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On a viscoelastic plate equation with a polynomial source term and $\vec{p}(x,t)-$ Laplacian operator in the presence of delay term

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Abstract. In this paper, the blow-up of solutions for the following Dirichlet-Neumann problem to initial nonlinear viscoelastic plate equation with a lower order perturbation of $\vec{p}(x,t)$-Laplacian operator in the presence of time delay is obtained

$$u_{tt} + \Delta^2 u + \Delta_{\vec{p}(x,t)} u - \int_0^t g(t-s)\Delta^2 u\,(s)\,ds - \mu_1 \Delta u_t - \mu_2 \Delta u_t(t-\tau) = u\,|u|^{q-2}.$$

Under suitable conditions on g and the variable exponent of the $\vec{p}(x,t)-$ Laplacian operator, we prove that any weak solution with nonpositive initial energy as well as positive initial energy blows up in a finite time.

Keywords: Blow-up, time delay, viscoelasticity, plate equation, nonstandard growth conditions, anisotropy.

2020 Mathematics Subject Classification: 39A10, 67B89.

1 Introduction

In this paper, we consider the Dirichlet–Neumann problem to the following initial nonlinear viscoelastic plate equation with a lower order perturbation of $\vec{p}(x,t)$-Laplacian operator and delay:

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta_{\vec{p}(x,t)} u - \int_0^t g(t-s)\Delta^2 u\,(s)\,ds - \mu_1 \Delta u_t & \\ -\mu_2 \Delta u_t(t-\tau) = u\,|u|^{q-2}, & x \in \Omega, t > 0, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u\,(x,0) = u_0(x), u_t\,(x,0) = u_1(x), & x \in \Omega, \\ u_t\,(x,t-\tau) = f_0\,(x,t-\tau), & x \in \Omega, t \in (0,\tau), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $n \geq 2$ with Lipschitz-continuous boundary $\Gamma = \partial\Omega$, and $q \geq 2$ is a positive constant

$$\Delta_{\overrightarrow{p}(x,t)} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i} \right), \overrightarrow{p} = (p_1, p_2, ..., p_n),$$

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is the $\overrightarrow{p}(x,t)$-Laplacian operator. The constant μ_1 is positive and μ_2 is a real number, $\tau > 0$ represents the time delay, $g > 0$ is a memory kernel and f is the forcing term.

In the absence of the viscoelastic term and delay term ($g = 0$ and $\mu_2 = 0$), and with the usual $p-$Laplacian operator $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$, $(p \geq 2)$ the equation in (1.1) reduces to the fourth order wave equation

$$u_{tt} + \Delta^2 u_t + div\left(|\nabla u|^{p-2}\nabla u\right) - \epsilon\Delta u_t = f(x,t,u,u_t), \tag{1.2}$$

which describes elastoplastic-microstructure flows. The problem (1.2) has been extensively studied (see [5,6,24]), and results concerning solutions existence, nonexistence, and long-time behavior have been proved.

The problem (1.2) without damping or forcing terms is related to the elastoplastic-microstructure models for longitudinal motion of an elastoplastic bar there arises the model equation

$$u_{tt} + u_{xxxx} = +a\left(u_x^2\right)_x + f(x),$$

where $a < 0$ is a constant (see [5]). I. Chueshov and I. Lasiecka in [8,9] discussed

$$u_{tt} + \Delta^2 u + div\left(|\nabla u|^2\nabla u\right) - ku_t = \sigma\Delta(u^2) + f(u),$$

and proved the existence of finite-dimensional global attractors. When $\epsilon = 0$ and in the presence of the viscoelastic term ($g \neq 0$) in (1.2), Jorge Silva and Ma [13], established exponential stability of solutions under the condition

$$g'(t) \leq -cg(t),\ \forall t \geq 0,\ c > 0.$$

Andrade and al. [1] proved exponential stability of solutions for the plate equation with finite memory and p-Laplacian. In the presence of the Kelvin–Voigt type dissipation ($\epsilon \neq 0$). In [18], Nakao obtained the existence of a global decaying solution for wave the equation with Kelvin–Voigt dissipation and a derivative nonlinearity. Pukach et al. [19] established sufficient conditions of the nonexistence of solution for a nonlinear hyperbolic equation with memory generalizing the Voigt–Kelvin model. Recently, Cavalcanti et al. [3] considered intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density.

In [2, 11], the authors improved the results from [1] by establishing local and global existence, as well as the uniqueness of the weak solution $u(x,t)$ to problem (1.1). To be more important, the authors of [2] and [11] established the local and global existence, uniqueness of weak solutions and the asymptotic behavior of solutions.

Time delays so often arise in many physical, chemical, biological, thermal, and economic phenomena because these phenomena depend not only on the present state but also on the system's history in a more complicated way. In recent years, many works have been published concerning the wave equation with delay. Kafini and Messaoudi [14] considered the following nonlinear wave equation with delay

$$u_{tt} - div(|\nabla u|^{m-2}\nabla u) + \mu_1 u_t + \mu_2 u_t(t-\tau) = b|u|^{p-2}u.$$

They proved the blow-up result of solutions with negative initial energy and $p > m$. Later, Kafini, Messaoudi, and Nicaise [15] considered the blow-up of solutions with negative initial energy for the second-order abstract evolution system with delay. Motivated by previous works, we study the blow-up of solutions. Recently, Shun-Tang WU [23] investigated the following nonlinear viscoelastic problem with delay

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = b|u|^{p-2}u.$$

He proved the blow-up result with nonpositive and positive initial energy by modifying the method in [14, 15]. Motivated by previous works. Moreover, we can mention some new related works (see [20]). In this paper, we investigate the problem (1.1) and prove a finite-time blow-up result of solutions. We will see that the direct method introduced and developed by Georgiev and Todorova [12], in 1994, and Salim A. Messaoudi [17] is efficient in our case. Combining this method with some necessary modifications due to the nature of the problem treated here. Our paper is organized as follows: in the next section, we prepare some material needed in our proofs. Section 3 is devoted to the statement and proof of the finite-time blow-up result.

$$1 \le p_i^- = const \le p_i(x,t) = p(x) \le p_i^+ = const < \infty, |p_{it}| \le C_{p_i}, i = 1, ..., n. \tag{1.3}$$

2 The functions space

Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be a bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$, $q \ge 2$ is a positive constant. We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with a compact support contained in Ω. The inner products and norms in $L^2(\Omega)$ and $H_0^1(\Omega)$ are represented by $(.,.)$, $\|.\|$ respectively and they are given by

$$(u,v)_\Omega = \int_\Omega u(x)v(x)dx \text{ and } \|u\|^2_{L^2(\Omega)} = \|\nabla u\|^2_{2,\Omega} = \int_\Omega u^2 dx\,,$$

$$\|u\|^2_{H_0^1(\Omega)} = \|\nabla u\|^2_{2,\Omega} = \int_\Omega |\nabla u|^2\,dx$$

We recall some known facts from the theory of the Sobolev spaces with variable exponent (see [4, 10]). Let $L^{p(.)}(\Omega)$ be the set of measurable functions f on Ω such that

$$A_{p(.)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

The set $L^{p(.)}(\Omega)$ equipped with the Luxemburg norm

$$\|f\|_{p(.),\Omega} = \|f\|_{p(.)} = \inf\left\{\lambda > 0; A_{p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}$$

is a Banach space. Let us list some basic properties of the spaces $L^{p(.)}(\Omega)$ used in the rest of this paper. It follows directly from the definition of the norm that

$$\min\left(\|f\|^{p^-}_{p(.)}, \|f\|^{p^+}_{p(.)}\right) \le A_{p(.)}(f) \le \max\left(\|f\|^{p^-}_{p(.)}, \|f\|^{p^+}_{p(.)}\right),$$

where

$$p^- = \inf_\Omega p(x), (p')^- = \inf_\Omega p'(x), p' = \frac{p(x)}{(p(x)-1)}.$$

We have following Holder-type inequality

$$\begin{aligned}\int_{\Omega}|fg|\,dx &\leq \left(\frac{1}{p^{-}}+\frac{1}{(p')^{-}}\right)\|f\|_{p(.)}\|g\|_{p'(.)}\\ &\leq 2\|f\|_{p(.)}\|g\|_{p'(.)},\end{aligned}$$

which holds for all $f \in L^{p(.)}(\Omega), g \in L^{p'(.)}(\Omega)$ with $p(x) \in (1,\infty)$. The Sobolev space $W_0^{1,p(.)}(\Omega)$ with $p(x) \in [p^-, p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) : |\nabla u|^{p(x)} \in L^1(\Omega), u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(.)}(\Omega)} = \sum_i \|D_i u\|_{p(.),\Omega} + \|u\|_{p(.),\Omega} \end{cases} \tag{2.1}$$

Let $p(x)$ be log-continuous in $\Omega, \forall x, y \in \Omega$ such that $|x-y| < \frac{1}{2}$

$$|p(x)-p(y)| \leq \omega(|x-y|) \text{ with } \overline{\lim}_{\tau\longrightarrow 0_+}\left(\omega(\tau)\ln\frac{1}{\tau}\right) = C < \infty \tag{2.2}$$

- Throughout the paper we use the following properties of the functions from the spaces $W_0^{1,p(.)}(\Omega)$:
- if condition (2.2) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(.)}(\Omega)$ and the space $W_0^{1,p(.)}(\Omega)$ can be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1) see [3,11,22,27],
- if $p(x) \in C^0\left(\overline{\Omega}\right)$, the the space $W_0^{1,p(.)}(\Omega)$ is separable and reflexive,
- if $1 < q(x) \leq \sup_\Omega q(x) < \inf_\Omega p_+(x)$ with

$$p_+(x) = \begin{cases} \frac{p(x)}{n-p(x)} \text{ if } p(x) < n, \\ \infty \text{ if } p(x) < n, \end{cases}$$

  then the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact if $q < p_+(x)$.

3 Anisotropic spaces of functions depending on x and t

Consider the cylinder

$$Q_T = \{z = (x,t) : x \in \Omega, t \in [0,T]\}$$

of a definite height T. Wherever it doesn't cause a confusion, we will use the notation $z = (x,t)$ for the points of the cylinder Q_T and drop the sub-index$T$. The lateral boundary of the cylinder Q is $\Gamma = \partial\Omega \times (0,T)$. If X is a Banach space, then we denote by $L^p(0,T,X), 1 \leq p \leq \infty$ the Banach space of measurable vector valued functions $u : (0,T) \longrightarrow X$, such that

$$\|u(t)\|_{L^p(0,T,X)} = \left[\int_0^T \|u(t)\|_X^p\,dt\right]^{\frac{1}{p}}, 1 \leq p < \infty,$$

$$\|u(t)\|_{L^p(0,T,X)} = ess \sup_{0\leq t\leq T} \|u(t)\|_X, p = \infty.$$

We will use the following function spaces (see [3,4])

$$W = W(Q_T) = \left\{u : u \in L^2\left(0,T,H_0^2(\Omega)\right); u_t \in L^2\left(0,T,H_0^1(\Omega)\right)\right\},$$

$$W^{\infty} = W^{\infty}(Q_T) = \left\{ u : u \in W(Q_T) ; u \in L^{\infty}\left(0, T, H_0^2(\Omega)\right) ; u_t \in L^{\infty}\left(0, T, L^2(\Omega)\right) \right\}$$

endowed with the norms

$$\|u\|_{W(Q)} = \|u\|_{W(Q_T)} + \|u\|_{L^{\infty}\left(0,T,H_0^2(\Omega)\right)} + \|u\|_{L^{\infty}(0,T,L^2(\Omega))}$$

Note that $\|u\|_{W(Q)}$ may be used in the equivalent form

$$\|u\|_{W(Q)} = \|u\|_{L^2(Q)} + \|\Delta u\|_{L^2(Q)} + \|\nabla u_t\|_{L^2(Q)}.$$

Let $p(z) = \overrightarrow{p} = (p_1(z), ..., p_n(z))$ be a vector-valued function defined on $Q = Q_T$. We assume that the components of $p(z)$ satisfy the conditions

$$\begin{cases} p_i(z) \text{ are measurable functions defined on } Q; \\ p_i(z) : Q \longrightarrow (1, \infty), \\ \text{there exist constants } p_i^{\pm}, p^{\pm} \text{such that} \\ p_i(z) \in \left[p_i^-, p_i^+\right] \subseteq \left[p^-, p^+\right] \subset (1, \infty). \end{cases} \tag{3.1}$$

For every fixed $t \in (0, T)$, we introduce the anisotropic Banach space

$$\begin{aligned} V_t(\Omega) &= \left\{ u(x) : u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |D_i u(x)|^{p_i(x,t)} \in L^1(\Omega) \right\}, \\ \|u\|_{V_t(\Omega)} &= \|u\|_{2,\Omega} + \sum_{i=1}^{n} \|D_i u\|_{p_i(.,t),\Omega} + \|\Delta u\|_{2,\Omega}. \end{aligned}$$

The elements of the space $V_t(\Omega)$ depend on $t \in (0, T)$as a parameter and the norms $\|u\|_{V_t(\Omega)}$ are functions of t by $V_t'(\Omega)$ we denote the dual space to $V_t(\Omega)$ with respect to the scalar product in $L^2(\Omega)$.
For every $t \in (0, T)$ the inclusion

$$V_t(\Omega) \subset X = W_0^{1,p^-}(\Omega) \cap L^2(\Omega),$$

holds. Thus, $V_t(\Omega)$ is reflexive and separable as a closed subspace of X.
By $W_{\overrightarrow{p}}(Q)$ we denote the Banach space

$$\begin{aligned} W_{\overrightarrow{p}}(Q) &= \left\{ u : (0, T) \longrightarrow V_t(\Omega) \middle| u \in L^2(\Omega), |D_i u(x)|^{p_i(x,t)} \in L^1(\Omega), u = 0 \text{ on}\Gamma \right\}, \\ \|u\|_{W_{\overrightarrow{p}}(Q)} &= \|u\|_{2,Q} + \sum_{i=1}^{n} \|D_i u\|_{p_i(.),Q}. \end{aligned}$$

$\left(W_{\overrightarrow{p}}(Q)\right)'$ is the dual of $W_{\overrightarrow{p}}(Q)$ (the space of linear functionals over $W(Q)$). We have the following characterization

$$\omega \in \left(W_{\overrightarrow{p}}(Q)\right)' \Leftrightarrow \begin{cases} \exists (\omega_0, \omega_1, ..., \omega_n), \omega_0 \in L^2(\Omega), \omega_i \in L^{p_i'(.)}(Q), \\ \forall \phi \in W(Q) \quad \langle\langle \omega, \phi \rangle\rangle = \int_Q \left(\omega_0 \phi + \sum_{i=1}^{n} \omega_i D_i \phi\right) dz. \end{cases}$$

The norm in $W'(Q)$ is defined by

$$\|u\|_{W'(Q)} = \sup\left\{ \langle u, \phi \rangle \middle| \phi \in W(Q) \ , \|\phi\|_{W(Q)} \leq 1 \right\}.$$

Let $v = (v_1, ..., v_n)$ be a vector-valued function defined in Q. Assume that $p_i(z)$ satisfy conditions 3.1. Introduce the modular

$$A_{p(.)}(v) = \sum_{i=1}^{n} \int_Q |v_i|^{p_i(z)}\, dz.$$

For the elements of $W_{\vec{p}}(Q)$ the following inequality

$$\begin{aligned} &\min\left\{ \sum_{i=1}^{n} \|D_i u\|^{p^-}_{p_i(.,.),Q}, \sum_{i=1}^{n} \|D_i u\|^{p^+}_{p_i(.,.),Q} \right\} \\ \leq\ & A_{p(.,.),Q}(\nabla u) \leq \max\left\{ \sum_{i=1}^{n} \|D_i u\|^{p^-}_{p_i(.,.),Q}, \sum_{i=1}^{n} \|D_i u\|^{p^+}_{p_i(.,.),Q} \right\}, \end{aligned} \tag{3.2}$$

holds. We also use the space

$$W^{\infty}_{\vec{p}}(Q) = \left\{ u : u \in W_{\vec{p}}(Q), |u_{x_i}|^{p_i(x,t)} \in L^{\infty}\left(0, T, L^1(\Omega)\right) \right\}.$$

Note that

$$W^{\infty}(Q) \subseteq W^{\infty}_{\vec{p}}(Q) \text{ if } p^+ \leq \frac{2n}{n-2}.$$

We introduce also the functional space

$$U(Q) = W(Q) \cap W_{\vec{p}}(Q),$$

endowed with the norm

$$\|u\|_{U(Q)} = \|u\|_{W(Q)} + \|u\|_{W_{\vec{p}}(Q)},$$

and

$$U^{\infty}(Q) = W^{\infty}(Q) \cap W^{\infty}_{\vec{p}}(Q).$$

For the exponents $p_i(x,t)$ depending on $(x,t) \in Q$ we will use the notation $p_i \in C_{\log}(Q)$ if p_i satisfies condition (3.1) in the cylinder Q and

$$C_{\log}(Q) := \left\{ p_i \in C^0\left(\overline{Q}\right) \,\middle|\, \begin{array}{c} \forall z = (x,t), \zeta = (y,\tau) \in Q \\ \text{such that } |x-y| + |t-\tau| < \frac{1}{2}, \\ |p_i(z) - p_i(z)| \leq \omega(|z-\zeta|) \end{array} \right\}, \tag{3.3}$$

with a continuous function ω satisfying the condition

$$\overline{\lim}_{\tau \longrightarrow 0^+} \omega(\tau) \ln \frac{1}{\tau} = C < +\infty.$$

4 Statement of the problem

We consider a class of nonlinear viscoelastic plate equations with delay and with $\vec{p}(x,t)$-Laplacian type

$$\left\{ \begin{array}{c} u_{tt} + \Delta^2 u(t) - \Delta_{\vec{p}(x,t)} u - \int_0^t g(t-s)\Delta^2 u(s)\, ds - \mu_1 \Delta u_t(t) \\ -\mu_2 \Delta z(1,t) = u|u|^{q-2}, \text{ in } \Omega \times \mathbb{R}^+, \end{array} \right. \tag{4.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ in } \partial\Omega \times \mathbb{R}^+, \tag{4.2}$$

$$u\left(x,0\right)=u_0(x),u_t\left(x,0\right)=u_1(x),\ x\in\Omega, \tag{4.3}$$

$$u_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right),\text{ in }\ \left(0,\tau\right), \tag{4.4}$$

here μ_1 is a positive constant, μ_2 is a real number, $\tau>0$ represents the time delay, and g is a positive function.

$$\Delta_{\overrightarrow{p}(x,t)}u=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\right),\ \overrightarrow{p}=\left(p_1,p_2,...,p_n\right),$$

under conditions

$$1\leq p_i^-=const\leq p_i\left(x,t\right)=p\left(x\right)\leq p_i^+=const<q<\infty,\left|p_{it}\right|\leq C_{p_i},i=1,...,n. \tag{4.5}$$

Assume that $g:\mathbb{R}_+\to\mathbb{R}_+$ is a nonincreasing and differentiable function satisfying

$$g(0)>0,g(s)\geq 0,1-\int_0^\infty g(s)ds:=\gamma>0, \tag{4.6}$$

and that μ_1 and μ_2 satisfy

$$\left|\mu_2\right|<\mu_1, \tag{4.7}$$

let $\beta>0$ be the constant satisfying

$$\left\|v\right\|_p\leq\beta\left\|\nabla v\right\|. \tag{4.8}$$

By using the direct calculations, we have

$$\begin{aligned}\int_0^t g(t-s)\left(\Delta u_t\left(t\right),\Delta u\left(s\right)\right)ds &= -\frac{1}{2}\frac{d}{dt}\left\{\left(g\circ\Delta u\right)\left(t\right)-\left(\int_0^t g(s)ds\right)\left\|\Delta u\right\|^2\right\}\\ &\quad -\frac{1}{2}g(t)\left\|\Delta u\right\|^2+\frac{1}{2}\left(g'\circ\Delta u\right)\left(t\right),\end{aligned} \tag{4.9}$$

where

$$\left(g\circ\Delta u\right)\left(t\right)=\int_0^t g(t-s)\left\|\Delta u\left(s\right)-\Delta u\left(t\right)\right\|^2ds.$$

We refer the reader to the work of Nicaise and Pignotti [19] for the existence of solutions to nonlinear problems with delay. Let us introduce the function

$$z\left(x,\rho,t\right)=u_t\left(x,t-\tau\rho\right),x\in\Omega,\rho\in\left(0,1\right),t>0.$$

Then, the problem (4.1)-(4.4) is equivalent to

$$\begin{cases} u_{tt}+\Delta^2u\left(t\right)-\Delta_{\vec{p}(x,t)}-\int_0^t g(t-s)\Delta^2u\left(s\right)ds-\mu_1\Delta u_t(t)-\mu_2\Delta z\left(1,t\right)=u\left|u\right|^{q-2}, & \\ \tau z_t\left(x,\rho,t\right)+z_\rho\left(x,\rho,t\right)=0, & \text{in }\left(0,1\right)\times\left(0,\infty\right),\\ z\left(0,t\right)=u_t(t), & \text{in }\left(0,\infty\right),\\ z\left(\rho,0\right)=f_0\left(-\tau\rho\right), & \text{in }\left(0,1\right),\\ u=\frac{\partial u}{\partial\nu}=0, & \text{on }\partial\Omega\times\left(0,\infty\right),\\ u(x,0)=u_0\left(x\right),\ u_t(x,0)=u_1\left(x\right), & \text{in }\Omega.\end{cases} \tag{4.10}$$

We first state a local existence theorem that can be established by combining the arguments of related works [7,8].

Theorem 4.1. *Let (4.6) and (4.7) hold and suppose that*

$$2 \le p_i < q \le \infty, \text{ for } n = 1,2.$$
$$2 \le p_i^- \le p_i \le p_i^+ < q < \frac{2n}{n-2}, \text{ for } n \ge 3 \tag{4.11}$$

Then, for every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0,1))$, there exists a unique local solution in the class $u \in C([0,T); H_0^1(\Omega))$, $u_t \in C([0,T); L^2(\Omega)) \cap L^2(\Omega \times [0,T))$ for some $T > 0$ Our aim is to investigate a blow-up result for problem (4.10). We define the energy associated with problem (4.10) by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \sum_{i=1}^n \int_\Omega \frac{1}{p_i}|u_{x_i}|^{p_i}dx \\ &+ \frac{1}{2}(g \circ \Delta u)(t) + \frac{\zeta}{2}\int_0^1 \|\nabla z(x,\rho,t)\|^2 d\rho - \frac{1}{q}\|u\|_q^q, \end{aligned} \tag{4.12}$$

where

$$\tau|\mu_2| < \zeta < \tau(2\mu_1 - |\mu_2|). \tag{4.13}$$

Note that this choice of ζ is possible from Assumption (4.7).

Lemma 4.2. *For u is the solution of (4.10), then there exists $C_0 \ge 0$ such that*

$$E'(t) \le -C_0\left(\|\nabla u_t\|^2 + \|\nabla z(1,t)\|^2\right) - \frac{1}{2}g(t)\|\Delta u\|^2 + \frac{1}{2}(g' \circ \Delta u)(t) \le 0 \tag{4.14}$$

Proof. Multiplying (4.10) by u_t and using (4.9), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big\{\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\left(1 - \textstyle\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \textstyle\sum_{i=1}^n \int_\Omega \frac{1}{p_i}|u_{x_i}|^{p_i}dx \\ &+ \tfrac{1}{2}(g \circ \Delta u)(t) - \tfrac{1}{q}\|u\|_q^q\Big\} \\ &= \tfrac{1}{2}(g' \circ \Delta u)(t) - \tfrac{1}{2}g(t)\|\Delta u\|_2^2 - \mu_1\|\nabla u_t\|_2^2 - \mu_2\textstyle\int_\Omega \nabla u_t \nabla z(1,t)dx \end{aligned} \tag{4.15}$$

From (4.12) and (4.15) ,we see that

$$\begin{aligned} E'(t) &= -\tfrac{1}{2}g(t)\|\Delta u\|_2^2 + \tfrac{1}{2}(g' \circ \Delta u)(t) - \mu_1\|\nabla u_t\|_2^2 \\ &- \mu_2\textstyle\int_\Omega \nabla u_t \nabla z(1,t)dx - \zeta\int_\Omega\int_0^1 \Delta z(x,\rho,t)z_t(x,\rho,t)d\rho dx \end{aligned} \tag{4.16}$$

We estimate the last terms of the right-hand side of (4.16) .From the second equation of (4.10), we get

$$\begin{aligned} \tfrac{\zeta}{\tau}\textstyle\int_\Omega\int_0^1 \Delta z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx &= \tfrac{\zeta}{2}\textstyle\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}|\nabla z(x,\rho,t)|^2 d\rho dx \\ &= \tfrac{\zeta}{2\tau}\left(\|\nabla u_t\|_2^2 - \|\nabla z(1,t)\|_2^2\right) \end{aligned} \tag{4.17}$$

Using Young's inequality, we have

$$\mu_2\int_\Omega \nabla u_t \nabla z(1,t)dx \le \frac{|\mu_2|}{2}\|\nabla u_t\|_2^2 + \frac{|\mu_2|}{2}\|\nabla z(1,t)\|_2^2 \tag{4.18}$$

Combining (4.16), (4.17), and (4.18) we obtain

$$E'(t) \le -C_0\left(\|\nabla u_t\|_2^2 + \|\nabla z(1,t)\|_2^2\right) - \frac{1}{2}g(t)\|\Delta u\|_2^2 + \frac{1}{2}(g' \circ \Delta u)(t),$$

where $C_0 = \min\left\{\mu_1 - \frac{|\mu_2|}{2} - \frac{\zeta}{2\tau}, \frac{\zeta}{2\tau} - \frac{|\mu_2|}{2}\right\}$, which is positive by (4.13) □

Next, we define the functionals

$$I(t)=\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2+\sum_{i=1}^n\int_\Omega |u_{x_i}|^{p_i}dx+(g\circ\Delta u)(t)-\|u\|_q^q$$
$$J(t)=\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2+\sum_{i=1}^n\int_\Omega \frac{1}{p_i}|u_{x_i}|^{p_i}dx+\frac{1}{2}(g\circ\Delta u)(t)-\frac{1}{q}\|u\|_q^q,$$

for $t\geq 0$, we denote

$$d(t)=\inf_{u\in H_0^1(\Omega),u\neq 0}\sup_{\lambda\geq 0}J(\lambda u).$$

Then, similar to the work of Liu and Yu [16], we can prove the following lemmas.

Lemma 4.3. *for $t\geq 0$,we have*

$$0<d_1\leq d(t)\leq\sup_{\lambda\geq 0}J(\lambda u),$$

where

$$\begin{cases} d_1=\sum_{i=1}^n\left[\frac{p_i^+(q-2)}{2qp_i^-}\left(\frac{1}{B_1^2}\right)^{\frac{qp_i^+}{qp_i^--2p_i^-}}+\frac{2q-2p_i^+}{2qp_i^-}\left(\frac{1}{B_2^2}\right)^{\frac{qp_i^+}{2q-2p_i^-}}\right] \\ d_2=\sup_{\lambda\geq 0}J(\lambda u)=\sum_{i=1}^n\frac{p_i^+(q-2)}{2qp_i^-}\left(\frac{\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2+(g\circ\Delta u)(t)}{\|u\|_q^q}\right)^{\frac{qp_i^+}{p_i^-(q-2)}} \\ +\sum_{i=1}^n\frac{2q-2p_i^+}{2qp_i^-}\left(\frac{\int_\Omega|u_{x_i}|^{p_i}dx}{\|u\|_q^q}\right)^{\frac{qp_i^+}{2q-2p_i^-}}. \end{cases}$$

Lemma 4.4. *Suppose that (4.6), (4.7), and (4.11) hold. For any fixed number $\delta<1$ assume that $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ and satisfy*

$$I(0)<0,E(0)\leq\delta d_1. \tag{4.19}$$

Assume further that g satisfies

$$\int_0^t g(s)ds<\sum_{i=1}^n\frac{p_i^+(q-2)}{\left[p_i^-(q-2)\right]+\frac{1}{\left[\left(1+\widehat{\delta}\right)^2\left(p_i^-(q-2)\right)+2p_i^-\left(1-\widehat{\delta}\right)\right]}}, \tag{4.20}$$

where $\widehat{\delta}=\max\{0,\delta\}$. Then, for some $T>0$,we have $I(t)<0$,for all $t\in[0,T)$, and

$$\begin{aligned} d_1&<\sum_{i=1}^n\frac{p_i^+(q-2)}{2qp_i^-}\left[\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2+(g\circ\Delta u)(t)\right] \\ &\quad+\sum_{i=1}^n\frac{2q-2p_i^+}{2qp_i^-}\left(\int_\Omega|u_{x_i}|^{p_i}dx\right) \\ &<\sum_{i=1}^n\left(\frac{p_i^+(q-2)}{2qp_i^-}+\frac{2q-2p_i^+}{2qp_i^-}\right)\|u\|_q^q \\ &<\sum_{i=1}^n\frac{q(p_i^++2)-4p_i^+}{2qp_i^-}\|u\|_q^q,t\in[0,T), \end{aligned} \tag{4.21}$$

we set

$$H(t)=\widehat{\delta}d_1-E(t). \tag{4.22}$$

Using (4.14), we see that

$$H'(t)=-E'(t)\geq C_0\left(\|\nabla u_t\|_2^2+\|\nabla z(1,t)\|_2^2\right)\geq 0, \tag{4.23}$$

and $H(t)$ is an increasing function. From (4.12), (4.19), and (4.21), we obtain

$$0 < H(0) \leq H(t) \leq \widehat{\delta} d_1 + \frac{1}{q} \|u\|_q^q \leq q_0 \|u\|_q^q, t \in [0, T), \tag{4.24}$$

where $q_0 = \sum_{i=1}^{n} \frac{[q(p_i^+ + 2) - 4p_i^+]\widehat{\delta}}{2qp_i^-} + \frac{1}{q}$. Moreover, similar to the work of Messaoudi [14] we can get the following lemma that is needed later.

Lemma 4.5. *Let the conditions of Lemma 3 hold. Then, we have, for any $2 \leq s \leq q$*

$$\begin{aligned} \|\nabla u\|_q^s \leq\ & C\left(-H(t) - \|u_t(t)\|_2^2 - \sum_{i=1}^{n} \int_\Omega |u_{x_i}|^{p_i} dx - (g \circ \Delta u)(t) \right. \\ & \left. + \|u\|_q^q - \int_0^1 \|\nabla z(x, \rho, t)\|_2^2 d\rho \right), \end{aligned} \tag{4.25}$$

where C is a positive constant depending on $p_i^{\pm}, \zeta, \gamma, B, q$,

Proof. From (4.8), there exists a positive constant c_0 such that

$$\|\nabla u\|_q^s \leq c_0 \left(\|u\|_q^q + \|\Delta u\|_2^2 \right), \textit{for any } 2 \leq s \leq q, \tag{4.26}$$

where $c_0 = \max\{1, B^2\}$. Using (4.6), (4.12), (4.22), and (4.24), we know that

$$\begin{aligned} \frac{\gamma}{2} \|\Delta u(t)\|_2^2 \leq\ & -H(t) - \frac{1}{2} \|u_t(t)\|_2^2 - \sum_{i=1}^{n} \int_\Omega \frac{1}{p_i} |u_{x_i}|^{p_i} dx - \frac{1}{2} (g \circ \Delta u)(t) \\ & - \frac{\zeta}{2} \int_0^1 \|\nabla z(x, \rho, t)\|_2^2 d\rho + q_o \|u\|_q^q \end{aligned} \tag{4.27}$$

From (4.26) and (4.27), we find that (4.25) holds. □

5 A blow-up result

In this section, we prove that the solutions for the problem (4.1)-(4.4) blow up in a finite time when the initial energy lies in nonpositive as well as positive. We use the improved method of Liu and Yu [16].

Theorem 5.1. *Let the conditions of Lemma 3 hold. Then, the solution of problem (4.1)-(4.4) blows up in a finite time.*

Proof. To prove this theorem, we adapt the idea given in the works of Messaoudi and Kafini [14,17]. Let us define

$$L(t) = H^{1-\sigma_i}(t) + \varepsilon (u(t), u_t(t)), \ \varepsilon > 0, \tag{5.1}$$

where

$$0 < \sigma_i < \min\left\{ \frac{q(p_i^+ + 2) - 4p_i^+}{2qp_i^-}, \frac{q(p_i^+ + 2) - 4p_i^+}{qp_i^-} \right\}. \tag{5.2}$$

From (4.10) and Young's inequality, we have

$$\begin{aligned} L'(t) =\ & (1 - \sigma_i) H^{-\sigma_i}(t) H'(t) + \varepsilon \|u_t(t)\|_2^2 - \varepsilon \|\Delta u\|_2^2 - \varepsilon \sum_{i=1}^{n} \int_\Omega |u_{x_i}|^{p_i} dx \\ & + \varepsilon \int_0^t g(t-s) (\Delta u(t), \Delta u(s)) ds - \varepsilon \mu_1 \int_\Omega \nabla u_t(t) \nabla u(t) dx \\ & + \varepsilon \|u\|_q^q - \varepsilon \mu_2 \int_\Omega \nabla u \nabla z(1, t) dx. \end{aligned} \tag{5.3}$$

Using Young's inequality and (4.23), we obtain, for any $\beta > 0$,

$$\begin{aligned}
&\mu_1\varepsilon \int_\Omega \nabla u_t(t)\nabla u(t)dx + \mu_2\varepsilon \int_\Omega \nabla u \nabla z\,(1,t)\,dx \\
\leq &\left(\frac{\mu_1^2\beta^2\varepsilon}{2} + \frac{\mu_2^2\beta^2\varepsilon}{2}\right)\|\nabla u\|_2^2 + \frac{\varepsilon}{2C_0}\beta^{-2}\left\{\|\nabla u_t\|_2^2 + \|\nabla z(1,t)\|_2^2\right\} \\
\leq &\left(\frac{\mu_1^2\beta^2\varepsilon}{2} + \frac{\mu_2^2\beta^2\varepsilon}{2}\right)\|\nabla u\|_2^2 + \frac{\varepsilon}{2C_0}\beta^{-2}H'(t).
\end{aligned} \tag{5.4}$$

Since, for some number $\eta > 0$,

$$\begin{aligned}
\int_0^t g(t-s)\left(\Delta u\,(t)\,,\Delta u\,(s)\right)ds &= \int_0^t g(t-s)\left(\Delta u\,(s) - \Delta u\,(t)\,,\Delta u\,(t)\right)ds \\
&\quad + \int_0^t g(t-s)\,\|\Delta u\|_2^2\,ds \\
&\geq \left(1 - \frac{1}{4\eta}\right)\int_0^t g\,(s)\,ds\,\|\Delta u\|_2^2 - \eta\,(g \circ \Delta u)\,(t)\,,
\end{aligned}$$

we get from (5.3) and (5.4) that

$$\begin{aligned}
L'(t) \geq\; &(1-\sigma_i)\,H^{-\sigma_i}(t)H'(t) + \varepsilon\,\|u_t\,(t)\|_2^2 - \varepsilon\,\|\Delta u\|_2^2 - \varepsilon\sum_{i=1}^n \int_\Omega |u_{x_i}|^{p_i}\,dx \\
&+\varepsilon\left(1 - \frac{1}{4\eta}\right)\int_0^t g\,(s)\,ds\,\|\Delta u\|_2^2 - \varepsilon\mu_1 \int_\Omega \nabla u_t(t)\nabla u(t)dx \\
&+\varepsilon\,\|u\|_q^q - \varepsilon\mu_2 \int_\Omega \nabla u\nabla z\,(1,t)\,dx.
\end{aligned}$$

Applying (4.12) and (4.22), we see that

$$\begin{aligned}
L'(t) \geq\; &\left\{(1-\sigma_i)\,H^{-\sigma_i}(t) - \frac{\varepsilon}{2C_0}\beta^{-2}\right\}H'(t) + \varepsilon\left(1 + \frac{q}{2}\right)\|u_t\,(t)\|_2^2 \\
&+\varepsilon\left(\frac{q}{2} - \eta\right)(g \circ \Delta u)\,(t) + \varepsilon\sum_{i=1}^n\left(\frac{q}{p_i} - 1\right)\int_\Omega |u_{x_i}|^{p_i}\,dx \\
&+\varepsilon\left\{\left(\frac{q}{2} - 1\right) - \left(-1 + \frac{1}{4\eta} + \frac{q}{2}\right)\int_0^t g(s)ds\right\}\|\Delta u\|_2^2 \\
&+\frac{\zeta\varepsilon q}{2}\int_0^1 \|\nabla z\,(x,\rho,t)\|^2\,d\rho - \left(\frac{\mu_1^2}{2} + \frac{\mu_2^2}{2}\right)\beta^2\varepsilon\,\|\nabla u\|_2^2 \\
&+\varepsilon q H(t) - \varepsilon q\widehat{\delta}d_1.
\end{aligned} \tag{5.5}$$

Using (4.20), (4.21), and (5.5), we find that, for some η with $0 < \eta < \lambda_1\left(\frac{q-2}{2\lambda_1} - \widehat{\delta}\right) + 1$,

$$\begin{aligned}
L'(t) \geq\; &\left\{(1-\sigma_i)\,H^{-\sigma_i}(t) - \frac{\varepsilon}{2C_0}\beta^{-2}\right\}H'(t) + \varepsilon\left(1 + \frac{q}{2}\right)\|u_t\,(t)\|_2^2 \\
&+\varepsilon\left[\lambda_1\left(\frac{q-2}{2\lambda_1} - \widehat{\delta}\right) + (1-\eta)\right](g \circ \Delta u)\,(t) \\
&+\varepsilon\left\{\lambda_1\left(\frac{q-2}{2\lambda_1} - \widehat{\delta}\right) - \left(\lambda_1\left(\frac{q-2}{2\lambda_1} - \widehat{\delta}\right) + \frac{1}{4\eta}\right)\int_0^t g(s)ds\right\}\|\Delta u\|_2^2 \\
&-\left(\frac{\mu_1^2}{2} + \frac{\mu_2^2}{2}\right)\beta^2\varepsilon\,\|\nabla u\|_2^2 + \varepsilon q H(t) + \frac{\zeta\varepsilon q}{2}\int_0^1 \|\nabla z\,(x,\rho,t)\|_2^2\,d\rho \\
&+\varepsilon\sum_{i=1}^n\left\{q\left(\frac{1}{p_i} - \frac{1}{2}\right) + \lambda_2\left(\frac{q-2}{2\lambda_2} - \widehat{\delta}\right)\right\}\int_\Omega |u_{x_i}|^{p_i}\,dx,
\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
= & \left\{(1-\sigma_i) H^{-\sigma_i}(t)-\frac{\varepsilon}{2C_0}\beta^{-2}\right\} H'(t)+\varepsilon\left(1+\frac{q}{2}\right)\|u_t(t)\|_2^2 \\
& +\varepsilon a_1(g\circ\Delta u)(t)+\varepsilon a_2\|\Delta u\|_2^2+\frac{\zeta\varepsilon q}{2}\int_0^1\|\nabla z(x,\rho,t)\|_2^2 d\rho \\
& -\left(\frac{\mu_1^2}{2}+\frac{\mu_2^2}{2}\right)\beta^2\varepsilon\|\nabla u\|_2^2+\varepsilon a_3\int_\Omega |u_{x_i}|^{p_i}dx+\varepsilon q H(t),
\end{aligned} \tag{5.7}
$$

where

$$
\left\{
\begin{array}{c}
a_1=\lambda_1\left(\frac{q-2}{2\lambda_1}-\widehat{\delta}\right)+(1-\eta)>0, \\
a_2=\lambda_1\left(\frac{q-2}{2\lambda_1}-\widehat{\delta}\right)-\left(\lambda_1\left(\frac{q-2}{2\lambda_1}-\widehat{\delta}\right)+\frac{1}{4\eta}\right)\int_0^t g(s)ds>0 \\
a_3=\sum_{i=1}^n\left\{q\left(\frac{1}{p_i^-}-\frac{1}{2}\right)+\lambda_2\left(\frac{q-2}{2\lambda_2}-\widehat{\delta}\right)\right\}>0 \\
\lambda_1=\sum_{i=1}^n\frac{p_i^+(q-2)}{2p_i^-}>0,\lambda_2=\sum_{i=1}^n\frac{2q-2p_i^+}{2p_i^-}>0
\end{array}
\right.
$$

Take β so that $\beta=(kH^{-\sigma_i}(t))^{-\frac{1}{2}}$, for large k to be specified later. Exploiting (4.24) and the inequality

$$\|u\|_{2,\Omega}^2\le C_1\|u\|_{q,\Omega}^2,$$

we see that

$$\beta^2\|\nabla u\|_2^2\le k^{-1}C_1 q_0^{\sigma_i}\|\nabla u\|_q^{q\sigma_i+2}$$

Substituting this into (5.6),

$$
\begin{aligned}
L'(t) \ \ge & \left\{(1-\sigma_i)-\frac{\varepsilon}{2C_0}k\right\} H^{-\sigma_i}(t)H'(t)+\varepsilon\left(1+\frac{q}{2}\right)\|u_t(t)\|_2^2 \\
& +\varepsilon a_1(g\circ\Delta u)(t)+\varepsilon a_2\|\Delta u\|_2^2+\varepsilon a_3\int_\Omega |u_{x_i}|^{p_i}dx \\
& +\frac{\zeta\varepsilon q}{2}\int_0^1\|\nabla z(x,\rho,t)\|_2^2 d\rho-\varepsilon k^{-1}C_2\|\nabla u\|_q^{q\sigma_i+2}+\varepsilon q H(t)
\end{aligned}
$$

where $C_2=\frac{(\mu_1^2+\mu_2^2)C_1 q_0^{\sigma_i}}{2}$.
From (5.2), and Lemma 4, for $S=q\sigma_i+2\le q$, we arrive at

$$\|\nabla u\|_q^{q\sigma_i+2}\le c\left(\begin{array}{c}-H(t)-\|u_t(t)\|_2^2-\sum_{i=1}^n\int_\Omega |u_{x_i}|^{p_i}dx-(g\circ\Delta u)(t)\\ +\|u\|_q^q-\int_0^1\|\nabla z(x,\rho,t)\|_2^2 d\rho\end{array}\right)$$

Then, we have

$$
\begin{aligned}
L'(t) \ \ge & \left\{(1-\sigma_i)-\frac{\varepsilon}{2C_0}k\right\} H^{-\sigma_i}(t)H'(t)+\varepsilon\left(\left(1+\frac{q}{2}\right)+ck^{-1}C_2\right)\|u_t(t)\|_2^2 \\
& +\varepsilon\left(a_1+ck^{-1}C_2\right)(g\circ\Delta u)(t)+\varepsilon\left(a_3+ck^{-1}C_2\right)\int_\Omega |u_{x_i}|^{p_i}dx \\
& +\varepsilon a_2\|\Delta u\|_2^2+\varepsilon\left(\frac{\zeta q}{2}+ck^{-1}C_2\right)\int_0^1\|\nabla z(x,\rho,t)\|_2^2 d\rho \\
& -\varepsilon ck^{-1}C_2\|u\|_q^q+\varepsilon\left(q+ck^{-1}C_2\right)H(t).
\end{aligned} \tag{5.8}
$$

Subtracting and adding $\varepsilon\theta H(t)$ on the right-hand side of (5.7) and using (4.12) and (4.22), we deduce

$$\begin{aligned} L'(t) \geq & \left\{(1-\sigma_i) - \frac{\varepsilon}{2C_0}k\right\} H^{-\sigma_i}(t)H'(t) + \varepsilon\left(1+\frac{q}{2}-\frac{\theta}{2}+ck^{-1}C_2\right)\|u_t(t)\|_2^2 \\ & +\varepsilon\left(a_1-\frac{\theta}{2}+ck^{-1}C_2\right)(g\circ\Delta u)(t) + \varepsilon\left(\frac{\theta}{q}-ck^{-1}C_2\right)\|u\|_q^q \\ & +\varepsilon\left\{a_2-\frac{\theta}{2}\left(1-\int_0^t g(s)ds\right)\right\}\|\Delta u\|_2^2 \\ & +\varepsilon\left(\frac{\zeta q}{2}-\frac{\theta\zeta}{2}+ck^{-1}C_2\right)\int_0^1 \|\nabla z(x,\rho,t)\|_2^2 d\rho + \varepsilon\left(q-\theta+ck^{-1}C_2\right)H(t) \\ & +\varepsilon\sum_{i=1}^{n}\left\{\left(a_3-\frac{\theta}{p_i^-}+ck^{-1}C_2\right)\int_\Omega |u_{x_i}|^{p_i}dx\right\} + \varepsilon\hat{\delta}d_1. \end{aligned} \tag{5.9}$$

First, we fix θ such that

$$0 < \theta < \min\{2a_1, 2a_2, q\}.$$

Second, we take $k > 0$ large enough such that

$$\frac{\theta}{q} - ck^{-1}C_2 > 0.$$

Once k is fixed, we select $\varepsilon > 0$ small enough so that

$$(1-\sigma_i) - \frac{\varepsilon}{2C_0}k > 0, H^{1-\sigma_i}(0) + \varepsilon\int_\Omega u_0u_1dx > 0.$$

Therefore, we obtain from (5.8), that

$$L'(t) \geq C\left(\|u_t(t)\|_2^2 + (g\circ\Delta u)(t) + \int_0^1 \|\nabla z(x,\rho,t)\|_2^2 d\rho + \sum_{i=1}^{n}\int_\Omega |u_{x_i}|^{p_i}dx + \|u\|_q^q + H(t)\right). \tag{5.10}$$

Here and in the sequel, C denotes a generic positive constant. On the other hand, using the similar arguments in the work of Messaoudi [17] we get

$$L^{\frac{1}{1-\sigma_i}}(t) \geq C\left(\|u_t(t)\|_2^2 + (g\circ\Delta u)(t) + \int_0^1 \|\nabla z(x,\rho,t)\|_2^2 d\rho + \sum_{i=1}^{n}\int_\Omega |u_{x_i}|^{p_i}dx + H(t) + \|u\|_q^q\right). \tag{5.11}$$

Combining (5.9), and (5.10), we find that

$$L'(t) \geq CL^{\frac{1}{1-\sigma_i}}(t), \text{for } t \geq 0.$$

A simple integration yields

$$L^{\frac{\sigma_i}{1-\sigma_i}}(t) \geq \frac{1}{L^{-\frac{\sigma_i}{1-\sigma_i}}(0) - \frac{C\sigma_i t}{1-\sigma_i}}, \text{for } t \geq 0.$$

Consequently, the solution of problem $(4.1)-(4.4)$ blows up in finite time T^* and

$$T^* \leq \frac{1-\sigma_i}{C\sigma_i L^{\frac{\sigma_i}{1-\sigma_i}}(0)}.$$

□

6 Conclusion

In this work, we are interested in a nonlinear problem of a viscoelastic plate equation with a polynomial source term in the presence of time delay. We show that the energy of any weak solution blows up in a finite time if the initial energy is nonpositive as well as positive. The delay effect is similar to memory processes that is important in the research of applied mathematics such as physics, and biological motivation. In future work, we will try to study the local existence of this problem with respect to some proposal conditions with a semi-groups method

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The authors declare that the study was realized in collaboration with equal responsibility.

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This work does not have any conflicts of interest.

References

[1] D. Andrade, M. A. Jorge Silva and T. F. Ma, *Exponential stability for a plate equation with $p-$Laplacian and memory terms*, Math. Methods Appl. Sci., **35**(4) (2012), 417–426. DOI

[2] S. Antontsev and J. Ferreira, *On a viscoelastic plate equation with strong damping and $\overrightarrow{p}(x,t)-$ Laplacian. Existence and uniqueness.* Diff. Integ. Equ., **27**(11/12) (2014), 1147–1170. URL

[3] S. Antontsev, *Wave equation with $p(x,t)-$Laplacian and damping term: Existence and blow-up*, Differ. Equ. Appl., **3**(4) (2011), 503–525. URL

[4] S. Antontsev and S. Shmarev, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat., **53**(2) (2009), 355–399. DOI

[5] L. J. An and A. Peirce, *The effect of microstructure on elastic–plastic models*, SIAM J. Appl. Math., **54**(3) (1994), 708–730. DOI

[6] L. J. An and A. Peirce, *A weakly nonlinear analysis of elastoplastic-microstructure models*, SIAM J. Appl. Math. **55** (1995), 136–155. DOI

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and C. M. Webler, *Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density*. Adv. Nonlinear Anal. **6**(2) (2017), 121–145 . DOI

[8] I. Chueshov and I. Lasiecka, *existence, uniqueness of weak solutions and global attractors for a class of nonlinear 2D Kirchhoff -Boussinesq models*, Discrete Contin. Dyn. Syst., **15** (2006), 777-809.

[9] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*. RI: AMS, vol.195. Providence, 2008. DOI

[10] L. Diening, P. Harjulehto, P. Hästö, and M. Rŭžİ čka,, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, 2011. Series: Lecture Notes in Mathematics, Undergraduate Texts in Mathematics, 1st Edition 2017. DOI

[11] J. Ferreira and S. A. Messaoudi, *On the general decay of a nonlinear viscoelastic plate equation with a strong damping and $\vec{p}(x,t)$-Laplacian*, Nonlinear Anal., **104** (2014), 40–49. DOI

[12] V. Georgiev and G. Todorova, *existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations, **109** (1994), 295-308. DOI

[13] M.A. Jorge Silva and T.F. Ma, *On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type*, IMA J. Appl. Math. **78** (2013), 1130–1146. DOI

[14] M. Kafini and S.A. Messaoudi, *A blow-up result in a nonlinear wave equation with delay*, Mediterr J. Math., **13**(1) (2016), 237-247. DOI

[15] M. Kafini, S. A Messaoudi and S. Nicaise,*A blow-up result in a nonlinear abstract evolution system with delay*, Nonlinear Differ. Equ. Appl.,**23**:13 (2016), 1–14. DOI

[16] W. J. Liu and J. Yu, *On decay and blow-up of the solution for a viscoelastic wave equation with boundary damping and source terms*, Nonlinear Anal., **74**(6)(2011), 2175-2190. DOI

[17] S. A. Messaoudi,*Blow-up of positive initial energy solutions of a nonlinear viscoelastic hyperbolic equation*, J. Math. Anal. Appl., **320**(2) (2006), 902-915. DOI

[18] M. Nakao, *Global solutions to the initial-boundary value problem for the quasilinear viscoelastic equation with a derivative nonlinearity*, Opusc. Math. **34**(3) (2014), 569–590. DOI

[19] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim.,**45**(5) (2006), 1561-1585. DOI

[20] E. Pişkin and H. Yüksekkaya, *Blow up of solution for a viscoelastic wave equation with m-Laplacian and delay terms*, Tbil. Math. J., SI (7) (2021), 21-32.

[21] P. Pukach, V. Il'kiv, Z. Nytrebych and M. Vovk, *On nonexistence of global in time solution for a mixed problem for a nonlinear evolution equation with memory generalizing the Voigt–Kelvin rheological model*, Opusc. Math., **37**(5) (2017), 735–753. DOI

[22] S. G. Samko, *Density*$C_0^\infty(\mathbb{R}^n)$ *in the generalized Sobolev spaces*$W^{m,p(x)}(\mathbb{R}^n)$, Dokl. Akad. Nauk, **369** (1999), 451-454.

[23] Wu and Z. Shun-Tang, *Blow-Up of Solution for A Viscoelastic Wave Equation with Delay*, Acta Math. Sci., **39**(1) (2019), 329–338. DOI

[24] Z. Yang, *Long-time behavior for a nonlinear wave equation arising in elasto-plastic flow*, Math. Methods Appl. Sci., 32(9) (2009), 1082–1104 . DOI

[25] V. V. Zhikov, *On Lavrentiev's effect*, Dokl., Ross. Akad. Nauk, **345**(1) (1995), 10–14.

[26] V. V. Zhikov, *On Lavrentiev's phenomenon*, Russian J. Math. Phys., **3**(2) (1995), 249-269.

[27] V. V. Zhikov, *On the density of smooth functions in Sobolev-Orlich spaces*, J. Math. Sci., **132** (2006), 285-294. DOI