercise 15(i), page 12] that $V(E) = V(I)$, where I is the ideal of R generated by E. The subspace of $Spec(R)$ consisting of all the maximal ideals of R with the induced topology is called the *maximal spectrum* of R and is denoted by $Max(R)$. The collection $\{V(I) \cap Max(R)|I$ varies over all ideals of $R\}$ is the collection of all closed sets of $Max(R)$. As in [15], we denote $V(I) \cap Max(R)$ by $M(I)$. Thus $M(I)$, as mentioned in the introduction, is the set of all maximal ideals $\mathfrak{m}$ of R such that $\mathfrak{m} \supseteq I$. As in [15], for an element $a \in R$, we denote $M(Ra)$ simply by $M(a)$ and $Max(R)\backslash M(a)$ by $D(a)$. Let $G = (V, E)$ be a simple graph. Let $v \in V$. Then the set of all $u \in V$ such that v is adjacent to u in G is called the set of *neighbours of* v in G and we use the notation $N_G(v)$ to denote the set of all neighbours of v in G.

Remark 4.1. *Let R be a ring such that $|Max(R)| \geq 2$. The following statements hold.*

(i) Let $I, J \in V(\mathscr{C}(R))$ be such that $I \perp J$ in $\mathscr{C}(R)$. Then $IJ \subseteq J(R)$.

(ii) If $\mathscr{C}(R)$ is complemented, then it is uniquely complemented.

Proof. *(i) Suppose that $IJ \not\subseteq J(R)$. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $IJ \not\subseteq \mathfrak{m}$. Hence, $I \not\subseteq \mathfrak{m}$ and $J \not\subseteq \mathfrak{m}$ and so, $I + \mathfrak{m} = J + \mathfrak{m} = R$. Now, $\mathfrak{m} \in V(\mathscr{C}(R))$ is such that $\mathfrak{m}$ is adjacent to both I and J in $\mathscr{C}(R)$. This is impossible, since $I \perp J$ in $\mathscr{C}(R)$. Therefore, $IJ \subseteq J(R)$.*

(ii) Let $I \in V(\mathscr{C}(R))$. We are assuming that $\mathscr{C}(R)$ is complemented. Hence, there exists at least one $J \in V(\mathscr{C}(R))$ such that $I \perp J$ in $\mathscr{C}(R)$. Let $J_1, J_2 \in V(\mathscr{C}(R))$ be such that $I \perp J_1$ and $I \perp J_2$ in $\mathscr{C}(R)$. We know from (i) that $IJ_i \subseteq J(R)$ for each $i \in \{1, 2\}$. Let $A \in V(\mathscr{C}(R))$ be such that J_1 is adjacent to A in $\mathscr{C}(R)$. Hence, $J_1 + A = R$. We claim that $J_2 + A = R$. Suppose that $J_2 + A \neq R$. Then there exists $\mathfrak{m} \in Max(R)$ such that $J_2 + A \subseteq \mathfrak{m}$. It follows from $I + J_2 = R$ that $I \not\subseteq \mathfrak{m}$. As $IJ_1 \subseteq J(R) \subseteq \mathfrak{m}$, we get that $J_1 \subseteq \mathfrak{m}$. Therefore, $J_1 + A \subseteq \mathfrak{m}$. This is impossible, since $J_1 + A = R$. Hence, $J_2 + A = R$. This shows that $N_{\mathscr{C}(R)}(J_1) \subseteq N_{\mathscr{C}(R)}(J_2)$. Similarly, it can be shown that $N_{\mathscr{C}(R)}(J_2) \subseteq N_{\mathscr{C}(R)}(J_1)$. Therefore, $N_{\mathscr{C}(R)}(J_1) = N_{\mathscr{C}(R)}(J_2)$. This proves that $\mathscr{C}(R)$ is uniquely complemented.

Lemma 4.2. *Let R be a ring such that $|Max(R)| \geq 2$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ is complemented.

(ii) $M(I)$ is a closed and open subset of $Max(R)$ for each $I \in V(\mathscr{C}(R))$.

Proof. We adapt an argument found in the proof of [15, Proposition 3.10].

$(i) \Rightarrow (ii)$ Assume that $\mathscr{C}(R)$ is complemented. Let $I \in V(\mathscr{C}(R))$. It is clear that $M(I)$ is a closed subset of $Max(R)$. Since $\mathscr{C}(R)$ is complemented, there exists $J \in V(\mathscr{C}(R))$ such that $I \perp J$ in $\mathscr{C}(R)$. Hence, I and J are adjacent in $\mathscr{C}(R)$ and there is no $A \in V(\mathscr{C}(R))$ such that A is adjacent to both I and J in $\mathscr{C}(R)$. That is, $I + J = R$ and there is no proper ideal A of R with $A + I = A + J = R$. Note that there exist $a \in I$ and $b \in J$ such that $a + b = 1$. We claim that $M(I) = D(b)$. Let $\mathfrak{m} \in M(I)$. As $\mathfrak{m} \supseteq I$, $a \in I$, and $a + b = 1$, it follows that $b \notin \mathfrak{m}$. Hence, $\mathfrak{m} \in D(b)$. This shows that $M(I) \subseteq D(b)$. Let $\mathfrak{m} \in D(b)$.

Hence, $\mathfrak{m} + J = R$. Since, $I \perp J$ in $\mathscr{C}(R)$, it follows that $I + \mathfrak{m} \neq R$ and so, $I \subseteq \mathfrak{m}$. That is, $\mathfrak{m} \in M(I)$. This proves that $D(b) \subseteq M(I)$ and so, $M(I) = D(b)$ is a closed and open subset of $Max(R)$.

$(ii) \Rightarrow (i)$ Let $I \in V(\mathscr{C}(R))$. By assumption, $M(I)$ is a closed and open subset of $Max(R)$. Hence, there exists an ideal J of R such that $M(I) = Max(R)\backslash M(J)$. This implies that $I + J = R$ and $IJ \subseteq J(R)$. If $A \in V(\mathscr{C}(R))$ is such that $A + I = A + J = R$, then $A + IJ = R$. This is impossible, since $IJ \subseteq J(R)$. Therefore, there is no $A \in V(\mathscr{C}(R))$ such that A is adjacent to both I and J in $\mathscr{C}(R)$. This proves that $I \perp J$ in $\mathscr{C}(R)$. Therefore, $\mathscr{C}(R)$ is complemented. □

Proposition 4.3. *Let R be a ring such that $|Max(R)| \geq 2$. If R is semiquasilocal, then $\mathscr{C}(R)$ is complemented.*

Proof. Let $\{\mathfrak{m}_i | i \in \{1, 2, \ldots, n\}\}$ denote the set of all maximal ideals of R. Note that $V(\mathscr{C}(R)) = \{I | I$ is a proper ideal of R with $I \not\subseteq J(R)\}$. Let $I \in V(\mathscr{C}(R))$. Let $i_1, \ldots, i_t \in \{1, 2, \ldots, n\}$ be such that $M(I) = \{\mathfrak{m}_{i_1}, \ldots, \mathfrak{m}_{i_t}\}$. It is clear that $1 \leq t < n$. Let us denote the set $\{1, 2, \ldots, n\}\backslash\{i_1, \ldots, i_t\}$ by $\{i_{t+1}, \ldots, i_n\}$. Consider the ideal $J = \cap_{j=t+1}^{n} \mathfrak{m}_{i_j}$. It is clear that $M(J) = \{\mathfrak{m}_{i_j} | j \in \{t+1, \ldots, n\}\}$. It follows from $I + J = R$ and $IJ \subseteq J(R)$ that $M(I) = Max(R)\backslash M(J)$ is a closed and open subset of $Max(R)$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Lemma 4.2 that $\mathscr{C}(R)$ is complemented. □

Corollary 4.4. *Let R be a ring such that $|Max(R)| \geq 2$. If R is semiquasilocal, then $\mathscr{C}(R)$ is uniquely complemented.*

Proof. This follows from Proposition 4.3 and Remark 4.1(ii). □

Let R be a ring. As in [12], we call the graph studied by P.K. Sharma and S.M. Bhatwadekar in [16] as the comaximal graph of R and as in [12], we denote it using the notation $\Gamma(R)$. It is useful to recall here that the vertex set of $\Gamma(R)$ is the set of all elements of R and distinct vertices a and b are joined by an edge in $\Gamma(R)$ if and only if $Ra + Rb = R$. Moreover, as in [12], we use the notation $\Gamma_1(R)$ to denote the subgraph of $\Gamma(R)$ induced on $U(R)$; we use $\Gamma_2(R)$ to denote the subgraph of $\Gamma(R)$ induced on $NU(R)$; for a ring R with $|Max(R)| \geq 2$, we use $\Gamma_2(R)\backslash J(R)$ to denote the subgraph of $\Gamma(R)$ induced on $NU(R)\backslash J(R)$. It is shown in [15, Proposition 3.11] that $\Gamma_2(R)\backslash J(R)$ is complemented if and only if $dim(\frac{R}{J(R)}) = 0$. We prove in Theorem 4.7 that for a ring R with $|Max(R)| \geq 2$, $\mathscr{C}(R)$ is complemented if and only if $\frac{R}{J(R)} \cong F_1 \times F_2 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$.

Lemma 4.5. *Let R be a ring such that $|Max(R)| \geq 2$. The following statements are equivalent:*

(i) *$\mathscr{C}(R)$ is complemented.*

(ii) *$\mathscr{C}(\frac{R}{J(R)})$ is complemented.*

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathscr{C}(R)$ is complemented. Observe that $J(\frac{R}{J(R)})$ is the zero ideal of $\frac{R}{J(R)}$. Let $\frac{I}{J(R)} \in V(\mathscr{C}(\frac{R}{J(R)}))$. Then it is clear that $I \in V(\mathscr{C}(R))$. As $\mathscr{C}(R)$ is complemented, there exists $J \in V(\mathscr{C}(R))$ such that $I \perp J$ in $\mathscr{C}(R)$. We know from the proof of $(i) \Rightarrow (ii)$ of Lemma 4.2 that there exists $b \in J$ such that $M(I) = D(b)$. It is not hard to verify that $M(\frac{I}{J(R)}) = D(b + J(R))$. Hence, for any $\frac{I}{J(R)} \in V(\mathscr{C}(\frac{R}{J(R)}))$, $M(\frac{I}{J(R)})$ is a closed and open subset of $Max(\frac{R}{J(R)})$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Lemma 4.2 that $\mathscr{C}(\frac{R}{J(R)})$ is complemented.

$(ii) \Rightarrow (i)$ We are assuming that $\mathscr{C}(\frac{R}{J(R)})$ is complemented. Let $I \in V(\mathscr{C}(R))$. Let us denote the ideal $I + J(R)$ by A. Then $\frac{A}{J(R)} \in V(\mathscr{C}(\frac{R}{J(R)}))$. Since $\mathscr{C}(\frac{R}{J(R)})$ is complemented, it follows from the proof of $(i) \Rightarrow (ii)$ of Lemma 4.2 that there exists $b \in R\backslash J(R)$ such that $M(\frac{A}{J(R)}) = D(b + J(R))$. Now, it is easy to verify that $M(I) = D(b)$. Thus for any $I \in V(\mathscr{C}(R))$, $M(I)$ is a closed and open subset of $Max(R)$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Lemma 4.2 that $\mathscr{C}(R)$ is complemented. □

Let R be a ring. Recall from [8, Exercise 16, page 111] that R is said to be *von Neumann regular* if for each element $a \in R$, there exists $b \in R$ such that $a = a^2b$. We know from $(a) \Leftrightarrow (d)$ of [8, Exercise 16, page 111] that R is von Neumann regular if and only if $dim R = 0$ and R is reduced. Hence, if R is von Neumann regular, then $J(R) = nil(R) = (0)$. Let R be a von Neumann regular ring. Let $a \in R$. We know from $(1) \Rightarrow (3)$ of [8, Exercise 24, page 113] that $a = ue$, where u is a unit of R and e is an idempotent element of R. Hence, any ideal of R is a radical ideal of R. Let I be any ideal R. Since the set of all prime ideals of R equals the set of all maximal ideals of R, it follows from [3, Proposition 1.14] that $I = r(I)$ is the intersection of all the maximal ideals $\mathfrak{m}$ of R such that $\mathfrak{m} \in M(I)$.

Lemma 4.6. *Let R be a von Neumann regular ring such that $|Max(R)| \geq 2$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ is complemented.

(ii) $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings for some $n \geq 2$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathscr{C}(R)$ is complemented. Since $J(R) = (0)$, it is clear that $V(\mathscr{C}(R))$ equals the set of all nonzero proper ideals of R. Let I be a nonzero proper ideal of R. As $\mathscr{C}(R)$ is complemented, we know from the proof of $(i) \Rightarrow (ii)$ of Lemma 4.2 that $M(I) = D(b)$ for some nonzero nonunit b of R. Note that $b = ue$, where u is a unit of R and e is an idempotent element of R. Therefore, $M(I) = D(b) = D(e) = M(1 - e)$. Hence, $I = \cap_{\mathfrak{m} \in M(I)} \mathfrak{m} = \cap_{\mathfrak{m} \in M(1-e)} \mathfrak{m} = R(1 - e)$. This proves that each ideal of R is finitely generated and so, R is Noetherian. Therefore, we obtain from [8, Exercise 21, page 112] that there exist $n \in \mathbb{N}$ and fields $F_1 \ldots, F_n$ such that $R \cong F_1 \times \cdots \times F_n$ as rings. Since $|Max(R)| \geq 2$, it follows that $n \geq 2$.

$(ii) \Rightarrow (i)$ Assume that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings for some $n \geq 2$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$. Let us denote the ring $F_1 \times F_2 \times \cdots \times F_n$ by T. Note that T is semilocal with $\{\mathfrak{m}_1 = (0) \times F_2 \times \cdots \times F_n, \mathfrak{m}_2 = F_1 \times (0) \times \cdots \times F_n, \ldots, \mathfrak{m}_n = F_1 \times \cdots \times F_{n-1} \times (0)\}$ as its set of all maximal ideals. We know from Proposition 4.3 that $\mathscr{C}(T)$ is complemented. As $R \cong T$ as rings, we get that $\mathscr{C}(R)$ is complemented. □

Theorem 4.7. *Let R be a ring such that $|Max(R)| \geq 2$. The following statements are equivalent:*

(i) $\mathscr{C}(R)$ is complemented.

(ii) $\frac{R}{J(R)} \cong F_1 \times F_2 \times \cdots \times F_n$ as rings for some $n \geq 2$, where F_i is field for each $i \in \{1, 2, \ldots, n\}$.

(iii) R is semiquasilocal.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathscr{C}(R)$ is complemented. We know from $(i) \Rightarrow (ii)$ of Lemma 4.5 that $\mathscr{C}(\frac{R}{J(R)})$ is complemented. Note that $J(\frac{R}{J(R)})$ equals the zero ideal of $\frac{R}{J(R)}$. Let $a \in R$ be such that $a + J(R)$ is a nonzero nonunit of $\frac{R}{J(R)}$. As $\mathscr{C}(\frac{R}{J(R)})$ is complemented, we obtain from $(i) \Rightarrow (ii)$ of Lemma 4.2 that $M(a + J(R))$ is a closed and open subset of $Max(\frac{R}{J(R)})$. Therefore, it follows from [15, Lemma 1.2] that $dim(\frac{R}{J(R)}) = 0$. Thus $\frac{R}{J(R)}$ is reduced and zero-dimensional and so, $\frac{R}{J(R)}$ is von Neumann regular. It now follows from $(i) \Rightarrow (ii)$ of Lemma 4.6 that $\frac{R}{J(R)} \cong F_1 \times F_2 \times \cdots \times F_n$ for some $n \geq 2$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$.

$(ii) \Rightarrow (iii)$ We are assuming that $\frac{R}{J(R)} \cong F_1 \times F_2 \times \cdots \times F_n$ as rings for some $n \geq 2$, where F_i is a field for each $i \in \{1, 2, \ldots, n\}$. Hence, $\frac{R}{J(R)}$ is semilocal. As $|Max(R)| = |Max(\frac{R}{J(R)})|$, we get that R is semiquasilocal.

$(iii) \Rightarrow (i)$ Since R is semiquasilocal, we obtain from Proposition 4.3 that $\mathscr{C}(R)$ is complemented. □

Let R be a ring with $|Max(R)| \geq 2$. We next discuss some results regarding the S-vertices of $\mathscr{C}(R)$.

Remark 4.8. *Let R be a ring such that $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. We know from (3) $\Rightarrow$ (1) of [18, Theorem 4.5] that $\mathscr{C}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2, where $V_i = \{I | M(I) = \{\mathfrak{m}_i\}\}$ for each $i \in \{1, 2\}$. Hence, it is clear that no vertex of $\mathscr{C}(R)$ is a S-vertex of $\mathscr{C}(R)$. Therefore, for a ring R with $|Max(R)| \geq 2$, in determining the S-vertices of $\mathscr{C}(R)$, we assume that $|Max(R)| \geq 3$.*

Proposition 4.9. *Let R be a ring such that $|Max(R)| \geq 3$. Let $I \in V(\mathscr{C}(R))$. Then the following statements are equivalent:*

(i) I is a S-vertex of $\mathscr{C}(R)$.

(ii) $|Max(R) \backslash M(I)| \geq 2$.

Proof. $(i) \Rightarrow (ii)$ Now, $I \in V(\mathscr{C}(R))$ and we are assuming that I is a S-vertex of $\mathscr{C}(R)$. Hence, there exist distinct $I_1, I_2, I_3 \in V(\mathscr{C}(R))$ such that $I - I_1, I - I_2, I_2 - I_3$ are edges of $\mathscr{C}(R)$, but there is no edge joining I_1 and I_3 in $\mathscr{C}(R)$. Since I_1 and I_3 are not adjacent in $\mathscr{C}(R)$, there exists $\mathfrak{m} \in Max(R)$ such that $I_1 + I_3 \subseteq \mathfrak{m}$. It follows from $I + I_1 = I_2 + I_3 = R$ that $I \not\subseteq \mathfrak{m}$ and $I_2 \not\subseteq \mathfrak{m}$. As I_2 is proper ideal of R, there exists $\mathfrak{m}' \in Max(R)$ such that $I_2 \subseteq \mathfrak{m}'$. It is clear that $\mathfrak{m} \neq \mathfrak{m}'$ and it follows from $I + I_2 = R$ that $I \not\subseteq \mathfrak{m}'$. Hence, $\{\mathfrak{m}, \mathfrak{m}'\} \subseteq Max(R) \backslash M(I)$ and so, $|Max(R) \backslash M(I)| \geq 2$.

$(ii) \Rightarrow (i)$ Let $I \in V(\mathscr{C}(R))$ be such that $|Max(R) \backslash M(I)| \geq 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\} \subset Max(R)$ be such that $I \not\subseteq \mathfrak{m}_i$ for each $i \in \{1, 2\}$. By hypothesis, $|Max(R)| \geq 3$. Hence, there exists $\mathfrak{m}_3 \in Max(R) \backslash \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Observe that $I - \mathfrak{m}_1, I - \mathfrak{m}_2, \mathfrak{m}_2 - \mathfrak{m}_1 \cap \mathfrak{m}_3$ are edges of $\mathscr{C}(R)$, but there is no edge of $\mathscr{C}(R)$ joining $\mathfrak{m}_1$ and $\mathfrak{m}_1 \cap \mathfrak{m}_3$. This proves that I s a S-vertex of $\mathscr{C}(R)$. $\square$

Acknowledgment: We are thankful to the referee for the support.

References

[1] D. F. Anderson, P. S. Livingston, The zero–divisor graph of a commutative ring, J. Algebra 217(2) (1999) 434–447.

[2] D. F. Anderson, R. Levy, J. Shapiro, Zero–divisor graphs, von Neumann regular rings, and Boolean Algebras, J. Pure Appl. Algebra 180(3) (2003) 221–241.

[3] M. F. Atiyah, I. G. Macdonald, Introduction to Commutative Algebra, Addison–Wesley, Reading, Massachusetts, 1969.

[4] R. Balakrishnan, K. Ranganathan, A Textbook of Graph Theory, Universitext, Springer, 2000.

[5] M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings I, J. Algebra Appl. 10(4) (2011) 727–739.

[6] M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings II, J. Algebra Appl. 10(4) (2011) 741–753.

[7] N. Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice Hall of India Private Limited, New Delhi, 1994.

[8] R. Gilmer, Multiplicative Ideal Theory, Marcel–Dekker, New York, 1972.

[9] M. I. Jinnah, S. C. Mathew, When is the comaximal graph split?, Comm. Algebra 40(7) (2012) 2400–2404.

[10] I. Kaplansky, Commutative Rings, The University of Chicago Press, Chicago, 1974.

[11] R. Levy, J. Shapiro, The zero–divisor graph of von Neumann regular rings, Comm. Algebra 30(2) (2002) 745–750.

[12] H. R. Maimani, M. Salimi, A. Sattari, S. Yassemi, Comaximal graph of commutative rings, J. Algebra 319(4) (2008) 1801–1808.

[13] S. M. Moconja, Z. Z. Petrovic, On the structure of comaximal graphs of commutative rings with identity, Bull. Aust. Math. Soc. 83(1) (2011) 11–21.

[14] A. M. Rahimi, Smarandache vertices of the graphs associated to the commutative rings, Comm. Algebra 41(5) (2013) 1989–2004.
[15] K. Samei, On the comaximal graph of a commutative ring, Canad. Math. Bull. 57(2) (2014) 413–423.
[16] P. K. Sharma, S. M. Bhatwadekar, A note on graphical representation of rings, J. Algebra 176(1) (1995) 124–127.
[17] S. Visweswaran, P. Sarman, On the planarity of a graph associated to a commutative ring and on the planarity of its complement, São Paulo J. Math. Sci. 11(2) (2017) 405–429.
[18] M. Ye, T. Wu, Co–maximal ideal graphs of commutative rings, J. Algebra Appl. 11(6) (2012) Article ID. 1250114 (14 pages).