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Recent progress in the conductivity reconstruction in Calderón's problem

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Abstract. In this work, we study a nonlinear inverse problem for an elliptic partial differential equation known as the Calderón problem or the inverse conductivity problem. We give a quick survey on the reconstruction question of conductivity from measurements on the boundary, by covering the main currently known results regarding the isotropic problem with full data in two and higher dimensions. We present Nachman's reconstruction procedure and summarize the theoretical progress of the technique to more recent results in the field. An open problem of significant interest is proposed to check whether extending the method for Lipschitz conductivities is possible.

Keywords: Calderón problem, inverse conductivity problem, Dirichlet-to-Neumann map, complex geometrical optics solutions, $\bar{\partial}$-method, boundary integral equation.

2020 Mathematics Subject Classification: 35R30.

1 Introduction

The present paper aims to summarize some reconstruction results from boundary measurements for less regular conductivities in the inverse conductivity problem, which has been developed for over 30 years and provides references for further research and practical applications on the topic. The Calderón problem [15] asks to recover a conductivity of a domain from measurements that are taken on the boundary. For a formal definition, let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and let γ be a positive real-valued function representing the electrical conductivity of Ω such that for almost every $x \in \Omega$ and for a constant $c_0 > 0$, the condition

$$\gamma(x) \geq c_0, \tag{1.1}$$

is satisfied. The application of a voltage $\psi \in H^{1/2}(\partial\Omega)$ on the boundary induces an electrical potential $w \in H^1(\Omega)$ in the interior of Ω, where w is the unique weak solution of the following elliptic boundary value problem

$$\begin{cases} \nabla \cdot \gamma \nabla w = 0 & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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In this case, the Dirichlet-to-Neumann map (DN map) relating the boundary voltage ψ (Dirichlet data) to the flux at the boundary $\gamma\frac{\partial w}{\partial \nu}$ (Neumann data) is defined as follows

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

$$\psi \mapsto \Lambda_\gamma(\psi) = \gamma \frac{\partial w}{\partial \nu}\bigg|_{\partial\Omega},$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative at $\partial\Omega$.

In this paper, we consider the Calderón problem of reconstructing a conductivity from measurements on the boundary. Since the motivation to reconstruct a conductivity comes from its uniqueness, we should first ask if it is possible to determine γ from the knowledge of Λ_γ, i.e., whether the map $\gamma \mapsto \Lambda_\gamma$ is injective? In 1980, Alberto Calderón, who proposed the problem, gave a positive answer. He proved in his pioneer paper [15] that for γ a perturbation of the identity, the injectivity of the linearized inverse problem holds. For $n \geq 3$, Sylvester and Uhlmann [42] were the first to show uniqueness for C^2 conductivities. They reduced the problem to a similar one for a Schrödinger equation. This reduction is based on the well-known Liouville transformation: if z is a weak solution of the conductivity equation $\nabla \cdot \gamma \nabla z = 0$, then $w = \gamma^{1/2} z$ is a solution to the Schrödinger equation $(-\Delta + q)w = 0$, where the potential $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. Under the standard assumption that 0 is not a Dirichlet eigenvalue for the Schrödinger equation, and for $q \in L^\infty(\Omega), \psi \in H^{1/2}(\partial\Omega)$, they considered the following Dirichlet problem

$$\begin{cases} -\Delta w + q w = 0 & \text{in } \Omega, \\ \qquad\quad w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The DN map associated with q is well-defined from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$ by $\psi \mapsto \Lambda_q(\psi) = \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega}$. The idea of Sylvester and Uhlmann was to look for special solutions $w(x,\zeta), \zeta \in \mathbb{C}^n, \zeta \cdot \zeta = 0$ satisfying $(-\Delta + q)w = 0$, which are asymptotically exponential, i.e., $w \sim e^{i\zeta.x}$ when $|\zeta| \to \infty$. The functions $w(x,\zeta) = e^{i\zeta.x}(1 + y_\zeta(x))$ are called complex geometrical optics solutions (CGOs), where $y_\zeta(x) \in H^1(\Omega)$ is a correction term that is needed to transit from an approximate solution to the exact one by taking $|\zeta| \to \infty$. Their result inspired many authors to find the lowest regularity condition on the conductivity under which uniqueness holds. More recent uniqueness results, and the used techniques are listed in table 1.1.

Table 1.1: Recent uniqueness results for $n \geq 3$.

n	γ	Techniques	Ref
≥ 3	$W^{3/2,2n+}$	Approximation argument	[14]
≥ 3	$C^1, W^{1,\infty}$ with $\|\|\nabla \log \gamma\|\|_{L^\infty}$ small	Bs, averaging argument	[23]
≥ 3	$W^{1,\infty}$	Bourgain's spaces (Bs)	[16]
3	$H^{3/2+}$	Standard Sobolev spaces	[36]
3,4	$W^{1,n}$	Bs, L^p harmonic analysis	[22]
5,6	$W^{1+(1-\theta)(1/2-2/n),n/(1-\theta)}, \theta \in [0,1)$	Bs, L^p harmonic analysis	[22]
5	$W^{41/40+,5}$	Bilinear estimate	[24]
6	$W^{11/10+,6}$	Bilinear estimate	[24]
≥ 5	$W^{1+\frac{n-5}{2p}+,p}, p \in [n,\infty)$	Bilinear estimate	[39]

The observation of the table makes us wonder how much it would be interesting to check whether it is possible to prove Brown's conjecture [11], which affirms that in three and higher dimensions $\gamma \in W^{1,n}$ is the minimum possible regularity for which uniqueness holds. Notice that the approaches used in [22,24,36] are not useful for reconstructing γ, because the proofs there are not constructive, meaning that they did not give a procedure to recover γ from Λ_γ.

The two-dimensional problem is also of significant interest but differs mainly from the higher dimensional one so that different techniques are used to address this case. Nachman [33] was the first who proved uniqueness for $\gamma \in W^{2,d}, d > 1$ in the plane. This last regularity assumption was relaxed by Brown-Uhlmann [12] to $\gamma \in W^{1,2+}$, and by Astala-Päivärinta [8] to $\gamma \in L^\infty$.

Once uniqueness holds, one can be interested in the reconstruction problem. In practice, Nachman's reconstruction procedure was widely applied in the implementation of algorithms [40]. For example, in medical imaging technology, the electrical impedance tomography (EIT) with several applications, including the detection of breast cancer and pulmonary imaging. See the review papers [11,25] for more detailed arguments on this technique.

While the current paper deals mainly with the entire data problem, we note that the partial data problem is subject to huge advances. The partial data type problem aims to reduce as much as possible the part of the boundary, where measurements are taken, and excitations on the studied body are imposed because, from a realistic view, it is not practical to consider measurements on the whole boundary of some domain. We refer the reader to the excellent survey paper [26] by Kenig and Salo on the recent progress in this problem. For the reconstruction results with partial data, we give further references [3,5,35]. When γ depends on direction, we are in the presence of the anisotropic Calderón problem. In the plane, uniqueness was shown for L^∞ anisotropic conductivities in [7]. For $n \geq 3$, this problem is also called Calderón's inverse problem on Riemannian manifolds, and as Lassas and Uhlmann pointed out in [30], this is a geometrical problem that has up to now remained open.

We aim to offer the interested reader a short introduction to the reconstruction problem. We hope that this work could inspire a different way of proposing a method of reconstructing the conductivity. We have not attempted to be exhaustive in this introduction. In particular, we have neglected stability and numerical results and closely related inverse problems. As the research field on the Calderón problem is too broad, we refer the reader to the review works [4,9,17,25,46] on the general problem.

The rest of this article is organized in the following way: the applied notation and background knowledge are summarized in Section 2. In Section 3, we give the precise statements of the known reconstruction results. Section 4 discusses the proof strategy. Section 5 contains an open problem.

2 Preliminaries

Throughout this article

- Ω denotes a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.
- $n \geq 2$ denotes the space dimension.

- $q : \Omega \to \mathbb{R}$ denotes an electrical potential.
- dS denotes the surface on $\partial\Omega$.
- $\mathcal{S}(\mathbb{R}^n)$ denotes Schwartz space.
- $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.
- $\langle , \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.
- $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.
- $B_R(0)$ denotes the closed ball with center 0 and radius $R > 0$.
- $a \lesssim b$ denotes that it exists a constant $c > 0$ such that $a \leq cb$.

2.1 Fourier transform and function spaces

For $\xi \in \mathbb{R}^n$, the applied notation for the Fourier transform is

$$\hat{w}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi . x} w(x) dx.$$

The inverse Fourier transform is noted by

$$\check{w}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi . x} w(\xi) d\xi.$$

For $s \in \mathbb{R}$, we define Sobolev spaces $H^s(\mathbb{R}^n)$ via Fourier transform as follows:

$$H^s(\mathbb{R}^n) = \{ w \in \ \mathcal{S}'(\mathbb{R}^n) : \ \ \langle \xi \rangle^s \widehat{w} \in L^2(\mathbb{R}^n) \},$$

where $\langle \xi \rangle = (|\xi|^2 + 1)^{1/2}$.
The associated norm is
$\|w\|_{H^s(\mathbb{R}^n)} = \|\langle \xi \rangle^s \widehat{w}\|_{L^2(\mathbb{R}^n)}$.
Recalling the Schrödinger equation from the problem (1.3), substituting with $w(x, \zeta) = e^{i\zeta . x}(1 + y_\zeta(x))$, we deduce an equivalent equation for y_ζ, precisely

$$\triangle_\zeta y_\zeta = (\Delta + 2i\zeta \cdot \nabla) y_\zeta = q(1 + y_\zeta) \ \text{ in } \Omega.$$

The right inverse of the differential operator $\triangle_\zeta$ is defined by

$$\widehat{\triangle_\zeta^{-1}} f(\xi) = p_\zeta(\xi)^{-1} \hat{f}(\xi). \tag{2.1}$$

with symbol

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi.$$

Using this symbol, we can define the space $\dot{X}_\zeta^b$ with the associated norm

$$||w||_{\dot{X}_\zeta^b} = |||p_\zeta(\xi)|^b \hat{w}(\xi)||_{L^2},$$

and the inhomogeneous spaces X_ζ^b with the associated norm

$$||w||_{X_\zeta^b} = ||(|\zeta| + |p_\zeta(\xi)|)^b \hat{w}(\xi)||_{L^2}.$$

In Section 5, we will only need to use the exponent $b = \pm 1/2$. Notice that those two spaces were firstly considered by Haberman and Tataru [23] in the spirit of Bourgain's spaces, see [10, 45].

2.2 DN map and integral identity

From the variational formulation of the problem (1.2), it follows the following Alessandrini identity [2].

$$\langle \Lambda_\gamma \psi, \phi \rangle = \left\langle \gamma \frac{\partial w}{\partial \nu}, \phi \right\rangle = \int_\Omega \gamma \nabla w \nabla z dx \ \ \forall \psi, \phi \in H^{1/2}(\partial\Omega),$$

where $z \in H^1(\Omega), z\big|_{\partial\Omega} = \phi$.

By recalling from the introduction the DN map Λ_q associated with (1.3), we can give another useful identity when $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. It is easy to check that the DN map Λ_q can be obtained from the DN map Λ_γ, where the explicit expression relating those two maps is given by

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2} f) + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu} f\bigg|_{\partial\Omega}. \tag{2.2}$$

One other important relation is the following integral identity that relates boundary measurements with interior potentials.

$$\left\langle (\Lambda_{q_1} - \Lambda_{q_2}) w_1\big|_{\partial\Omega}, w_2\big|_{\partial\Omega} \right\rangle = \int_\Omega (q_1 - q_2) w_1 w_2 \, dx, \tag{2.3}$$

for $q \in L^\infty(\Omega)$ and $w_j \in H^1$ uniquely solve $-\Delta w_j + q_j w_j = 0$, for $j = 1, 2$.

2.3 Faddeev's Green's function and layer operator

While the equation (2.1) implicitly gives the right inverse G_ζ of $\triangle_\zeta$, the following explicit functions

$$g_\zeta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i\xi.x}}{p_\zeta(\xi)} d\xi, \qquad G_\zeta(x) = e^{i\zeta.x} g_\zeta(x), \tag{2.4}$$

are the Faddeev's Green's functions for $(\Delta + 2i\zeta.\nabla)$ and the Laplacian, respectively.

Now, we introduce some useful operators, which will be needed later in Section 4. Using the family G_ζ of Green's functions for $x \in \mathbb{R}^n \backslash \partial\Omega$, we define the following layer potentials.
Single layer potential:

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x,y) f(y) dS(y). \tag{2.5}$$

Double layer potential:

$$D_\zeta f(x) = \int_{\partial\Omega} \frac{\partial G_\zeta(x,y)}{\partial\nu(y)} f(y) dS(y).$$

We define also for $x \in \partial\Omega$, the boundary double layer potential:

$$B_\zeta f(x) = p.v. \int_{\partial\Omega} \frac{\partial G_\zeta(x,y)}{\partial\nu(y)} f(y) dS(y). \tag{2.6}$$

3 Reconstruction results

Throughout this section, we try to give precise statements of the known reconstruction results. We will split the section into two subsections, depending on the study domain dimension. Notice that the used approach for the two-dimensional problem, which is essentially based on complex analysis, is quite different from the higher-dimensional problem. Thus, we first present the known reconstruction results in the plane.

3.1 Reconstruction in two dimensions

For the two-dimensional problem, Novikov and Nachman were the first to answer the reconstruction question in [38] and [33]. Nachman's result is presented as follows:

Theorem 3.1. *[33] Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, smooth domain, and let $\gamma \in W^{2,p}(\Omega), p > 1$. Then there is a procedure to reconstruct γ uniquely from Λ_γ.*

Inspired by the uniqueness proof of Brown and Uhlmann [12], Knudsen and Tamasan [28] applied the $\bar{\partial}$-method to produce a reconstruction algorithm for $\gamma \in W^{1,p}, p > 2$. Their result is considered as a sharp improvement over the last one due to Nachman.

Theorem 3.2. *[28] Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth domain, and let $0 < \varsigma < 1$ with $\gamma \in W^{1+\varsigma,p}(\Omega), p > 2$ satisfying (1.1). Then γ can be reconstructed on Ω from the knowledge of Λ_γ.*

In 2018 Lytle, Perry, and Siltanen [31] proved that Nachman's reconstruction method still holds for L^∞ conductivities, which are 1 in a neighborhood of the boundary. Here we present their main Theorem, and further details on their work are given in Section 4.

Theorem 3.3. *[31] Let $\gamma \in L^\infty(\mathbb{D})$ satisfying (1.1), and suppose that the condition*

$$\text{there is a } x_0 \in (0,1) \text{such that } \gamma = 1 \text{ for } |x| \geq x_0, \tag{3.1}$$

holds. Then, for each $\zeta \in \mathbb{C}$, there exists a unique $w|_{\partial\mathbb{D}} \in H^{1/2}(\partial\mathbb{D})$ such that

$$w|_{\partial\mathbb{D}} = e^{i\zeta.x}|_{\partial\mathbb{D}} - S_\zeta(\Lambda_\gamma - \Lambda_1)w|_{\partial\mathbb{D}}. \tag{3.2}$$

By abuse of notation, the map $\Lambda_1 = \Lambda_0$ is the DN map for harmonic functions on $\mathbb{D}$ that correspond to $q = 0$ and $\gamma = 1$.

3.2 Reconstruction in higher dimensions

In 1988 for higher dimensions, Nachman [34] and Novikov [37] were also the first who provided a constructive procedure to recover $\gamma \in C^{1,1}$ from the knowledge of Λ_γ.

Theorem 3.4. *[34] Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with a $C^{1,1}$ boundary, and let $\gamma \in C^{1,1}(\bar{\Omega})$ satisfying (1.1). Then there is a procedure to reconstruct γ uniquely from Λ_γ.*

Novikov [37] has independently shown a similar result to the previous one given by Nachman. He was the first who introduced the key ingredient of the boundary integral equation, which will be explained later in the next section.

Based on the uniqueness result of Haberman and Tataru [23], Nachman's procedure was followed by García and Zhang in [20] to reconstruct C^1, or Lipschitz conductivities with $|\nabla \log \gamma|$ sufficiently small.

Theorem 3.5. *[20] Let Ω be a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, and let γ be a strictly positive real-valued function on Ω satisfying (1.1).*

1. *$\gamma \in C^1(\bar{\Omega})$.*

2. *$\gamma \in Lip(\Omega)$, such that $|\nabla \log \gamma(x)| < \delta_{\Omega,n}$ with $\delta_{\Omega,n}$ a constant.*

If 1 or 2 is satisfied, then γ can be reconstructed on Ω from the knowledge of Λ_γ.

In 2020, Tarikere extended the uniqueness result of Brown and Torres [14] to prove the validity of Nachman's method for $W^{3/2,2n}$ conductivities.

Theorem 3.6. *[44] Let Ω be a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, and let $\gamma \in W^{3/2,2n}(\Omega)$ be a strictly positive real-valued function on Ω satisfying (1.1) with $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$. Then γ can be reconstructed from Λ_γ.*

While all the previous results concern the full data problem, Nachman was also interested in the reconstruction of the partial data type problem. Based on the well-known Carleman estimate approach in [27], Nachman and Street obtained a reconstruction proof with partial data measurements on a slightly overlapping partition of the boundary $\partial\Omega$. The reader is referred to ([35], Theorem 1.3) for the precise result. Their result was recently approved by Garde [21] to piecewise constant layered conductivities. Grade's reconstruction method only relies on the monotonicity principles of the local DN map, and therefore lends well to efficient numerical implementation models.

4 Proof strategy

In the present section, we briefly review the proof of the reconstruction results described above and the main theoretical tools used therein. The two-dimensional problem is quite different from the higher dimensional case. For example, it is no longer over-determined. To show that, we propose the following explanation. Since it is a linear operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, the DN map Λ_γ can be expressed in terms of the Schwartz kernel $K : \partial\Omega \times \partial\Omega \longrightarrow \mathbb{R}$ by

$$\Lambda_\gamma f(x) = \int_{\partial\Omega} K(x,y) f(y) dS(y). \tag{4.1}$$

From one side, it is known that the dimension of $\partial\Omega$ is $n-1$. Then, the kernel K is a function of $2(n-1)$ variables. On the other side, the conductivity γ, which we wish to recover, is defined in an n-dimensional domain. Thus, for $n = 2$, the Calderón problem in the plane is formally well-determined and fairly well-understood.
From (4.1), it is clear that for $n \geq 3$, the inverse problem is formally over-determined since the known data has more degree of freedom than the quantity γ, which we are trying to recover. That means that sometimes (but certainly not always) the problem may be easier to manipulate in higher dimensions.
The precedent motivates in some way that, to deal with the two-dimensional problem, we need to invoke a different technique than the one used when $n \geq 3$.

4.1 Preliminary reductions

To simplify the problem, we use the following two types of reductions. On the one hand, Nachman ([33], Section 6) proceeds to a reduction of γ in a neighborhood of $\partial\Omega$. His idea was to reduce the Calderón problem to a problem having a constant $\gamma \equiv 1$ near $\partial\Omega$, then to extend γ outside the study domain Ω such that the initial regularity assumption is conserved. Thus, solving the extended problem on the large domain means that the original problem on Ω is implicitly solved.

The main idea behind this reduction is based on the following step of reconstructing the boundary value of the unknown conductivity and its derivative from the DN map.

4.1.1 Reconstruction at the boundary

From identity (2.2), it is clear that to find the value of Λ_q, we need a procedure to recover the values of γ and $\frac{\partial\gamma}{\partial\nu}$ on the boundary $\partial\Omega$ from Λ_γ. Thus, we deduce the importance of boundary determination, which depends on the regularity of both the domain boundary and the conductivity itself. For the case of smooth conductivities in smooth domains, Kohn and Vogelius [29] proved that Λ_γ determines γ and all its normal derivatives on the boundary. More results and approaches to boundary determination of the conductivity were shown in [1, 43]. In particular, Brown [13] proved that we could recover the boundary values of a $W^{1,1}$, or a C^0 conductivity from the knowledge of Λ_γ.

In the appendix of [20], the gradient at the boundary of a C^1 conductivity in a Lipschitz domain was recovered by Brown in collaboration with García and Zhang. In all ways, this boundary determination is based on testing the DN map against highly oscillatory functions at the domain boundary.

On the other hand, we saw in the introduction that the conductivity problem (1.2) could be reduced to the Schrödinger problem (1.3) by a well-known transformation under the condition that the conductivities are sufficiently regular (which is the case here). The desired conclusion behind those reductions is to possess a potential q having a compact support in Ω.

4.2 Nachman's method

After reducing the inverse conductivity problem to the inverse problem for a Schrödinger equation, the reconstruction method of Nachman could be decomposed into three steps. First, we extend q to be 0 in $\mathbb{R}^2$ outside the study domain. The second step consists of computing the scattering transform t of the Schrödinger equation associated with the extended potential q from the given DN map. Finally, the $\bar\partial$-method permits solving the scattering problem, which is used to calculate the value of γ.

Below, we will give a discerption of the reconstruction process in the plane [33].

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. For $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, Nachman used Faddeev's [18] CGOs in the problem (1.3) to get

$$\begin{cases} -\Delta w + qw = 0, \\ \lim_{|x|\to\infty} e^{-i\zeta . x} w(x,\zeta) - 1 = 0. \end{cases} \tag{4.2}$$

We define the useful complex derivative operators $\bar\partial$ and ∂ as follows:

$$\bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \qquad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

By substituting with $a(x,\zeta) = e^{-i\zeta.x}w(x,\zeta)$ in (4.2), we get

$$\begin{cases} \bar{\partial}(\partial + ix)a = \frac{1}{4}qa, \\ \lim_{|x|\to\infty} a = 1. \end{cases} \tag{4.3}$$

Then, one can use (4.3) to define the scattering transform t

$$t(\zeta) = \int_{\mathbb{R}^2} b_\zeta(x)q(x)a(x,\zeta)dx, \tag{4.4}$$

where $b_\zeta(x) = e^{i(\zeta.x+\overline{\zeta.x})}$.
Nachman showed that the solutions $a(x,\zeta)$ solve

$$\begin{cases} \bar{\partial}_\zeta a = \frac{t(\zeta)}{4\pi\zeta} b_{-\zeta}(x)\bar{a}, \\ \lim_{|\zeta|\to\infty} a = 1. \end{cases} \tag{4.5}$$

Since we know from the preceding subsection that the used reduction guarantees that q has a compact support in Ω, then (4.3) and (4.4) can be reduced to the following boundary integral equations, respectively.

$$w|_{\partial\Omega} = e^{i\zeta.x}|_{\partial\Omega} - S_\zeta(\Lambda_q - \Lambda_0)w|_{\partial\Omega}. \tag{4.6}$$

$$t(\zeta) = \int_{\partial\Omega} e^{i\bar{\zeta}.\bar{x}}(\Lambda_q - \Lambda_0)w|_{\partial\Omega}\, dS. \tag{4.7}$$

Where S_ζ is defined in (2.5). As was mentioned in Section 3, the boundary integral identity (4.6) was developed for the first time by Novikov [37].
Finally, by giving the value of t from (4.7), we can solve (4.5) to recover the conductivity from the identity

$$\gamma(x) = a(x,0)^2. \tag{4.8}$$

In the plane, we recapitulate Nachman's reconstruction method for $\gamma \in W^{2,p}$ in the following four steps.

1. Solve (4.6) for $w|_{\partial\Omega}$.
2. Calculate the value of t from (4.7).
3. Solve the $\bar{\partial}_\zeta$-equation (4.5).
4. Recover γ from (4.8).

Remark 4.1.
- The Knudsen-Tamasan result in Theorem 3.2 for a less regular γ was proposed by following the uniqueness proof of Brown and Uhlmann [12], and by making every step in their proof constructive.
- The reconstruction algorithm of Knudsen-Tamasan [28] is a generalization of the above-summarized one, and the proof steps are almost the same. For other kinds of algorithms based on a linearized or iterative schema, see [9,17].

4.3 Beltrami equation

The construction of CGOs viewed before relies on the available regularity assumption on γ. Another construction that requires no smoothness on γ was introduced in [8] for $\gamma \in L^\infty$ strictly positive, using the Beltrami equation approach.
Next, we describe the analysis steps of the reconstruction process proposed by Lytle, Perry, and Siltanen in Theorem 3.3.
Without loss of generality, we assume that the domain Ω is the unit disc $\mathbb{D}$ and $\gamma = 1$ in a neighborhood of $\mathbb{D}$. More precisely, we consider that condition (3.1) holds. The Beltrami coefficient μ used by Astala and Päivärinta [8] is defined by

$$\mu = \frac{1-\gamma}{1+\gamma},$$

satisfying $|\mu(x)| < 1$, and having a compact support since that the conductivity is set to be equal to one outside a compact set. Then, for any function $w \in H^1(\mathbb{D})$ that solves the conductivity equation given in (1.2), there exists $\tilde{w} \in H^1(\mathbb{D})$ a real-valued function named the conjugate harmonic function of w such that the Beltrami equation

$$\bar{\partial}\dot{w} = \mu\partial\dot{w} \tag{4.9}$$

has a solution $\dot{w} = w + i\tilde{w}$.
The key ingredient in the analysis in [31] is this last Beltrami equation (4.9), which admits CGOs. Those CGOs can be used to define an associated scattering transform, which is identified as a natural analog of Nachman's one (4.7). This transform remains well-defined under the weaker regularity assumption $\mu \in L^\infty(\Omega)$ by Theorem 4.2 from [8]. Theorem 3.3 combined with Corollary 18.1.2 from [6] about the uniqueness of CGOs for the conductivity equation, establish the unique solvability of the integral equation (3.2).
Notice that the followed strategy to prove Theorem 3.3 is to show the compactness of the integral operator $T_\zeta = S_\zeta(\Lambda_\gamma - \Lambda_1)$ from $H^{1/2}(\partial\mathbb{D})$ to $H^{-1/2}(\partial\mathbb{D})$. Then, to prove that the integral equation (3.2) is uniquely solvable, it suffices by Fredholm theory, to show that the only vector $v \in H^{1/2}(\partial\mathbb{D})$ with $T_\zeta v = -v$ is the zero vector.
For more efficient algorithms for the computation of CGOs $\dot{w}$, and numerical examples, see ([32], Chapter 14, page: 215-221). Interested readers are referred to ([32], Chapter 15), and the references therein for readings on the D-bar method, which is based on Nachman's result [33].

4.4 Boundary integral equation

In the present subsection, we will describe more carefully each step in the reconstruction procedure in higher dimensions. For $n \geq 3$, the valuable tool of CGOs, which was presented in the introduction to show the uniqueness in Calderón problem in the work of Sylvester and Uhlmann [42], was used later by Nachman in Theorem 3.4 and by Novikov in [37] independently to reconstruct the conductivity γ. We will describe Nachman's idea [34] as follows. As it was already seen in subsection 4.1, we can give the boundary reconstruction of γ and $\frac{\partial\gamma}{\partial\nu}$ from the DN map. Then, if Λ_γ is knew, Λ_q is calculated from identity (2.2). Hence, the problem is reduced to the reconstruction of q from Λ_q. Once we have the value of $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, we can solve the following problem to deduce γ.

$$\begin{cases} -\Delta w + qw = 0 & \text{in } \Omega, \\ w = \gamma^{1/2} & \text{on } \partial\Omega. \end{cases}$$

Now, let $q_1 = q$, $q_2 = 0$ in the integral identity (2.3). Then we get

$$\int_\Omega q w_1 w_2 \, dx = \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(w_1|_{\partial\Omega}) w_2|_{\partial\Omega} \, dS, \tag{4.10}$$

where $w_1, w_2 \in H^1(\Omega)$ solves $-\Delta w_1 + q w_1 = 0$, and $-\Delta w_2 = 0$, respectively.
In the following, we use expression (4.10) and appropriate CGOs to reconstruct the Fourier transform of q. We consider $\xi \in \mathbb{R}^n, \xi \neq 0$, and we define the set B by $B = \{\zeta_j \in \mathbb{C}^n : \zeta_j \cdot \zeta_j = 0,\ |\zeta_1| = |\zeta_2| = h,\ \zeta_1 + \zeta_2 = \xi, j = 1, 2\}$. The application of the argument from [42] ensures the existence of CGOs $w_1 = e^{i\zeta_1 . x}(1 + y_{\zeta_1})$ for $-\Delta w_1 + q w_1 = 0$, with the correction term y_{ζ_1} decaying to zero when $|\zeta_1| \to \infty$. Furthermore, the appropriate choice of $\zeta_2 . \zeta_2 = 0$ implies that $\Delta e^{i\zeta_2 . x} = 0$.
By substituting in (4.10) and by using the decay property of y_{ζ_1}, we have

$$\hat{q}(\xi) = \lim_{h\to\infty} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(w_1|_{\partial\Omega}) e^{i\zeta_2 . x}|_{\partial\Omega} \, dS. \tag{4.11}$$

From (4.11), we deduce that the Fourier transform of q for $\xi \neq 0$ can be recovered from the DN map if $w_1|_{\partial\Omega}$ is knew. We know that q is compactly supported, then $\hat{q}(\xi)$ is continuous so that $\hat{q}(0)$ can be determined by continuity [41]. Hence, $\hat{q}(\xi)$ is known as a tempered distribution, and the potential q can be recovered in $\mathbb{R}^n$ by simply inverting the Fourier transform. Therefore, it is a question to get the value of $w_1|_{\partial\Omega}$ to recover $\hat{q}(\xi)$.
The aim now is to find a method to calculate $w_1\big|_{\partial\Omega}$. The idea is to look at the exterior problem, which means that we extend q to $\mathbb{R}^n$ to be $q = 0$ outside the study domain Ω. Since $q = 0$ in $\mathbb{R}^n \backslash \bar{\Omega}$, the equation $(-\Delta + q)w_1 = 0$ in $\mathbb{R}^n$ becomes $-\Delta w_1 = 0$ in $\mathbb{R}^n \backslash \bar{\Omega}$. Therefore, the function w_1 is a solution to the following exterior problem.

$$\begin{cases} -\Delta w_1 = 0 \text{ in } \mathbb{R}^n \backslash \bar{\Omega}, \\ w_1|_{\partial\Omega} = f_\zeta, \\ \frac{\partial w_1}{\partial \nu}|_{\partial\Omega} = \Lambda_q f_\zeta. \end{cases} \tag{4.12}$$

For a fixed $R > R_0$ such that $\Omega \subset B_R(0)$, it is known from [34] that if w_1 satisfies the following analog of Sommerfeld radiation condition

$$\lim_{R\to\infty} \int_{|y|=R} \left(G_\zeta(x,y) \frac{\partial (w_1 - e^{i\zeta . x})}{\partial \nu(y)} - (w_1 - e^{i\zeta . x}) \frac{\partial G_\zeta(x,y)}{\partial \nu(y)} \right) dS(y) = 0, \tag{4.13}$$

then, by using Green's formula in (4.12), we can show that the boundary value $w_1|_{\partial\Omega}$ can be characterized as the unique solution f_ζ of the following boundary integral equation of Fredholm type.

$$e^{i\zeta . x} - S_\zeta(\Lambda_q - \Lambda_0) f_\zeta = f_\zeta \quad \text{on } \partial\Omega. \tag{4.14}$$

As we notice that the operator on the left-hand side of the boundary integral equation (4.14), depends on the DN map and other known quantities, we can recover the value of $w_1|_{\partial\Omega}$ by solving (4.14). Moreover, (4.14) is an inhomogeneous integral equation for f_ζ having a unique solution $f_\zeta \in H^{3/2}(\partial\Omega)$. By Fredholm alternative, the uniqueness of the solution follows from the fact that the homogeneous equation

$$-S_\zeta(\Lambda_q - \Lambda_0) f_\zeta = f_\zeta \quad \text{on } \partial\Omega,$$

only has the zero solution, which follows by its turn from the uniqueness of the CGOs.

Remark 4.2.
- Nachman derived the slight different type of boundary integral equation:

$$e^{i\zeta.x} - (S_\zeta \Lambda_q - B_\zeta - \frac{1}{2}I)f_\zeta = f_\zeta \quad \text{on } \partial\Omega, \tag{4.15}$$

  where the operator B_ζ is defined in (2.6). Since we can easily show that $S_\zeta\Lambda_0 = B_\zeta + \frac{1}{2}I$, it is clear that the expressions (4.15) and (4.14) are equivalent.

- Because it is complicated to check that the condition (4.13) is satisfied by w_1, Nachman's idea was to construct from (4.15) CGOs to the Schrödinger equation $(-\Delta + q)w = 0$ in $\mathbb{R}^n$, that automatically satisfy condition (4.13), then to prove that those CGOs coincide with the ones constructed by Sylvester and Uhlmann [42].

Now, we turn to give the sketch of the proof of theorems 3.5 and 3.6. Mainly, the strategy used there was to follow the discussed Nachman's method for Theorem 3.4.

Due to the weak assumption regularity on γ in Theorem 3.5 ($\gamma \in C^1$ or γ Lipschitz with $|\nabla \log \gamma(x)| < \delta_{\Omega,n}$) and Theorem 3.6 ($\gamma \in W^{3/2,2n}$), some changes are made in the above steps. The proof outline consists of constructing CGOs to the conductivity equation or the Schrödinger equation in $\mathbb{R}^n$, respectively, from the boundary integral-equation on the boundary. Then, to show that these solutions coincide with the ones constructed by Haberman-Tataru [23] and Brown-Torres [14], respectively. Note that the reconstruction presentation in [44] follows mainly the analysis and notations from ([19], Chapter 4.7), which focuses on reconstructing $\gamma \in C^2(\Omega)$.
We know that by plugging $w(x,\zeta) = e^{i\zeta.x}(1 + y_\zeta(x))$ in the Schrödinger equation, we get

$$(-\Delta - 2i\zeta \cdot \nabla)y_\zeta(x) + q(x)y_\zeta(x) = -q(x) \quad \text{in } \mathbb{R}^n. \tag{4.16}$$

By convolving (4.16) with g_ζ which is defined in (2.4), we obtain the Lippmann-Schwinger-Faddeev integral equation

$$(I + g_\zeta * q)y_\zeta(x) = g_\zeta * q. \tag{4.17}$$

The last equation (4.17) is equivalent to the following integral equation

$$w(x,\zeta) + \int_{\mathbb{R}^n} G_\zeta(x,y)q(y)w(y,\zeta)dy = e^{i\zeta.x}, \tag{4.18}$$

where G_ζ is defined in (2.4). It is clear that the combination of (2.3) and (4.18) gives (4.14) for w. Moreover, the homogenous version of (4.18) is

$$w(x,\zeta) = \int_{\mathbb{R}^n} G_\zeta(x,y)q(y)w(y,\zeta)dy. \tag{4.19}$$

The analysis in [20] and [44] showed that the operator at the right-hand side of (4.19) is a contraction, provided the corresponding CGOs are constructed for sufficiently large $|\zeta|$. Finally, the problem is reduced to a fixed point problem.

5 Open problem, conjecture, and discussion

In the precedent sections, some methods for conductivity reconstruction were reviewed. Those methods were analyzed, compared, and their steps were summarized. The results show that all the cited methods are in some way a generalization of Nachman's (or Novikov's) method. Besides, those results can provide a reference to the reconstruction subject of the problem.

Under the broad research field of Calderón's problem, we wrote this note to motivate and draw more attention to the reconstruction topic. Therefore, we hope that something might lie beyond this paper. In this final section, we propose the following open question and discuss plausibly research extensions that can be subject to new results in the reconstruction direction of the problem.

Question. (Reconstruction of Lipschitz conductivities) *If Ω is a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, $\gamma \in Lip(\Omega)$ a strictly positive real-valued function on Ω satisfying (1.1), with $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$, show that γ can be reconstructed on Ω from the knowledge of Λ_γ.*

Recently, Caro and Rogers [16] used Bourgain's spaces to prove the uniqueness of Lipschitz conductivities in three and higher dimensions. Their result makes us wonder how much it would be interesting to check whether it is possible to use this uniqueness proof to generalize Nachman's method to Lipschitz conductivities by taking off the smallness condition on $|\nabla \log \gamma|$ to improve the results of Theorem 3.5. The key ingredient in the uniqueness proof in [16] for Lipschitz conductivities without a smallness condition is the following a priori estimate:

$$||w||_{X_\zeta^{1/2}} \lesssim ||(-\Delta + 2\zeta \cdot \nabla + q)w||_{X_\zeta^{-1/2}},$$

for a function $w \in \mathcal{S}(\mathbb{R}^n)$ with support in Ω, and the function spaces $X_\zeta^{\pm 1/2}$ were defined in Section 2. From the last estimate and a standard functional analysis argument, it follows a key bound on the potential q

$$||y_\zeta||_{X_\zeta^{1/2}(\Omega)} \lesssim ||q||_{X_\zeta^{-1/2}},$$

for some corrector function y_ζ. The occurring complication is that the solutions here are local, but in our case, we need to extend them in some way to $\mathbb{R}^n$. Therefore, we conjecture that the techniques used until now, which have been reviewed in this survey, have reached some sort of limit. Thus, we can not follow the contraction mapping approach to apply the fixed point argument used in the above methods. However, it is straightforward that this problem seems more complicated and may require new ideas beyond the known techniques to overcome its difficulties.

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Conflict of Interest

The author has no conflicts of interest to disclose.

References

[1] G. Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, J. Differential Equations. **84**(2) (1990), 252–272.

[2] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27**(1-3) (1988), 153–172.

[3] H. Ammari and G. Uhlmann, *Reconstruction of the potential from partial Cauchy data for the Schrödinger equation,* Indiana Univ. Math. J. **53**(1) (2004), 169–183.

[4] M. Aoudj, *On the main aspects of the inverse conductivity problem*, Journal of Nonlinear Modeling and Analysis. **4**(1) (2022), 1–17.

[5] Y. M. Assylbekov, *Reconstruction in the partial data Calderón problem on admissible manifolds*, Inverse Probl. Imaging. **1**1(3) (2017), 455–476.

[6] K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, Princeton, 2009.

[7] K. Astala, M. Lassas and L. Päivärinta, *Calderón's inverse problem for anisotropic conductivity in the plane,* Comm. Partial Differential Equations. **30**(1-2) (2005), 207–224.

[8] K. Astala and L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. of Math. **163**(1) (2006), 265–299.

[9] L. Borcea, *Electrical impedance tomography,* Inverse Problems. **18**(6) (2002), R99–R136.

[10] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations,* Geom. Funct. Anal. **3**(2) (1993), 107–156.

[11] B. H. Brown, *Electrical impedance tomography (eit): a review,* J. Med. Eng. Technol. **27**(3) (2003), 97–108.

[12] R. Brown and G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions,* Comm. Partial Differential Equations. **22**(5-6) (1997), 1009–1027.

[13] R. Brown, *Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result,* J. Inverse Ill-Posed Probl. **9**(6) (2001), 567–574.

[14] R. M. Brown and R. H. Torres, *Uniqueness in the inverse conductivity problem for conductivities with* $3/2$ *derivatives in* $L^p, p > 2n$, J. Fourier Anal. Appl. **9**(6) (2003), 563–574.

[15] A. P. Calderón, On an inverse boundary value problem, in:*Seminar on Numerical Analysis and its Applications to Continuum Physics*, 1980, pp. 65–73.

[16] P. Caro and K. Rogers, *Global uniqueness for the Calderón problem with Lipschitz conductivities,* Forum Math. Pi. **4**(e2) 28 (2016).

[17] M. Cheney, D. Isaacson and J. C. Newell, *Electrical impedance tomography,* SIAM Review. **41**(1) (1999), 85–101.

[18] L. D. Faddeev, *Growing solutions to Schrödinger's equation,* Doklady Akademii nauk SSSR. **165**(3) (1965), 514–517.

[19] J. Feldman, M. Salo and G. Uhlmann, *The Calderón Problem – An Introduction to Inverse Problems*, Spring-Verlag, New York, 2010.

[20] A. García and G. Zhang, *Reconstruction from boundary measurements for less regular conductivities*, Inverse Problems. **32** (11) (2016), Article ID 115015, 20 pages.

[21] H. Garde, *Reconstruction of piecewise constant layered conductivities in electrical impedance tomography,* Comm. Partial Differential Equations. **45**(9) (2020), 1118–1133.

[22] B. Haberman, *Uniqueness in Calderón's problem for conductivities with unbounded gradient,* Comm. Math. Phys. **340**(2) (2015), 639–659.

[23] B. Haberman and D. Tataru, *Uniqueness in Calderón problem with Lipschitz conductivities*, Duke Math. **162**(3) (2013), 497–516.

[24] S. Ham, Y. Kwon and S. Lee, *Uniqueness in the Calderón problem and bilinear restriction estimates*, J. Funct. Anal. **281**(8) (2021), Article ID 109119, 58 pages.

[25] M. Hanke and M. Bröhl, *Recent progress in electrical impedance tomography,* Inverse Problems. **19**(6) (2003), S65–S90.

[26] C. Kenig and M. Salo, *Recent progress in the Calderón problem with partial data,* Inverse Problems and Applications. **165**(2) (2017), 567–591.

[27] C. Kenig, J. Sjöstrand and G. Uhlmann, *The Calderón problem with partial data.* Ann. of Math. **165**(2) (2007), 567–591.

[28] K. Knudsen and A. Tamasan, *Reconstruction of less regular conductivities in the plane*, Comm. Partial Differential Equations. **29**(3-4) (2005), 361–381.

[29] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements,* Comm. Pure Appl. Math. **37**(3)(1984), 289–298.

[30] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. Ec. Norm. Supér. **34**(5) (2001), 771–787.

[31] G. Lytle, P. Perry and S. Siltanen, *Nachman's reconstruction method for the Calderón problem with discontinuous conductivities*, Inverse problems. **36**(3) (2019), 035018. 15 pages. DOI:10.1088/1361-6420/ab5a12

[32] J. L. Mueller and S. Siltanen, *Linear and Nonlinear Inverse Problems with Practical Applications*, Computational Science & Engineering, SIAM, Philadelphia, PA, 2012. DOI:10.1137/1.9781611972344

[33] A. I. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem,* Ann. of Math. **143**(2) (1996), 71–96.

[34] A. I. Nachman, *Reconstructions from boundary measurements*, Ann. of Math. **128**(3) (1988), 531–576.

[35] A. Nachman and B. Street, *Reconstruction in the Calderón problem with partial data,* Comm. Partial Differential Equations. **35**(2) (2010), 375–390.

[36] H. M. Nguyen and D. Spirn, *Recovering a potential from cauchy data via complex geometrical optics solutions*, (2014) preprint arXiv:1403.2255

[37] R. G. Novikov, *A multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. Funct. Anal. Appl. **22**(4) (1988), 263–272.

[38] R. G. Novikov and G. M. Khenkin, *The* ∂*-equation in the multi-dimensional inverse scattering problem*, Russian Math. Surveys. **42**(3) (1987), 109–180.

[39] F. Ponc-Vanegas, *The bilinear strategy for Calderón's problem*, Rev. Mat. Iberoam. **37**(6) (2021), 2119–2160.

[40] S. Siltanen, J. L. Mueller and D. Isaacson, *An implementation of the reconstruction algorithm of A. Nachman for the 2D inverse conductivity problem*, Inverse Problems. **16**(3) (2000), 681–699.

[41] E. M. Stein and R. Shakarchi, *Fourier Analysis an Introduction*, Princeton University Press, Princeton, Oxford, 2003.

[42] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. **125**(1) (1987), 153–169.

[43] J. Sylvester and G. Uhlmann, *Inverse boundary value problems at the boundary-continuous dependence,* Comm. Pure Appl. Math. **41**(2) (1988), 197–219.

[44] A. Tarikere, *Reconstruction of rough conductivities from boundary meuserments*, (2020), preprint arXiv:2001.05155v1

[45] D. Tataru, *The* X^s_θ *spaces and unique continuation for solutions to the semilinear wave equation*, Comm. Partial Differential Equations. **21**(5-6) (1996), 841–887.

[46] G. Uhlmann, *Electrical impedance tomography and Calderón's problem,* Inverse Problems. **25**(12) (2009), 123011, 42 pages.