Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 31–39. n2t.net/ark:/49935/jiamcs.v1i1.5

http://jiamcs.centre-univ-mila.dz/

Existence and asymptotic stability of continuous solutions for integral equations of product type

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Received 14 November 2021, Accepted 27 December 2021, Published 29 December 2021

Abstract. In this paper, we study the existence of a continuous solution for a nonlinear integral equation of a product type. The analysis uses the techniques of measures of noncompactness and Darbo's fixed point Theorem. Our results are obtained under rather general assumptions. Moreover, the method used in the proof allows us to obtain the asymptotic stability of the solutions.

Keywords: Integral equation of a product type, measure of weak noncompactness, fixed point theorem, continuous solutions.

2020 Mathematics Subject Classification: 45D05, 47H30.

1 Introduction

In this paper, we consider the following nonlinear integral equation of product type

$$x(t) = f(t, x(t)) + \left[p(t) + \int_0^t u(t, s, x(s))ds\right] \times \left[q(t) + \int_0^t v(t, s, x(s))ds\right], \, t \in \mathbb{R}^+, \tag{1.1}$$

where f, p, q, u, v are continuous functions and $x(t) \in C(\mathbb{R}^+, \mathbb{R})$ is an unknown function.
A variety of problems in physics and biology have their mathematical setting as integral equations of product type. In particular, in the study of the spread of an infectious disease that does not induce permanent immunity (see, for example [3, 10, 11, 16]).
Recently, there has been a growing interest in integral equations of product type. In [12] Gripenberg studied the qualitative behavior of solutions of the following integral equation of product type

$$x(t) = k\left[p(t) + \int_0^t A(t-s)x(s)ds\right] \times \left[q(t) + \int_0^t B(t-s)x(s)ds\right]. \tag{1.2}$$

More exactly, the author studied the existence and uniqueness of a bounded continuous and nonnegative solution of (1.2). Moreover, Pachpatte [15], Abdeldaim [1] and Li et al. [13] studied the boundedness, the asymptotic behavior and continuous solutions of (1.2).

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ISSN (electronic): 2773-4196

Bellour et al. [8] studied the existence of an integrable solution of (1.1) on the interval $[0, 1]$.
On the other hand, Ardjouni and Djoudi [2] studied the existence and approximation of solutions of the initial value problems of nonlinear hybrid Caputo fractional integro-differential equations, which can be transformed to the following integral equation of product type

$$x(t) = \left[p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s)) ds\right] \times \left[\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds\right],$$

on a bounded interval $[0, a]$.
In the paper [14], Olaru studied the existence and the uniqueness of the continuous solution of the following integral equation

$$x(t) = \prod_{i=1}^{m} \left(g_i(t) + \int_a^t K_i(t, s, x(s)) ds \right), \tag{1.3}$$

on a bounded interval $[a, b]$, where $K_i, i = 1, ..., n$ are continuous functions satisfying Lipschitz conditions with respect to the last variable.
Later, Boulfoul et al. [9] studied the existence of an integrable solution of a generalization of (1.3) on $\mathbb{R}^+$.

The purpose of the present work is to study the existence of a continuous solution and bounded solution to (1.1) under fairly simple conditions. Moreover, the method used in the proof allows us to obtain the asymptotic stability of the solutions. An example is presented to show the importance and the applicability of our results.

2 Auxiliary facts and results

In this section, we provide some notations, definitions and auxiliary facts which will be needed for stating our results. Denote by $BC(\mathbb{R}^+, \mathbb{R})$ the Banach space of all real functions defined, continuous and bounded on $\mathbb{R}^+$. It is equipped with the standard norm

$$\|x\| = \sup_{t \in \mathbb{R}^+} |x(t)|.$$

For later use, we assume that X be a Banach space. Let $\mathcal{B}(X)$ denote the family of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all relatively compact subsets of X. Finally, let B_r denote the closed ball centered at 0 with radius r.
Recall the following definition of the concept of the axiomatic measure of noncompactness.

Definition 2.1. [6]. A function $\mu : \mathcal{B}(X) \longrightarrow \mathbb{R}^+$ is said to be a measure of noncompactness if it satisfies the following conditions:

(1) The family $ker(\mu) = \{M \in \mathcal{B}(X) : \mu(M) = 0\}$ is nonempty and $ker(\mu) \subset \mathcal{W}(X)$.

(2) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.

(3) $\mu(co(M)) = \mu(M)$, where $co(M)$ is the convex hull of M.

(4) $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$ for $\lambda \in [0, 1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$ such that $\lim_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.
A measure μ is said to be sublinear if it satisfies the following two conditions:

(6) $\mu(\lambda M) = |\lambda|\mu(M)$ for $\lambda \in \mathbb{R}$.

(7) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

The family $ker(\mu)$ described in (1) is called the kernel of the measure of noncompactness μ. More information about measures of noncompactness and their properties can be found in [5]. For our purposes, we will only need the following fixed point theorem [5].

In what follows, we will use a measure of noncompactness in the space $BC(\mathbb{R}^+, \mathbb{R})$ which was introduced in [5]. In order to recall the definition of this measure let us fix a nonempty bounded subset $X \in BC(\mathbb{R}^+, \mathbb{R})$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon > 0$, let us define the following quantities (cf. [5]):

$$\omega^T(x, \varepsilon) = \sup \{|x(s) - x(t)| : t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega^T(X, \varepsilon) = \sup \left\{\omega^T(x, \varepsilon) : x \in X\right\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \longrightarrow 0} \omega^T(X, \varepsilon), \quad \omega_0(X) = \lim_{T \longrightarrow \infty} \omega_0^T(X).$$

For a fixed number $t \geq 0$, we denote

$$d(X(t)) = \sup \{|x(t) - y(t)| : x, y \in X\}.$$

and

$$d(X) = \limsup_{t \longrightarrow \infty} d(X(t)).$$

Finally, the function μ is defined by putting

$$\mu(X) = \omega_0(X) + d(X).$$

It can be shown [5] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}^+, \mathbb{R})$ with the kernel $ker(\mu)$ consisting of all nonempty and bounded sets X such that functions from X are equicontinuous and nondecreasing on $\mathbb{R}^+$. For other properties of μ, see [5].

3 Main result

We will use the following fixed point theorem.

Theorem 3.1. *[4] Let $\mathcal{Q}$ be nonempty bounded closed convex subset of the space E and let $F : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a continuous operator such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of $\mathcal{Q}$, where $k \in [0, 1)$ is a constant. Then F has a fixed point in the set $\mathcal{Q}$.*

Equation (1.1) will be studied under the following assumptions:

(i) The functions $p, q : \mathbb{R}^+ \to \mathbb{R}$ are continuous and bounded functions on $\mathbb{R}^+$. Let $\|p\|$ be the norm of p in $BC(\mathbb{R}^+, \mathbb{R})$ and $\|q\|$ be the norm of q in $BC(\mathbb{R}^+, \mathbb{R})$.

(ii) The function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is Lipschizian with respect to the second variable with a Lipschitz constant α, that is, $|f(t, x) - f(t, y)| \leq \alpha|x - y|$ for all $t \in \mathbb{R}^+$ and all $x, y \in \mathbb{R}$. Let $\beta(t) = |f(t, 0)| \in BC(\mathbb{R}^+, \mathbb{R})$.

(iii) The function $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a positive constant b_1 and a function $a_1 \in BC(\mathbb{R}^+, \mathbb{R})$ such that $|u(t,s,x)| \leq k_1(t,s)\,[a_1(s) + b_1\,|x|]$ for $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, where $k_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is measurable function and the linear Volterra operator K_1 generated by k_1,

$$(K_1x)\,(t) = \int_0^t k_1(t,s)x(s)ds, \tag{3.1}$$

transforms the space $BC(\mathbb{R}^+, \mathbb{R})$ into itself. Let $\|K_1\|$ be the norm of this operator.

(iv) The function $v : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $a_2 \in BC(\mathbb{R}^+, \mathbb{R})$ such that $|v(t,s,x)| \leq k_2(t,s)a_2(s)$ for $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, where $k_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is measurable function and the linear Volterra operator K_2 generated by k_2,

$$(K_2x)\,(t) = \int_0^t k_2(t,s)x(s)ds, \tag{3.2}$$

transforms the space $BC(\mathbb{R}^+, \mathbb{R})$ into itself. Let $\|K_2\|$ be the norm of this operator.

(v) $\lim\limits_{t \longrightarrow +\infty} (K_i1)\,(t) = \lim\limits_{t \longrightarrow +\infty} \int_0^t k_i(t,s)ds = 0$, for $i = 1, 2$.

(vi) $\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|) < 1$.

To prove our main result, we need the following lemma.

Lemma 3.2. *Under the assumptions (i)-(v) the operators*

$$(Ux)(t) = p(t) + \int_0^t u(t,s,x(s))ds,$$
$$(Vx)(t) = q(t) + \int_0^t v(t,s,x(s))ds.$$

map $BC(\mathbb{R}^+, \mathbb{R})$ continuously into itself.

Proof. We prove only that U maps $BC(\mathbb{R}^+, \mathbb{R})$ continuously into itself and the proof of V is similarly.
It is clear that the operator U maps $BC(\mathbb{R}^+, \mathbb{R})$ into $C(\mathbb{R}^+, \mathbb{R})$. Moreover, let $x \in BC(\mathbb{R}^+, \mathbb{R})$, since

$$|\,(K_1x)\,(t)| \leq \|x\|\,(K_11)\,(t).$$

On the other hand, from the assumption (v), there exists $T > 0$ such for all $t \geq T$

$$(K_11)\,(t) \leq 1.$$

Hence, from the assumption (iii), we have for all $t \geq T$

$$|(Ux)(t)| \leq (\|a_1\| + b_1\|x\|)\,(K_11)\,(t) \leq \|a_1\| + b_1\|x\|.$$

On the other hand, (Ux) is bounded on $[0, T]$, we deduce that U maps $BC(\mathbb{R}^+, \mathbb{R})$ into itself.
Now, to prove that U is continuous, let $\{x_n\}$ be an arbitrary sequence in $BC(\mathbb{R}^+, \mathbb{R})$ which converges to $x \in BC(\mathbb{R}^+, \mathbb{R})$.
Then, for $\varepsilon > 0$ there exist $n_1 \in \mathbb{N}$ and $T > 0$, such that for all $n \geq n_1$ and $t \geq T$, we have

$$\|x_n\| \leq 1 + \|x\|, (K_11)\,(t) \leq \frac{\varepsilon}{2\|a_1\| + b_1(2\|x\| + 1)}.$$

It follows that, for $n \geq n_1$ and $t \geq T$, we have

$$|(Ux_n - Ux)(t)| \leq (2\|a_1\| + b_1(2\|x\| + 1))\,(K_1 1)\,(t) \leq \varepsilon. \tag{3.3}$$

On the other hand, since u is uniformly bounded on the compact set $[0, T] \times [0, T] \times [-1 - \|x\|, 1 + \|x\|]$, hence there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$\sup\{|u(t, s, x_n(s)) - u(t, s, x(s))|, (t, s) \in [0, T] \times [0, T], n \geq n_2\} \leq \frac{\varepsilon}{T},$$

which implies that, for all $n \geq n_2$ and $t \in [0, T]$

$$|(Ux_n - Ux)(t)| \leq \varepsilon. \tag{3.4}$$

Then, from (3.3) and (3.4), we deduce that, for all $n \geq n_0 = \max(n_1, n_2)$

$$\|Ux_n - Ux\| \leq \varepsilon.$$

Thus, U maps $BC(\mathbb{R}^+, \mathbb{R})$ continuously into itself. □

Remark 3.3. [7] The concept of the asymptotic stability of a solution $x = x(t)$ of Eq. (1.1) is understood in the following sense.
For any $\varepsilon > 0$ there exist $T > 0$ and $r > 0$ such that if $x = x(t), y = y(t)$ are solutions of (1.1) then $|x(t) - y(t)| \leq \varepsilon$ for $t \geq T$.

Now we are able to state our main result.

Theorem 3.4. *Under the assumptions above the nonlinear integral equation (1.1) has at least an asymptotically stable solution $x \in BC(\mathbb{R}^+, \mathbb{R})$.*

Proof. Solving Eq. (1.1) is equivalent to finding a fixed point of the operator A, where $Ax(t) = f(t, x(t)) + (Ux)(t) \times (Vx)(t)$. We will show that A satisfies the conditions of Theorem 3.1. The proof is split into four steps.
Step 1. We first show that there exists B_{r_0} from $BC(\mathbb{R}^+, \mathbb{R})$ such that $A(B_{r_0}) \subset B_{r_0}$. To see this, let $x \in B_r$. Then

$$\begin{aligned}
\|Ax\| \leq& \|f(t, x(t))\| + \|(Ux)(t) \times (Vx)(t)\| \\
\leq& \alpha\|x\| + \|\beta\| + (\|p\| + \|K_1(a_1 + b_1 x)\|) \times (\|q\| + \|K_2(a_2)\|) \\
\leq& \alpha\|x\| + \|\beta\| + (\|p\| + \|K_1\|(\|a_1\| + b_1\|x\|)) \times (\|q\| + \|K_2\|\|a_2\|) \\
\leq& \alpha r + \|\beta\| + (\|p\| + \|K_1\|\|a_1\| + b_1\|K_1\|r) \times (\|q\| + \|K_2\|\|a_2\|) \\
\leq& (\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|))r + \|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|).
\end{aligned}$$

Since $\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|) < 1$, we deduce that the operator A transforms the ball B_{r_0} into itself for $r_0 = \frac{\|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|)}{1 - (\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|))}$.
Step 2. The operator A maps B_{r_0} continuously into itself. To see this, take an arbitrary number $\epsilon > 0$ and a convergent sequence (x_n) to (x) in B_{r_0}.
Hence, by Lemma 3.2, there exists n_0 such that for all $n \geq n_0$, we have

$$\begin{aligned}
\|x_n - x\| \leq& \frac{\epsilon}{3\alpha}, \|Ux_n - Ux\| \leq \frac{\epsilon}{3(\|q\| + \|K_2\|\|a_2\|)}, \\
\|Vx_n - Vx\| \leq& \frac{\epsilon}{3(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))}.
\end{aligned}$$

Which implies, for all $n \geq n_0$,

$$\begin{aligned}\|Ax_n - Ax\| \leq & \alpha\|x_n - x\| + \|(Ux_n) \times (Vx_n) - (Ux) \times (Vx)\| \\ \leq & \alpha\|x_n - x\| + \|Ux_n\|\|Vx_n - Vx\| + \|Vx\|\|Ux_n - Ux\| \\ \leq & \alpha\|x_n - x\| + (\|p\| + \|K_1\|(\|a_1\| + b_1\|x_n\|))\|Vx_n - Vx\| \\ & + (\|q\| + \|K_2\|\|a_2\|)\|Ux_n - Ux\| \\ \leq & \alpha\|x_n - x\| + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|Vx_n - Vx\| \\ & + (\|q\| + \|K_2\|\|a_2\|)\|Ux_n - Ux\| \\ \leq & \epsilon.\end{aligned}$$

We deduce that the operator A maps B_{r_0} continuously into itself.
Step 3. We illustrate that there exists $\gamma \in [0, 1)$ such that $\mu(AX) \leq \gamma\mu(X)$ for all subset X of B_{r_0}. To see this, take an arbitrary number $t \geq 0$. Then for any $x, y \in X$, we have

$$\begin{aligned}|Ax(t) - Ay(t)| \leq & \alpha|x(t) - y(t)| + |Ux(t)||Vx(t) - Vy(t)| + |Vy(t)||Ux(t) - Uy(t)| \\ \leq & \alpha\|x(t) - y(t)\| + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))|Vx(t) - Vy(t)| \\ & + (\|q\| + \|K_2\|\|a_2\|)|Ux(t) - Uy(t)| \\ \leq & \alpha\|x(t) - y(t)\| + 2(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|a_2\|K_2 1(t) \\ & + 2(\|q\| + \|K_2\|\|a_2\|)(\|a_1\| + b_1 r_0))K_1 1(t).\end{aligned}$$

Which implies that

$$\begin{aligned}d(AX(t)) \leq & \alpha d(X(t)) + 2(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|a_2\|K_2 1(t) \\ & + 2(\|q\| + \|K_2\|\|a_2\|)(\|a_1\| + b_1 r_0))K_1 1(t).\end{aligned}$$

Now, taking into account the assumption (v) we obtain the following estimate:

$$d(AX) \leq \alpha d(X). \tag{3.5}$$

Further, let us fix arbitrarily numbers $T > 0$, $\varepsilon > 0$, let $x \in X$ and take $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$.
Then, in view of our assumptions, we have

$$\begin{aligned}|Ax(t_2) - Ax(t_1)| \leq & \alpha|x(t_2) - x(t_1)| + |Ux(t_2)||Vx(t_2) - Vx(t_1)| \\ & + |Vx(t_1)||Ux(t_2) - Ux(t_1)| \\ \leq & \alpha|x(t_2) - x(t_1)| + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))|Vx(t_2) - Vx(t_1)| \\ & + (\|q\| + \|K_2\|\|a_2\|)|Ux(t_2) - Ux(t_1)|.\end{aligned} \tag{3.6}$$

Now, from the assumption *(iii)*, we have

$$\begin{aligned}|Ux(t_2) - Ux(t_1)| \leq & \int_0^{t_2} |u(t_2, s, x(s)) - u(t_1, s, x(s))|ds \\ & + \int_{t_1}^{t_2} |u(t_1, s, x(s))|ds \\ \leq & T\overline{\omega}^T(u, \varepsilon) + |t_2 - t_1|\overline{u} \\ \leq & T\overline{\omega}^T(u, \varepsilon) + \varepsilon\overline{u},\end{aligned} \tag{3.7}$$

where,

$$\begin{aligned}\overline{\omega}^T(u,\varepsilon) &= \sup\{|u(t_2,s,x)-u(t_1,s,x)|, t_1,t_2,s\in[0,T], |t_2-t_1|\le\varepsilon, |x|\le r_0\},\\ \overline{u} &= \sup\{|u(t,s,x)|, t,s\in[0,T], |x|\le r_0\}.\end{aligned}$$

Similarly, from the assumption $(i\mathrm{v})$, we obtain

$$|Vx(t_2)-Vx(t_1)|\le T\overline{\omega}^T(v,\varepsilon)+\varepsilon\overline{v}, \tag{3.8}$$

where,

$$\begin{aligned}\overline{\omega}^T(v,\varepsilon) &= \sup\{|v(t_2,s,x)-v(t_1,s,x)|, t_1,t_2,s\in[0,T], |t_2-t_1|\le\varepsilon, |x|\le r_0\},\\ \overline{v}^T &= \sup\{|v(t,s,x)|, t,s\in[0,T], |x|\le r_0\}.\end{aligned}$$

Hence, from (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}\omega^T(Ax,\varepsilon)\le&\alpha\omega^T(x,\varepsilon)+(\|p\|+\|K_1\|(\|a_1\|+b_1r_0))(T\overline{\omega}^T(v,\varepsilon)+\varepsilon\overline{v})\\ &+(\|q\|+\|K_2\|\|a_2\|)(T\overline{\omega}^T(u,\varepsilon)+\varepsilon\overline{u}).\end{aligned}$$

Since $\lim\limits_{\varepsilon\longrightarrow 0}\overline{\omega}^T(u,\varepsilon)=\lim\limits_{\varepsilon\longrightarrow 0}\overline{\omega}^T(v,\varepsilon)=0$, then

$$\omega_0(AX)\le\alpha\omega_0(X). \tag{3.9}$$

We deduce, from (3.5) and (3.9), that

$$\mu(AX)\le\alpha\mu(X).$$

Hence the third step is completed by taking $\gamma=\alpha<1$.
Finally, using Theorem 3.1, we can see that (1.1) has at least one solution $x\in BC(\mathbb{R}^+,\mathbb{R})$.
Step 4. The solution x is asymptotically stable on $\mathbb{R}^+$.
Let $\varepsilon>0$, and taking $r=r_0$, then, for any other solution $y\in B_{r_0}$, we have from Step 3

$$\begin{aligned}|Ax(t)-Ay(t)|\le&\alpha\|x(t)-y(t)\|+2(\|p\|+\|K_1\|(\|a_1\|+b_1r_0))\|a_2\|K_21(t)\\ &+2(\|q\|+\|K_2\|\|a_2\|)(\|a_1\|+b_1r_0))K_11(t).\end{aligned}$$

Since $\alpha<1$, we obtain

$$\begin{aligned}|Ax(t)-Ay(t)|\le&\frac{2(\|p\|+\|K_1\|(\|a_1\|+b_1r_0))\|a_2\|}{1-\alpha}K_21(t)\\ &+\frac{2(\|q\|+\|K_2\|\|a_2\|)(\|a_1\|+b_1r_0)}{1-\alpha}K_11(t).\end{aligned}$$

By using Assumption (v), we deduce that there exists $T>0$ such that for all $t\ge T$

$$|Ax(t)-Ay(t)|\le\varepsilon.$$

Which implies that the solution is asymptotically stable on $\mathbb{R}^+$. □

4 Example

Consider the following integral equation

$$x(t) = t\exp(-t) + 1 + \frac{1}{2}x(t) + \left(\frac{1}{5+t} + \int_0^t \frac{\cos(s+t)}{t+\lambda}\ln(1+x^2(s))ds\right) \times \left(\exp(-t) + \int_0^t \frac{\sin(t)}{(1+2t-s+|x(s)|)^2}ds\right), \tag{4.1}$$

where $t \in \mathbb{R}^+$ and λ is a positive number.
Set

$$f(t,x) = t\exp(-t) + 1 + \frac{1}{2}x, p(t) = \frac{1}{5+t}, q(t) = \exp(-t), k_1(t,s) = \frac{|\cos(s+t)|}{t+\lambda},$$

and

$$k_2(t,s) = \frac{1}{(1+2t-s)^2}, a_1(s) = 0, b_1 = 1, a_2(s) = |\sin(t)|.$$

Using the notations of Theorem 3.4, we can easily show that

$$\alpha = \frac{1}{2}, \|p\| = \frac{1}{5}, \|q\| = \|a_2\| = 1, K_1 1(t) \leq \frac{2}{t+\lambda}, K_2 1(t) \leq \frac{1}{1+t},$$

and

$$\|K_1\| \leq \frac{2}{\lambda}, \|K_2\| \leq 1.$$

Then the assumption (v) is satisfied, therefore, the inequality (vi) takes the form

$$\frac{1}{2} + \frac{4}{\lambda} < 1 \Longleftrightarrow \lambda > 8.$$

Then by Theorem 3.4, we conclude that the integral equation (4.1) has an asymptotically stable solution $x \in BC(\mathbb{R}^+, \mathbb{R})$ whenever $\lambda > 8$.

5 Conclusion

In this paper, we have considered a general form of integral equations of product type on the half-axis. The existence of a continuous solution and its asymptotic stability have been investigated using the measures of non-compactness and Darbo's fixed point theorem. Finally, an example is provided to illustrate our main result.

Conflict of Interest

The authors have no conflicts of interest to disclose.

References

[1] A. Abdeldaim, *On some new Gronwall-Bellman-Ou-Iang type integral inequalities to study certain epidemic models*, J. Integral Equations Appl. **24** (2012), 149–166.

[2] A. Ardjouni and A. Djoudi, *Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle*, Ural Math. J. **5** (2019), 3–12.

[3] N. T. J. Bailey, *The mathematical theory of infectious diseases and its applications*, Hafner, New York, 1975.

[4] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, 1980.

[5] J. Banas and L. Olszowy, *Measures of noncompactness related to monotonicity*, Comment. Math. **41** (2001), 13–23.

[6] J. Banas and J. Rivero, *On measures of weak noncompactness*, Ann. Mat. Pura Appl. **151** (1988), 213–224.

[7] J. Banas and B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl. **284**(1) (2003), 165–173.

[8] A. Bellour, M. Bousselsal and M. A. Taoudi, *Integrable solutions of a nonlinear integral equation related to some epidemic models*, Glas. Mat. **49**(2) (2014), 395–406.

[9] B. Boulfoul, A. Bellour and S. Djebali, *Solvability of nonlinear integral equations of product type*, Electron. J. Differ. Equ. **19** (2018), 1–20.

[10] O. Diekmann, *A note on the asymptotic speed of propagation of an epidemic*, J. Differ. Equations **33**(1) (1979), 58–73.

[11] G. Gripenberg, *Periodic solutions of an epidemic model*, J. Math. Biol. **10** (1980), 271–280.

[12] G. Gripenberg, *On some epidemic models*, Quart. Appl. Math. **39**(3) (1981), 317–327.

[13] L. Li, F. Meng and P. Ju, *Some new integral inequalities and their applications in studying the stability of nonlinear integro-differential equations with time delay*, J. Math. Anal. Appl. **377** (2011), 853–862.

[14] I. M. Olaru, *Generalization of an integral equation related to some epidemic models*, Carpathain J. Math. **26**(1) (2010), 92–96.

[15] B. G. Pachpatte, *On a new inequality suggested by the study of certain epidemic models*, J. Math. Anal. Appl. **195**(1) (1995), 638–644.

[16] P. Waltman, *Deterministic threshold models in the theory of epidemics*, Lecture Notes in Biomathematics, vol. 1, Springer-Verlag, New York, 1974.