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Measure of noncompactness for nonlinear Hilfer fractional differential equation with mixed fractional integral boundary conditions in Banach space

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Abstract. The aim of this work is to study the existence of solutions to a class of fractional differential equations with a mixed of fractional integral boundary conditions involving the Hilfer fractional derivative. The proof is based on Mönch's fixed point theorem and the technique of measures of noncompactness. Two examples illustrating the main results are also constructed.

Keywords: Fractional differential equation, Hilfer fractional derivative, Erdélyi-Kober fractional integral, Katugampola fractional integral, Mönch fixed point theorems, Kuratowski measure of noncompactness.
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1 Introduction

In recent years, the extensive study of fractional differential equations owes to widespread applications of the subject in engineering and technical sciences. Crucial phenomena in economics, physics, chemical technology, biology, control theory, signal and image processing, electromagnetics, acoustics, viscoelasticity, and material science are well described by differential equations of fractional order. In contrast to classical differential and integral operators, fractional-order operators are nonlocal and account for the memory and hereditary properties of many phenomena and processes in nature. Much of the literature on fractional-order boundary problems involves classical Riemann-Liouville or Hadamard type integral boundary conditions. Besides the conditions mentioned above, there are other types of integral boundary conditions which contain Erdélyi-Kober and Katugampola fractional integral operators (introduced by Arthur Erdélyi and Hermann Kober [16] in 1940 and U.N. Katugampola [21] in 2011). Such operators play an important role in solving single, dual, and triple integral equations possessing special functions of mathematical physics in their kernels. For details and applications of the Erdélyi-Kober and Katugampola fractional integrals, for instance, see [3,4,24]. The main reason for the success of fractional calculus can be shown by the accuracy of new fractional-order models than integer-order ones. One of the most important

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fractional derivatives (and integrals) properties is the nonlocal property. Most fractional operators consider the entire history of the being considered process, thus being able to model the nonlocal and distributed effects often encountered in natural and technical phenomena. For recent developments and various applications of different fractional operators in the fractional modeling, we refer to [1,6,7,9–12,14,15,25,26,30,32] and the references therein.

In [4] authors gave sufficient criteria for the existence of solutions for the following Caputo fractional differential equation:

$$\mathbb{D}^q x(\tau) = \mathbb{F}(\tau, x(\tau)), \qquad \tau \in (0, \mathrm{T}),$$

subject to nonlocal generalized Riemann-Liouville (Katugampola) fractional integral boundary conditions of the form

$$x(0) = \gamma \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\xi} \frac{s^{\rho-1} x(s)}{(\xi^\rho - s^\rho)^{1-\alpha}} ds := \gamma\, {}^\rho\mathbb{I}^\alpha x(\xi),$$
$$x(\mathrm{T}) = \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\varepsilon} \frac{s^{\rho-1} x(s)}{(\varepsilon^\rho - s^\rho)^{1-\alpha}} ds := \delta\, {}^\rho\mathbb{I}^\beta x(\varepsilon), 0 < \xi, \varepsilon < \mathrm{T},$$

where $\mathbb{D}^q$ denotes the Caputo fractional derivative of order q, ${}^\rho\mathbb{I}^z$, $z \in \{\alpha, \beta\}$, is the generalized Riemann-Liouville fractional integral of order $z > 0$, $\rho > 0$, ξ and ε are arbitrary real constants, with $\xi,\ \varepsilon \in (0, \mathrm{T})$, $\gamma,\ \delta \in \mathbb{R}$ and $\mathbb{F} : [0, \mathrm{T}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [33] authors established the existence of solutions for the following nonlinear Riemann-Liouville fractional differential equation subject to nonlocal Erdelyi-Kober fractional integral conditions:

$$\mathbb{D}^q x(\tau) = \mathbb{F}(\tau, x(\tau)), \qquad \tau \in (0, \mathrm{T}),$$
$$x(0) = 0, \quad \alpha x(\mathrm{T}) = \sum_{i=1}^{m} \beta_i \mathbb{I}_{\eta_i}^{\gamma_i, \delta_i} x(\xi_i),$$

where $1 < q \leq 2$, $\mathbb{D}^q$ is the standard Riemann-Liouville fractional derivative of order q, $\mathbb{I}_{\eta_i}^{\gamma_i, \delta_i}$ is the Erdelyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2, ..., m$, $\mathbb{F} : [0, \mathrm{T}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha,\ \beta_i \in \mathbb{R},\ \xi_i \in (0, \mathrm{T}), i = 1, 2, ..., m$, are given constants.

In [3], authors studied a new class of boundary value problems of Caputo fractional differential equations:

$${}^c\mathbb{D}^q x(\tau) = \mathbb{F}(\tau, x(\tau)), \qquad \tau \in [0, \mathrm{T}],$$

supplemented with Riemann-Liouville and Erdelyi-Kober fractional integral boundary conditions at the left and right end points of the interval $[0, \mathrm{T}]$ respectively, that is,

$$x(0) = \alpha \frac{1}{\Gamma(p)} \int_0^{\zeta} (\zeta - s)^{p-1} ds := \alpha \mathbb{I}^p x(\zeta),$$
$$x(\mathrm{T}) = \beta \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^{\xi} \frac{s^{\eta\gamma+\eta-1} x(s)}{(\xi^\eta - s^\eta)^{1-\delta}} ds := \beta \mathbb{I}_\eta^{\gamma,\delta} x(\xi), 0 < \xi, \zeta < \mathrm{T},$$

where ${}^c\mathbb{D}^q$ is the Caputo fractional derivative of order $1 < q \leq 2$, $\mathbb{F} : [0, \mathrm{T}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathbb{I}^p$ denotes Riemann-Liouville fractional integral of order $p > 0$, and $\mathbb{I}_\eta^{\gamma,\delta}$ denotes Erdélyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$, $\gamma \in \mathbb{R}$.

In [13] authors gave the sufficient conditions of existence solutions of nonlocal boundary conditions for the following nonlinear fractional differential equation of order $\alpha \in]2, 3]$:

$${}^c\mathbb{D}^q x(\tau) = \mathbb{F}(\tau, x(\tau)), \qquad \tau \in [0, \mathrm{T}],$$

subject to nonlocal Erdélyi-Kober fractional integral boundary conditions of the form:

$$\begin{aligned} x(T) &= \sum_{i=1}^{m} a_i J_{\eta_i}^{\gamma_i,\delta_i} x(\beta_i), \quad 0<\beta_i<T, \\ x'(T) &= \sum_{i=1}^{m} b_i J_{\eta_i}^{\gamma_i,\delta_i} x'(\sigma_i), \quad 0<\sigma_i<T, \\ x''(T) &= \sum_{i=1}^{m} d_i J_{\eta_i}^{\gamma_i,\delta_i} x''(\varepsilon_i), \quad 0<\varepsilon_i<T, \end{aligned} \tag{1.1}$$

where $\mathbb{D}^{\alpha}$ is the Caputo fractional derivative of order $2<\alpha\leq 3$ and $\mathbb{I}_{\eta_i}^{\gamma_i,\delta_i}$ denotes Erdélyi-Kober fractional integral of order $\delta_i>0$, $\eta_i>0$, $\gamma_i\in\mathbb{R}$, $\mathbb{F}:[0,\mathrm{T}]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Motivated by the studies above, in this paper, we concentrate on the following boundary value problem of nonlinear Hilfer fractional differential equation

$$\mathbb{D}_{0^+}^{\alpha,\beta} x(\tau) = \mathbb{F}(\tau, x(\tau)), \qquad \text{for a.e.} \qquad \tau \in \mathbb{J} := [0,\mathrm{T}], \tag{1.2}$$

with the mixed fractional integral boundary conditions:

$$\begin{aligned} \mathbb{I}^{1-\gamma} x(\mathrm{T}) &= \lambda_1 \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{\varepsilon} \frac{s^{\rho-1} x(s)}{(\varepsilon^{\rho}-s^{\rho})^{1-q}} ds := \lambda_1\, {}^{\rho}\mathbb{I}^{q} x(\varepsilon), \\ \mathbb{I}^{2-\gamma} x'(\mathrm{T}) &= \lambda_2 \frac{\eta \xi^{-\eta(\delta+\gamma_1)}}{\Gamma(\delta)} \int_0^{\xi} \frac{s^{\eta\gamma_1+\eta-2} x(s)}{(\xi^{\eta}-s^{\eta})^{2-\delta}} ds := \lambda_2 \mathbb{I}_{\eta}^{\gamma_1,\delta} x'(\xi), 0<\xi,\varepsilon<\mathrm{T}, \end{aligned} \tag{1.3}$$

where $\mathbb{D}_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative such that $0<\alpha<1,\ 0\leq\beta\leq 1$, $\gamma=\alpha+\beta-\alpha\beta$, ${}^{\rho}\mathbb{I}^{q}$ is the Katugampola integral of $q>0$ and $\mathbb{I}_{\eta}^{\gamma,\delta}$ denote Erdélyi-Kober fractional integral of order $\delta>0$, $\eta>0$, $\gamma_1\in\mathbb{R}$, and let $\mathbb{E}$ be a reflexive Banach space with norm $\|.\|$, $\mathbb{F}:\mathbb{J}\times\mathbb{E}\to\mathbb{E}$ is a continuous function, $\lambda_i, i=1,2$ are real constants.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and introduce the integral operator associated with the given problem. In Section 3, existence results, which rely on Mönchs fixed point theorem and its related Kuratowski measure of noncompactness, are presented. Finally, in Section 4, we provide two examples to show the applicability of our main results.

2 Preliminaries and lemmas

In what follows, we introduce definitions, notations, and preliminary facts used in the sequel. For more details, we refer to [5,17,22,27,31].
Let $\mathbb{E}$, be a Banach space. Denote by $\mathbb{C}(\mathbb{J},\mathbb{E})$ the Banach space of continuous functions $\varpi:\mathbb{J}\to\mathbb{E}$, with the usual supremum norm

$$\|\varpi\|_{\infty} = \sup\Big\{\|\varpi(\tau)\|, \tau\in\mathbb{J}\Big\}.$$

Let $L^1(\mathbb{J},\mathbb{E})$ be the Banach space of measurable functions $\varpi:\mathbb{J}\to\mathbb{E}$ which are Bochner integrable, equipped with the norm

$$\|\varpi\|_{L^1} = \int_{\mathbb{J}} \|\varpi(\tau)\| dt.$$

$AC^1(\mathbb{J},\mathbb{E})$ denotes the space of functions $\varpi:\mathbb{J}\to\mathbb{E}$, whose first derivative is absolutely continuous.

Definition 2.1. Let $\mathbb{J} = [0, T]$ be a finite interval and $0 \leq \gamma < 1$, we introduce the weighted space $\mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E})$ of continuous functions ϖ on $(0, T]$

$$\mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E}) = \Big\{\varpi : (0, T] \to \mathbb{E} : (\tau - a)^{1-\gamma}\varpi(\tau) \in \mathbb{C}(\mathbb{J}, \mathbb{E})\Big\}.$$

In the space $\mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E})$, we define the norm

$$\|\varpi\|_{\mathbb{C}_{1-\gamma}} = \|(\tau - a)^{1-\gamma}\varpi(\tau)\|_{\mathbb{C}}.$$

Definition 2.2. Let $0 < \alpha < 1,\ 0 \leq \beta \leq 1$, the weighted space $\mathbb{C}^{\alpha,\beta}_{1-\gamma}(\mathbb{J}, \mathbb{E})$ is defined by

$$\mathbb{C}^{\alpha,\beta}_{1-\gamma}(\mathbb{J}, \mathbb{E}) = \Big\{\varpi : (0, T] \to \mathbb{R} : \mathbb{D}^{\alpha,\beta}_{0^+}\varpi \in \mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E})\Big\}, \gamma = \alpha + \beta - \alpha\beta,$$

and

$$\mathbb{C}^{1}_{1-\gamma}(\mathbb{J}, \mathbb{E}) = \Big\{\varpi : (0, T] \to \mathbb{R} : \varpi' \in \mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E})\Big\}, \gamma = \alpha + \beta - \alpha\beta,$$

with the norm

$$\|\varpi\|_{\mathbb{C}^{1}_{1-\gamma}} = \|\varpi\|_{\mathbb{C}} + \|\varpi'\|_{\mathbb{C}_{1-\gamma}}. \tag{2.1}$$

Moreover, $\mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E})$ is complete metric space of all continuous functions mapping $\mathbb{J}$ into $\mathbb{E}$ with the metric d defined by

$$d(\varpi_1, \varpi_2) = \Big\|\varpi_1 - \varpi_2\Big\|_{\mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{E})} := \max_{\tau\in\mathbb{J}} \Big|(\tau - a)^{1-\gamma}\Big[\varpi_1(\tau) - \varpi_2(\tau)\Big]\Big|.$$

Now, we give some results and properties of fractional calculus.

Definition 2.3. ([22]) The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a continuous function $\varpi : (0, \infty) \to \mathbb{R}$ is defined by

$$\mathbb{I}^{\alpha}_{0^+}\varpi(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau - s)^{\alpha-1}\varpi(s)ds, \quad \tau > 0, \tag{2.2}$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler's Gamma function.

Definition 2.4. ([22]) The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}^+$ of a continuous function $\varpi : (0, \infty) \to \mathbb{R}$ is defined by

$$\mathbb{D}^{\alpha}_{0^+}\varpi(\tau) = \frac{1}{\Gamma(n - \alpha)}\frac{d^n}{dt^n}\int_0^{\tau}(\tau - s)^{n-\alpha-1}\varpi(s)ds, \quad n - 1 < \alpha < n, \tag{2.3}$$

where $n = [\alpha] + 1$, and $[\alpha]$ means the integral part of α, provided the right hand side is point-wise defined on $(0, \infty)$.

Definition 2.5. ([22]) The Caputo derivative of order α for a function $\varpi : [0, \infty) \to \mathbb{R}$ can be written as

$${}^{C}\mathbb{D}^{\alpha}\varpi(\tau) = \mathbb{D}^{\alpha}_{0^+}\left(\varpi(\tau) - \sum_{k=0}^{n-1}\frac{\tau^k}{k!}\varpi^{(k)}(0)\right), \quad \tau > 0,\ n - 1 < \alpha < n. \tag{2.4}$$

Remark 2.6. If $\varpi(\tau) \in \mathbb{C}^n[0,\infty)$, then

$$\begin{aligned} {}^{C}\mathbb{D}^{\alpha}\varpi(\tau) &= \frac{1}{\Gamma(n-\alpha)}\int_0^{\tau}(\tau-s)^{n-\alpha-1}\varpi^{(n)}(s)ds \\ &= \mathbb{I}^{n-p}\varpi^{(n)}(\tau),\ \tau>0,\ n-1<\alpha<n. \end{aligned} \tag{2.5}$$

Definition 2.7. ([21]) Katugampola integral of order $q>0$ and $\rho>0$, of a function $\varpi(\tau)$, for all $0<\tau<\infty$, is defined as

$${}^{\rho}\mathbb{I}^{q}\varpi(\tau) = \frac{\rho^{1-q}}{\Gamma(q)}\int_0^{\tau}\frac{s^{\rho-1}\varpi(s)}{(\tau^{\rho}-s^{\rho})^{1-q}}ds, \tag{2.6}$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Remark 2.8. ([21]) The above definition corresponds to the one for Riemann-Liouville fractional integral of order $q>0$ when $\rho=1$, while the famous Hadamard fractional integral follows for $\rho\to 0$, that is,

$$\lim_{\rho\to 0}{}^{\rho}\mathbb{I}^{q}\varpi(\tau) = \frac{1}{\Gamma(q)}\int_0^{\tau}\left(\log\frac{\tau}{s}\right)^{q-1}\frac{\varpi(s)}{s}ds. \tag{2.7}$$

Lemma 2.9. *([4]) Let $\rho, q>0$ and $p>0$ be the given constants. Then the following formula holds:*

$${}^{\rho}\mathbb{I}^{q}\tau^{p} = \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})}\frac{\tau^{p+\rho q}}{\rho^{q}}. \tag{2.8}$$

Definition 2.10. ([16]) The Erdélyi-Kober fractional integral of order $\delta>0$ with $\eta>0$ and $\gamma\in\mathbb{R}$ of a continuous function $\varpi:(0,\infty)\to\mathbb{R}$ is defined by

$$\mathbb{I}_{\eta}^{\gamma,\delta}\varpi(\tau) = \frac{\eta\tau^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^{\tau}\frac{s^{\eta\gamma+\eta-1}\varpi(s)}{(\tau^{\eta}-s^{\eta})^{1-\delta}}ds, \tag{2.9}$$

provided the right side is point-wise defined on $\mathbb{R}^{+}$.

Remark 2.11. For $\eta=1$ the above operator is reduced to the Kober operator

$$\mathbb{I}_{1}^{\gamma,\delta}\varpi(\tau) = \frac{\tau^{-(\delta+\gamma)}}{\Gamma(\delta)}\int_0^{\tau}\frac{s^{\gamma}\varpi(s)}{(\tau-s)^{1-\delta}}ds,\ \ \gamma,\ \delta>0, \tag{2.10}$$

that was introduced for the first time by Kober in [23]. For $\gamma=0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight:

$$\mathbb{I}_{1}^{0,\delta}\varpi(\tau) = \frac{\tau^{-\delta}}{\Gamma(\delta)}\int_0^{\tau}\frac{\varpi(s)}{(\tau-s)^{1-\delta}}ds,\ \delta>0. \tag{2.11}$$

Lemma 2.12. *Let $\delta,\ \eta>0$ and $\gamma,\ q\in\mathbb{R}$. Then we have*

$$\mathbb{I}_{\eta}^{\gamma,\delta}\tau^{q} = \frac{\tau^{q}\Gamma(\gamma+(q/\eta)+1)}{\Gamma(\gamma+(q/\eta)+\delta+1)}. \tag{2.12}$$

In [17], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [18–20]).

Definition 2.13. ([17]) The Hilfer fractional derivative $\mathbb{D}_{0^+}^{\alpha,\beta}$ of order $\alpha \ (n-1<\alpha<n)$ and type $\beta \ (0\leq\beta\leq 1)$ is defined by

$$\mathbb{D}_{0^+}^{\alpha,\beta}=\mathbb{I}_{0^+}^{\beta(n-\alpha)}\mathbb{D}^{n}\mathbb{I}_{0^+}^{(1-\beta)(n-\alpha)}\varpi(\tau), \tag{2.13}$$

where $\mathbb{I}_{0^+}^{\alpha}$ and $\mathbb{D}_{0^+}^{\alpha}$ are Riemann-Liouville fractional integral and derivative defined by (2.2) and (2.3), respectively.

Remark 2.14. ([17]) Hilfer fractional derivative interpolates between the Riemann-Liouville ((2.3), if $\beta=0$) and Caputo ((2.4), if $\beta=1$) fractional derivatives since

$$\mathbb{D}_{0^+}^{\alpha,0}={}^{R-L}\,\mathbb{D}_{0^+}^{\alpha} \text{ and } \mathbb{D}^{\alpha,1}={}^{C}\,\mathbb{D}_{0^+}^{\alpha}.$$

Lemma 2.15. *([17]) Let $0<\alpha<1, 0\leq\beta\leq 1, \gamma=\alpha+\beta-\alpha\beta$, and $\varpi\in L^1(\mathbb{J},\mathbb{E})$. The operator $\mathbb{D}_{0^+}^{\alpha,\beta}$ can be written as*

$$\mathbb{D}_{0^+}^{\alpha,\beta}\varpi(\tau)=\left(\mathbb{I}_{0^+}^{\beta(1-\alpha)}\frac{d}{dt}\mathbb{I}_{0^+}^{(1-\gamma)}\varpi\right)(\tau)=\mathbb{I}_{0^+}^{\beta(1-\alpha)}\mathbb{D}^{\gamma}\varpi(\tau),\ \textit{for a.e. } \tau\in\mathbb{J}.$$

Moreover, the parameter γ satisfies

$$0<\gamma\leq 1,\ \ \gamma\geq\alpha,\ \ \gamma<\beta,\ 1-\gamma<1-\beta(1-\alpha).$$

Lemma 2.16. *([17]) Let $0<\alpha<1,\ \ 0\leq\beta\leq 1,\ \ \gamma=\alpha+\beta-\alpha\beta$, If $\mathbb{D}_{0^+}^{\beta(1-\alpha)}\varpi$ exists and in $L^1(\mathbb{J},\mathbb{E})$, then*

$$\mathbb{D}_{0^+}^{\alpha,\beta}\mathbb{I}_{0^+}^{\alpha}\varpi(\tau)=\mathbb{I}_{0^+}^{\beta(1-\alpha)}\mathbb{D}_{0^+}^{\beta(1-\alpha)}\varpi(\tau),\quad \textit{for a.e. } \tau\in\mathbb{J}.$$

Furthermore, if $\varpi\in\mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{E})$ and $\mathbb{I}_{0^+}^{1-\beta(1-\alpha)}\varpi\in\mathbb{C}_{1-\gamma}^{1}(\mathbb{J},\mathbb{E})$, then

$$\mathbb{D}_{0^+}^{\alpha,\beta}\mathbb{I}_{0^+}^{\alpha}\varpi(\tau)=\varpi(\tau), \textit{for a.e. } \tau\in\mathbb{J}.$$

Lemma 2.17. *Let $0<\alpha<1,\ 0\leq\beta\leq 1,\ \gamma=\alpha+\beta-\alpha\beta$, and $\varpi\in L^1(\mathbb{J},\mathbb{E})$. If $\mathbb{D}_{0^+}^{\gamma}\varpi$ exists and in $L^1(\mathbb{J},\mathbb{E})$, then*

$$\mathbb{I}_{0^+}^{\alpha}\mathbb{D}_{0^+}^{\alpha,\beta}\varpi(\tau)=\mathbb{I}_{0^+}^{\gamma}\mathbb{D}_{0^+}^{\gamma}\varpi(\tau)=\varpi(\tau)-\frac{\mathbb{I}_{0^+}^{1-\gamma}\varpi(0^+)}{\Gamma(\gamma)}(\tau-a)^{\gamma-1}, \textit{for a.e. } \tau\in\mathbb{J}.$$

Lemma 2.18. *([22]) For $\tau>a$, we have*

$$\begin{aligned}\mathbb{I}_{0^+}^{\alpha}(\tau-a)^{\beta-1}(\tau)&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\tau-a)^{\beta+\alpha-1},\\ \mathbb{D}_{0^+}^{\alpha}(\tau-a)^{\beta-1}(\tau)&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\tau-a)^{\beta-\alpha-1}.\end{aligned} \tag{2.14}$$

Lemma 2.19. *Let $\alpha>0,\ 0\leq\beta\leq 1$, so the homogeneous differential equation with Hilfer fractional order*

$$\mathbb{D}_{0^+}^{\alpha,\beta}h(\tau)=0, \tag{2.15}$$

has a solution

$$h(\tau)=c_0\tau^{\gamma-1}+c_1\tau^{\gamma+2\beta-2}+c_2\tau^{\gamma+2(2\beta)-3}+...+c_n\tau^{\gamma+n(2\beta)-(n+1)}.$$

Proposition 2.20. *for a given set V of functions $v : \mathbb{J} \to \mathbb{E}$, let us denote by*

$$V(\tau) = \Big\{ v(\tau) : \ v \in V \Big\}, \ \ \tau \in \mathbb{J},$$

and

$$V(\mathbb{J}) = \Big\{ v(\tau) : \ v \in V, \ \ \tau \in \mathbb{J} \Big\}.$$

Definition 2.21. A map $\varpi : \mathbb{J} \times \mathbb{E} \to \mathbb{E}$ is said to be Caratheodory if
(i) $\tau \mapsto \varpi(\tau, u)$ is measurable for each $u \in \mathbb{E}$,
(ii) $u \mapsto \varpi(\tau, u)$ is continuous for almost all $\tau \in \mathbb{J}$.

For convenience, we recall the definitions of the Kuratowski measure of noncompactness and summarize the main properties of this measure.

Definition 2.22. ([5,8]). Let $\mathbb{E}$ be a Banach space and $\Omega_{\mathbb{E}}$ the bounded subsets of $\mathbb{E}$. The Kuratowski measure of noncompactness is the map $\mu : \Omega_{\mathbb{E}} \to [0, \infty]$ defined by

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_{\mathbb{E}}.$$

This measure of noncompactness satisfies some important properties [5,8]:

(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact),

(b) $\mu(B) = \mu(\overline{B})$,

(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,

(d) $\mu(A + B) \leq \mu(A) + \mu(B)$,

(e) $\mu(cB) = |c|\mu(B); c \in \mathbb{R}$,

(f) $\mu(convB) = \mu(B)$.

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.23. *([2,31]). Let $\mathbb{D}$ be a bounded, closed and convex subset of a Banach space such that $0 \in \mathbb{D}$, and let N be a continuous mapping of $\mathbb{D}$ into itself. If the implication*

$$V = \overline{conv}N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{2.16}$$

holds for every subset V of $\mathbb{D}$, then N has a fixed point.

Lemma 2.24. *([31]) Let $\mathbb{D}$ be a bounded, closed and convex subset of the Banach space $C(\mathbb{J}, \mathbb{E})$, G a continuous function on $\mathbb{J} \times \mathbb{J}$ and ϖ a function from $\mathbb{J} \times \mathbb{E} \longrightarrow \mathbb{E}$ which satisfies the Caratheodory conditions, and suppose there exists $p \in L^1(\mathbb{J}, \mathbb{R}^+)$ such that, for each $\tau \in \mathbb{J}$ and each bounded set $B \subset \mathbb{E}$, we have*

$$\lim_{h \to 0^+} \mu(\varpi(\mathbb{J}_{\tau,h} \times B)) \leq p(\tau)\mu(B); \text{ here } \mathbb{J}_{\tau,h} = [\tau - h, \tau] \cap \mathbb{J}.$$

If V is an equicontinuous subset of $\mathbb{D}$, then

$$\mu\left(\left\{\int_{\mathbb{J}} G(s,\tau)\varpi(s, y(s))ds : y \in V\right\}\right) \leq \int_{\mathbb{J}} \|G(\tau, s)\| p(s)\mu(V(s))ds.$$

3 Main results

Let us start by defining what we meant by a solution to the problem (1.2)-(1.3).

Definition 3.1. A function $x \in \mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{E})$ is said to be a solution of the problem (1.2)-(1.3) if x satisfies the equation $\mathbb{D}^{\alpha,\beta}_{0^+}x(\tau) = \mathbb{F}(\tau, x(\tau))$ on $\mathbb{J}$, and the conditions (1.3).

Lemma 3.2. *For any $x \in \mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{R})$, x is a solution of the linear fractional differential equation*

$$\mathbb{D}^{\alpha,\beta}x(\tau) = y(\tau), \tag{3.1}$$

supplemented with the boundary conditions (1.3) if and only if

$$\begin{aligned} x(\tau) = \mathbb{I}^{\alpha}y(\tau) &+ \frac{\tau^{\gamma-1}}{\Lambda}\Big\{v_4(\lambda_1^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T})) + v_2(\lambda_2\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T}))\Big\} \\ &+ \frac{\tau^{\gamma+2\beta-2}}{\Lambda}\Big\{v_1(\lambda_2\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T})) - v_3(\lambda_1^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T}))\Big\}, \end{aligned} \tag{3.2}$$

we can write

$$\begin{aligned} x(\tau) = \mathbb{I}^{\alpha}y(\tau) + \frac{\tau^{\gamma-1}}{\Lambda}\Big\{&\lambda_1(v_4 - v_3\tau^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) + \lambda_2(v_2 + v_1\tau^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) \\ &+ ((v_3 - v_1)\tau^{2\beta-1} - (v_4 + v_2))\mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T})\Big\}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &\Lambda = v_1v_4 + v_2v_3 \neq 0, \\ &v_1 = \Gamma(\gamma) - \lambda_1\frac{\Gamma(\frac{\gamma+\rho-1}{\rho})\varepsilon^{\gamma+\rho q-1}}{\Gamma(\frac{\gamma+\rho q+\rho-1}{\rho})\rho^{q}}, \\ &v_2 = \lambda_1\frac{\Gamma(\frac{\gamma+2\beta+\rho-2}{\rho})\varepsilon^{\gamma+2\beta+\rho q-2}}{\Gamma(\frac{\gamma+2\beta+\rho q+\rho-2}{\rho})\rho^{q}} - \frac{\Gamma(\gamma+2\beta-1)\mathrm{T}^{2\beta-1}}{\Gamma(2\beta)}, \\ &v_3 = \Gamma(\gamma) - \lambda_2(\gamma-1)\frac{\Gamma(\gamma_1 + \frac{\gamma-2}{\eta} + 1)\xi^{\gamma-2}}{\Gamma(\gamma_1 + \frac{\gamma-2}{\eta} + \delta + 1)}, \\ &v_4 = \frac{\Gamma(\gamma+2\beta-1)}{\Gamma(2\beta)}\mathrm{T}^{2\beta-1} - \lambda_2(\gamma+2\beta-2)\frac{\Gamma(\gamma_1 + \frac{\gamma+2\beta-3}{\eta} + 1)\xi^{\gamma+2\beta-3}}{\Gamma(\gamma_1 + \frac{\gamma+2\beta-3}{\eta} + \delta + 1)}. \end{aligned} \tag{3.4}$$

Proof. It is well known that the general solution of the fractional differential equation (3.1) can be written as

$$x(\tau) = \mathbb{I}^{\alpha}y(\tau) + c_1\tau^{\gamma-1} + c_2\tau^{\gamma+2\beta-2}, \tag{3.5}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions (1.3) in (3.5) together with Lemma 2.9, Lemma 2.12 and Lemma 2.18 we obtain a system of equations in c_1 and c_2 given by

$$\begin{aligned} v_1c_1 - v_2c_2 &= \lambda_1^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T}), \\ v_3c_1 + v_4c_2 &= \lambda_2\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T}). \end{aligned} \tag{3.6}$$

Solving the system (3.6), we get

$$c_1 = \frac{1}{\Lambda}\left\{v_4(\lambda_1^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T})) + v_2(\lambda_2\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T}))\right\},$$

$$c_2 = \frac{1}{\Lambda}\left\{v_1(\lambda_2\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T})) - v_3(\lambda_1^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) - \mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T}))\right\},$$

where v_1, v_2, v_3, and v_4 are given by (3.4). Substituting the values of c_0, c_1 in (3.5), we obtain (3.2). Conversely, it can easily be shown by direct computation that the integral solution (3.2) satisfies the equation (3.1) and boundary conditions (1.3). This completes the proof. □

In order to present and prove our main results, we consider the following hypotheses:
(H1) $\mathbb{F} : \mathbb{J} \times \mathbb{E} \to \mathbb{E}$ satisfies the Caratheodory conditions.
(H2) There exists $p \in L^1(\mathbb{J}, \mathbb{R}^+)$, such that,

$$\|\mathbb{F}(\tau, x)\| \leq p(\tau)\|x\|, \text{ for } \tau \in \mathbb{J} \text{ and each } x \in \mathbb{E}.$$

(H3) For each $\tau \in \mathbb{J}$ and each bounded set $B \subset \mathbb{E}$, we have

$$\lim_{h\to 0^+} \mu(\mathbb{F}(\mathbb{J}_{\tau,h} \times B)) \leq \tau^{1-\gamma}p(\tau)\mu(B); \text{ here } \mathbb{J}_{\tau,h} = [\tau - h, \tau] \cap \mathbb{J}.$$

Now, we shall prove the following theorem concerning the existence of solutions of (1.2)-(1.3). Let

$$p^* = \sup_{\tau\in\mathbb{J}} p(\tau).$$

Theorem 3.3. *Assume that the hypotheses (H1)-(H3) hold. If*

$$p^* M < 1, \tag{3.7}$$

then the problem (1.2)-(1.3) *has at least one solution defined on* $\mathbb{J}$ *.*

In the sequel, we use the following expressions:

$$\mathbb{I}^{\alpha}\mathbb{F}(s, x(s))(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau - s)^{\alpha-1}\mathbb{F}(s, x(s))ds, \qquad \tau \in (0, \mathrm{T}),$$

$$^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}\mathbb{F}(s, x(s))(\varepsilon) = \frac{\rho^{1-q}}{\Gamma(\alpha)\Gamma(q)}\int_0^{\varepsilon}\int_0^{r}\frac{r^{\rho-1}(r-s)^{\alpha-1}}{(\varepsilon^{\rho} - r^{\rho})^{1-q}}\mathbb{F}(s, x(s))dsdr, \qquad \varepsilon \in (0, \mathrm{T}),$$

$$\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}\mathbb{F}(s, x(s))(\xi) = \frac{\eta\tau^{-\eta(\delta+\gamma_1)}}{\Gamma(\alpha-1)\Gamma(\delta)}\int_0^{\xi}\int_0^{r}\frac{s^{\eta\gamma_1+\eta-1}(r-s)^{\alpha-2}}{(\xi^{\eta} - r^{\eta})^{1-\delta}}\mathbb{F}(s, x(s))dsdr, \qquad \xi \in (0, \mathrm{T}).$$

For convenience, we set a constant

$$\begin{aligned} M := \frac{\mathrm{T}^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|}\Bigg\{&\left(\frac{|\lambda_1|(|v_4| + |v_3|\mathrm{T}^{2\beta-1})}{\Gamma(\alpha+1)}\right)\left(\frac{\Gamma(\frac{\alpha+\rho}{\rho})\varepsilon^{\alpha+\rho q}}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})\rho^{q}}\right) + \left(\frac{|\lambda_2|(|v_2| + |v_1|\mathrm{T}^{2\beta-1})}{\Gamma(\alpha)}\right) \\ &\times\left(\frac{\Gamma(\gamma_1 + \frac{\alpha-1}{\eta} + 1)\xi^{\alpha-1}}{\Gamma(\gamma_1 + \frac{\alpha-1}{\eta} + \delta + 1)}\right) + ((|v_3| + |v_1|)\mathrm{T}^{2\beta-1} + (|v_4| + |v_2|))\left(\frac{\mathrm{T}^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\right)\Bigg\}. \end{aligned}$$

Proof. Transform the problem (1.2)-(1.3) into a fixed point problem. Consider the operator $\mathbb{Q} : \mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E}) \to \mathbb{C}_{1-\gamma}(\mathbb{J}, \mathbb{E})$ defined by

$$\begin{aligned} \mathbb{Q}x(\tau) = \mathbb{I}^{\alpha}y(\tau) + \frac{\tau^{\gamma-1}}{\Lambda}\Big\{&\lambda_1(v_4 - v_3\tau^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}y(\varepsilon) + \lambda_2(v_2 + v_1\tau^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}y(\xi) \\ &+ ((v_3 - v_1)\tau^{2\beta-1} - (v_4 + v_2))\mathbb{I}^{\alpha-\gamma+1}y(\mathrm{T})\Big\}. \end{aligned} \tag{3.8}$$

Clearly, the fixed points of the operator $\mathbb{Q}$ are solutions of the problem (1.2)-(1.3).

$$\mathbb{D}=\Big\{x\in\mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{E}):\|x\|\leq R\Big\},$$

where R satisfies inequality (3.7).
Notice that the subset $\mathbb{D}$ is closed, convex, and equicontinuous. We shall show that the operator $\mathbb{Q}$ satisfies all the assumptions of Mönch's fixed point theorem.
The proof will be given in three steps.

Step 1: $\mathbb{Q}$ is continuous:

Let x_n be a sequence such that $x_n\to x$ in $\mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{E})$.
Then, for each $\tau\in\mathbb{J}$,

$$\begin{aligned}
&\|\tau^{1-\gamma}((\mathbb{Q}x_n)(\tau)-(\mathbb{Q}x)(\tau))\|\\
&\leq\tau^{1-\gamma}\mathbb{I}^{\alpha}\|\mathbb{F}(s,x_n(s))-\mathbb{F}(s,x(s))\|(\tau)\\
&+\frac{1}{\Lambda}\Big\{|\lambda_1|(|v_4|+|v_3|\tau^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}\|\mathbb{F}(s,x_n(s))-\mathbb{F}(s,x(s))\|(\varepsilon)\\
&+|\lambda_2|(|v_2|+|v_1|\tau^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}\|\mathbb{F}(s,x_n(s))-\mathbb{F}(s,x(s))\|(\xi)\\
&+((|v_3|+|v_1|)\tau^{2\beta-1}+(|v_4|+|v_2|))\mathbb{I}^{\alpha-\gamma+1}\|\mathbb{F}(s,x_n(s))-\mathbb{F}(s,x(s))\|(\mathrm{T})\Big\}\\
&\leq\Big\{\mathrm{T}^{1-\gamma}\mathbb{I}^{\alpha}(1)(\mathrm{T})+\frac{1}{\Lambda}\Big\{|\lambda_1|(|v_4|+|v_3|\mathrm{T}^{2\beta-1})\mathrm{T}^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}(1)(\varepsilon)\\
&+|\lambda_2|(|v_2|+|v_1|\mathrm{T}^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}(1)(\xi)+((|v_3|+|v_1|)\mathrm{T}^{2\beta-1}+(|v_4|+|v_2|))\\
&\mathbb{I}^{\alpha-\gamma+1}(1)(\mathrm{T})\Big\}\Big\}\|\mathbb{F}(s,x_n(s))-\mathbb{F}(s,x(s))\|.
\end{aligned}$$

Since $\mathbb{F}$ is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|\mathbb{Q}(x_n)-\mathbb{Q}(x)\|_{\infty}\to 0 \text{ as } n\to\infty.$$

Step 2: $\mathbb{Q}$ maps $\mathbb{D}$ into itself :

Take $x\in\mathbb{D}$, by (H2), we have, for each $\tau\in\mathbb{J}$ and assume that $\mathbb{Q}x(\tau)\neq 0$.

$$
\begin{aligned}
&\|\tau^{1-\gamma}(\mathbb{Q}x)(\tau)\|\\
\leq&\tau^{1-\gamma}\mathbb{I}^{\alpha}\|\mathbb{F}(s,x(s))\|(\tau)+\frac{1}{\Lambda}\Big\{|\lambda_1|(|v_4|+|v_3|\tau^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}\|\mathbb{F}(s,x(s))\|(\varepsilon)\\
&+|\lambda_2|(|v_2|+|v_1|\tau^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}\|\mathbb{F}(s,x(s))\|(\xi)\\
&+((|v_3|+|v_1|)\tau^{2\beta-1}+(|v_4|+|v_2|))\mathbb{I}^{\alpha-\gamma+1}\|\mathbb{F}(s,x(s))\|(\mathrm{T})\Big\}\\
\leq&\tau^{1-\gamma}\mathbb{I}^{\alpha}\|x\|p(s)(\tau)+\frac{1}{\Lambda}\Big\{|\lambda_1|(|v_4|+|v_3|\mathrm{T}^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}\|x\|p(s)(\varepsilon)+|\lambda_2|(|v_2|+|v_1|\mathrm{T}^{2\beta-1})\\
&\times\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}\|x\|p(s)(\xi)+((|v_3|+|v_1|)\mathrm{T}^{2\beta-1}+(|v_4|+|v_2|))\mathbb{I}^{\alpha-\gamma+1}\|x\|p(s)(\mathrm{T})\Big\}\\
\leq&p^*R\Big[\mathrm{T}^{1-\gamma}\mathbb{I}^{\alpha}(1)(\tau)+\frac{1}{\Lambda}\Big\{|\lambda_1|(|v_4|+|v_3|\mathrm{T}^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}(1)(\varepsilon)\\
&+|\lambda_2|(|v_2|+|v_1|\mathrm{T}^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}(1)(\xi)+((|v_3|+|v_1|)\mathrm{T}^{2\beta-1}+(|v_4|+|v_2|))\mathbb{I}^{\alpha-\gamma+1}(1)(\mathrm{T})\Big\}\Big]\\
\leq&p^*R\left[\frac{\mathrm{T}^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{1}{|\Lambda|}\left\{\left(\frac{|\lambda_1|(|v_4|+|v_3|\mathrm{T}^{2\beta-1})}{\Gamma(\alpha+1)}\right)\left(\frac{\Gamma(\frac{\alpha+\rho}{\rho})\varepsilon^{\alpha+\rho q}}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho}\rho^{q}}\right)\right.\right.\\
&+\left(\frac{|\lambda_2|(|v_2|+|v_1|\mathrm{T}^{2\beta-1})}{\Gamma(\alpha)}\right)\times\left(\frac{\Gamma(\gamma_1+\frac{\alpha-1}{\eta}+1)\xi^{\alpha-1}}{\Gamma(\gamma_1+\frac{\alpha-1}{\eta}+\delta+1)}\right)\\
&\left.\left.+((|v_3|+|v_1|)\mathrm{T}^{2\beta-1}+(|v_4|+|v_2|))\left(\frac{\mathrm{T}^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+2)}\right)\right\}\right]\\
:=&p^*RM\\
\leq&R.
\end{aligned}
$$

Next, we show that $\mathbb{Q}(\mathbb{D})$ is equicontinuous :
By Step 2, it is obvious that $\mathbb{Q}(\mathbb{D})\subset\mathbb{C}_{1-\gamma}(\mathbb{J},\mathbb{E})$ is bounded. For the equicontinuity of $\mathbb{Q}(\mathbb{D})$, let $\tau_1,\tau_2\in\mathbb{J}$, $\tau_1<\tau_2$ and $x\in\mathbb{D}$, so $\tau_2^{1-\gamma}\mathbb{Q}x(\tau_2)-\tau_1^{1-\gamma}\mathbb{Q}x(\tau_1)\neq0$. Hence,

$$
\begin{aligned}
\|\tau_2^{1-\gamma}\mathbb{Q}x(\tau_2)-\tau_1^{1-\gamma}\mathbb{Q}x(\tau_1)\|&\leq\mathbb{I}^{\alpha}\left|\tau_2^{1-\gamma}\mathbb{F}(s,x(s))(\tau_2)-\tau_1^{1-\gamma}\mathbb{F}(s,x(s))(\tau_1)\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]\|\mathbb{F}(s,x(s))\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}\|\mathbb{F}(s,x(s))\|ds\\
&\leq\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]\|x\|p(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}\|x\|p(s)ds\\
&\leq p^*R\left\{\frac{(\tau_2^{\alpha-\gamma+1}-\tau_1^{\alpha-\gamma+1})}{\Gamma(\alpha+1)}\right\}.
\end{aligned}
$$

As $\tau_1\to\tau_2$, the right hand side of the above inequality tends to zero.
Hence $\mathbb{Q}(\mathbb{D})\subset\mathbb{D}$.

Step 3: The implication (2.16) holds.

Now let V be a bounded and equicontinuous subset of $\mathbb{D}$. Hence $\tau \mapsto v(\tau) = \mu(V(\tau))$ is continuous on $\mathbb{J}$ such that $V \subset \overline{conv}(0 \cup \mathbb{Q}(V))$. Clearly, $V(\tau) \subset \overline{conv}(\{0\} \cup \mathbb{Q}(V))$ for al $\tau \in \mathbb{J}$. Hence $\mathbb{Q}V(\tau) \subset \mathbb{Q}\mathbb{D}(\tau)$, $\tau \in \mathbb{J}$ is bounded in $\mathbb{E}$. By assumption (H3), and the properties of measure μ , we have, for each $\tau \in \mathbb{J}$,

$$
\begin{aligned}
&\tau^{1-\gamma}v(\tau) \leq \mu(\tau^{1-\gamma}\mathbb{Q}(V)(\tau) \cup \{0\})) \leq \mu(\tau^{1-\gamma}(\mathbb{Q}V)(\tau))\\
\leq & \mu\Big\{\tau^{1-\gamma}\mathbb{I}^{\alpha}\mathbb{F}(s,V(s))(\tau) + \frac{1}{|\Lambda|}\Big\{|\lambda_1|(|v_4| + |v_3|\tau^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}\mathbb{F}(s,V(s))(\varepsilon)\\
&+ \lambda_2(v_2 + v_1\tau^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}\mathbb{F}(s,V(s))(\xi) + ((v_3 - v_1)\tau^{2\beta-1} - (v_4 + v_2))\mathbb{I}^{\alpha-\gamma+1}\mathbb{F}(s,V(s))(\mathrm{T})\Big\}\Big\}\\
\leq & \tau^{1-\gamma}\mathbb{I}^{\alpha}\mu(\mathbb{F}(s,V(s)))(\tau) + \frac{1}{|\Lambda|}\Big\{|\lambda_1|(|v_4| + |v_3|\tau^{2\beta-1})^{\rho}I^{q}\mathbb{I}^{\alpha}\mu(\mathbb{F}(s,V(s)))(\varepsilon) + |\lambda_2|(|v_2| + |v_1|\tau^{2\beta-1})\\
&\times \mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}\mu(\mathbb{F}(s,V(s)))(\xi) + ((|v_3| + |v_1|)\tau^{2\beta-1} + (|v_4| + |v_2|))\mathbb{I}^{\alpha-\gamma+1}\mu(\mathbb{F}(s,V(s)))(\mathrm{T})\Big\}\\
\leq & \mathrm{T}^{1-\gamma}\mathbb{I}^{\alpha}p(s)v(s)(\tau) + \frac{1}{|\Lambda|}\Big\{|\lambda_1|(|v_4| + |v_3|\mathrm{T}^{2\beta-1})^{\rho}I^{q}\mathbb{I}^{\alpha}p(s)v(s)(\varepsilon) + |\lambda_2|(|v_2| + |v_1|\mathrm{T}^{2\beta-1})\\
&\times \mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}p(s)v(s)(\xi) + ((|v_3| + |v_1|)\mathrm{T}^{2\beta-1} + (|v_4| + |v_2|))\mathbb{I}^{\alpha-\gamma+1}p(s)v(s)(\mathrm{T})\Big\}\\
\leq & p^*\|v\|\Big[\mathrm{T}^{1-\gamma}\mathbb{I}^{\alpha}(1)(\tau) + \frac{1}{|\Lambda|}\Big\{|\lambda_1|(|v_4| + |v_3|\mathrm{T}^{2\beta-1})^{\rho}\mathbb{I}^{q}\mathbb{I}^{\alpha}(1)(\varepsilon)\\
&+ |\lambda_2|(|v_2| + |v_1|\mathrm{T}^{2\beta-1})\mathbb{I}_{\eta}^{\gamma_1,\delta}\mathbb{I}^{\alpha-1}(1)(\xi) + ((|v_3| + |v_1|)\mathrm{T}^{2\beta-1} + (|v_4| + |v_2|))\mathbb{I}^{\alpha-\gamma+1}(1)(\mathrm{T})\Big\}\Big]\\
\leq & p^*\|v\|M.
\end{aligned}
$$

This means that

$$\|v\|(1 - p^*M) \leq 0.$$

By (3.7) it follows that $\|v\| = 0$, that is $v(\tau) = 0$ for each $\tau \in \mathbb{J}$, and then $V(\tau)$ is relatively compact in $\mathbb{E}$. In view of the Ascoli-Arzela theorem, V is relatively compact in $\mathbb{D}$. Applying now Theorem 2.23, we conclude that $\mathbb{Q}$ has a fixed point which is a solution of the problem (1.2)-(1.3).

□

4 Examples

In this section, we present an example to illustrate our results.
Let $\mathbb{E} = l^1 = \{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with the norm

$$\|x\|_{\mathbb{E}} = \sum_{n=1}^{\infty} |x_n|.$$

Example 4.1. Consider the following nonlinear Hilfer fractional differential equation with a mixed of fractional integral conditions

$$
\begin{cases}
\mathbb{D}_{0^+}^{1/2,1/2}x(\tau) &= \frac{(\sin\tau+1)e^{-\tau}}{24}\left(\frac{x^2(\tau)}{1+|x(\tau)|}\right), \qquad \tau \in \mathbb{J} = [0,\pi],\\
\mathbb{I}^{1/4}x(\pi) &= \frac{2}{3}^{1/2}I^{1/3}x(\pi/3), \quad \mathbb{I}^{5/4}x'(\pi) = \frac{4}{5}I_{3/8}^{4/9,2/7}x'(\pi/2).
\end{cases}
\tag{4.1}
$$

Here

$$\begin{array}{cccc} \alpha = 1/2, & \beta = 1/2, & \gamma = 3/4, & \lambda_1 = 2/3, \\ \lambda_2 = 4/5, & q = 1/3, & \rho = 1/2, & \gamma_1 = 4/9, \\ \delta = 2/7, & \eta = 3/8, & \xi = \pi/2, & \varepsilon = \pi/3. \end{array}$$

Form the given data, we get

$$\begin{array}{ccc} v_1 = 1.4622, & v_2 = 0.2838, & v_3 = 1.4750, \\ v_4 = 3.1361, & \Lambda = 5.0042, & M = 7.3159. \end{array}$$

and

$$\mathbb{F}(\tau, x) = (((\sin\tau + 1)e^{-\tau})/24)(x^2/(1+|x|)).$$

Further,

$$|\mathbb{F}(\tau, x)| \leq \frac{1}{12}|x|.$$

With

$$p^* = \frac{1}{12}.$$

Hence

$$p^* M \simeq 0.6097 < 1.$$

Therefore, we deduce from the conclusion of Theorem 3.3 that the problem (4.2) has a solution on $[0, \pi]$.

Example 4.2. Consider the following nonlinear Hilfer fractional differential equation:
Firstly, we fixed $\beta = 1$, on other hand, reduce Problem (4.2) into Caputo fractional differential equation

$$\begin{cases} \mathbb{D}^{1/2;1} u(t) = \frac{\sqrt{3}|u|\cos^2(2\pi t)}{3(27-t)}, \\ t \in J = [0,1], \\ \\ \mathbb{I}^0 x(1) = \frac{2}{5}^{1/2} I^{1/4} x(1/3), \quad \mathbb{I}^1 x'(1) = \frac{7}{9} I_{3/7}^{4/9,2/5} x'(2/3). \end{cases} \tag{4.2}$$

Here

$$\begin{array}{ccc} v_1 = 1.0133, & v_2 = 0.3005, & v_3 = 1.0482, \\ v_4 = 2.9344, & \Lambda = 3.1152, & M = 6.2147. \end{array}$$

Clearly, the function $\mathbb{F}$ is continuous. For each $u \in \mathbb{E}$ and $t \in [0,1]$, we have

$$|\mathbb{F}(t, \omega)| \leq \frac{\sqrt{3}}{81} |\omega|.$$

Hence, the hypothesis (H2) is satisfied with $p^* = \frac{\sqrt{3}}{81}$. We shall show that condition (3.7) holds with $\mathbb{J}$. Indeed,

$$p^* M \simeq 0.1335 < 1.$$

Simple computations show that all conditions of Theorem (3.3) are satisfied. It follows that the problem (4.2) has at least one weak solution defined on $\mathbb{J}$.

5 Conclusion

The fractional differential equations have been preferred over integer-order differential equations for their ability to describe the dynamical behaviors of numerous processes in the scientific and engineering fields. To confirm such a claim, one can observe different investigations done with the aid of researchers in the literature. In this research article, we investigate the existence of solutions of a nonlinear fractional differential equation involving the Hilfer fractional operator with a mixed of fractional integral boundary conditions (Erdélyi-Kober fractional integral, Katugampola fractional integral). To achieve the goals, we use a method involving a measure of noncompactness and a fixed point theorem of the Monch type. Though the technique applied to establish the existence results for the problem at hand is a standard one, its exposition in the present framework is new. Two examples are presented to guarantee the viability of our obtained results. Our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem. Especially, problem (1.2) is formulated in general form that combines both fractional Caputo problems and Riemann-Liouville problems, the choices of $\beta = 1$ on the one hand and $\beta = 0$ on the other hand, reduce the problem (1.2) into the Caputo fractional differential equation and Riemann-Liouville fractional differential equation, respectively.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Conflict of interest

The authors declare that they have no competing interests.

References

[1] N. Adjimi, A. Boutiara, M. S. Abdo and M. Benbachir, *Existence results for nonlinear neutral generalized Caputo fractional differential equations*, J. Pseudo-Differ. Oper. Appl. **12**(2):25 (2021), 1–17. DOI

[2] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics (141), Cambridge University Press, Cambridge, 2001. DOI

[3] B. Ahmad, S. K. Ntouyas, J. Tariboon and A. Alsaedi, *Caputo Type Fractional Differential Equations with Nonlocal Riemann-Liouville and Erdélyi-Kober Type Integral Boundary Conditions*, Filomat **31**(14) (2017), 4515-4529. DOI

[4] B. Ahmad, S. K. Ntouyas and J. Tariboon, *Nonlocal fractional-order boundary value problems with generalized Riemann-Liouville integral boundary conditions*, J. Comput. Anal. Appl. **23**(7) (2017), 1281-1296. URL

[5] R. R. Akhmerov, M. I. Kamenskii, A. S. Patapov, A. E. Rodkina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Operator Theory: Advances and Applications, Birkhäuser, Basel, 1992. DOI

[6] A. Alsaedi, D. Baleanu, S. Etemad and S. Rezapour, *On coupled systems of times-fractional differential problems by using a new fractional derivative*, J. Funct. Sp. **2016**:4626940 (2016), 1–8. DOI

[7] F. Si Bachir, S. Abbas, M. Benbachir and M. Benchohra, *Successive approximations for random coupled Hilfer fractional differential systems*, Arab. J. Math. **10** (2021), 301–310. DOI

[8] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, Volume 60, Marcel Dekker, New York, 1980. URL

[9] H. Belbali and M. Benbachir, *Existence results and Ulam-Hyers stability to impulsive coupled system fractional differential equations*, Turk. J. Math. **45** (2021), 1368-1385. URL

[10] S. Ben Chikh, A. Amara, S. Etemad and S. Rezapour, *On Ulam-Hyers-Rassias stability of a generalized Caputo type multi-order boundary value problem with four-point mixed integro-derivative conditions*, Adv. Differ. Equ. **2020**, 680 (2020).

[11] S. Bourafa, M-S. Abdelouahab and R. Lozi, *On periodic solutions of fractional-order differential systems with a fixed length of sliding memory*, J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 64–78. ARK

[12] A. Boutiara and M. Benbachir, *Implicit Fractional Differential Equation Involving ψ-Caputo with Boundary Conditions*, Bull. Inst. Math. Acad. Sin. **16**(1) (2021), 1–19. URL

[13] A. Boutiara, M. Benbachir and K. Guerbati, *Caputo type fractional differential equation with nonlocal Erdélyi-Kober type integral boundary conditions in Banach spaces*, Surv. Math. Appl. **15** (2020), 399–418. URL

[14] A. Boutiara, K. Guerbati and M. Benbachir, *Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces*, AIMS Mathematics, **5**(1), (2020), 259–272. DOI

[15] A. Boutiara, M. M. Matar, M. K. Kaabar, F. Martnez, S. Etemad and S. Rezapour, *Some Qualitative Analysis of Neutral Functional Delay Differential Equation with Generalized Caputo Operator*, J. Funct. Spaces, (2021). DOI

[16] A. Erdélyi and H. Kober, *Some remarks on Hankel transforms*, Q. J. Math. **11**(1) (1940), 212–221. DOI

[17] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000. DOI

[18] R. Hilfer, *Threefold Introduction to Fractional Derivatives*, Wiley VCH Verlag GmbH, 2008, Chapter 2, pages 17–73. URL

[19] R. Hilfer, Y. Luchko and Z. Tomovski, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivative*, Fractional Calculus Appl. Anal. **12** (2009), 289–318. URL

[20] R. Kamocki and C. Obcznnski, *On fractional Cauchy-type problems containing Hilfer derivative*, Electronic Journal of Qualitative of Differential Equations, **50** (2016), 1–12. DOI

[21] U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218**(3) (2011), 860–865. DOI

[22] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. URL

[23] H. Kober, *On fractional integrals and derivatives*, Quart. J. Math., Oxford, second Ser. **ll** (1940), 193-211. DOI

[24] N. I. Mahmudov and S. Emin, *Fractional-order boundary value problems with Katugampola fractional integral conditions*, Adv. Differ. Equ. **2018**:81 (2018), 1–17. DOI

[25] H. Mohammadi, D. Baleanu, S. Etemad and S. Rezapour, *Criteria for existence of solutions for a Liouville-Caputo boundary value problem via generalized Gronwall's inequality*, J. Inequal. Appl. **36** (2021), 1-19. DOI

[26] H. Mohammadi, S. Kumar, S. Rezapour and S. Etemad, *A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control*, Chaos Solitons Fractals, **144**:110668 (2021), 1–13. DOI

[27] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal. **4**(5) (1980), 985-999. DOI

[28] H. Mönch and G. F. Von Harten, *On the Cauchy problem for ordinary differential equations in Banach spaces*, Archiv. Math. Basel **39** (1982), 153-160. DOI

[29] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York (1999). URL

[30] S. Rezapour, S. Etemad and H. Mohammadi, *A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals*, Adv. Differ. Equ. **2020**:481 (2020), 1–30. DOI

[31] S. Szufla, *On the application of measure of noncompactness to existence theorems*, Rend. Sem. Mat. Univ. Padova **75** (1986), 1-14. URL

[32] S. T. M. Thabet, S. Etemad, and S. Rezapour, *On a new structure of the pantograph inclusion problem in the Caputo conformable setting*, Bound. Value Probl. **2020**:171 (2020), 1–21. DOI

[33] N. Thongsalee, S. K. Ntouyas and J. Tariboon, *Nonlinear Riemann-Liouville fractional differential equations with nonlocal Erdelyi-Kober fractional integral conditions*, Fract. Calc. Appl. Anal. **19**(2) (2016), 480-497. DOI