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A New Special 15-Step Block Method for Solving General Fourth Order Ordinary Differential Equations

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Abstract

A new higher-implicit block method for the direct numerical solution of fourth order ordinary differential equation is derived in this research paper. The formulation of the new formula which is 15-step, is achieved through interpolation and collocation techniques. The basic numerical properties of the method such as zero-stability, consistency and A-stability have been examined. Investigation showed that the new method is zero stable, consistent and A-stable, hence convergent. Test examples from recent literature have been used to confirm the accuracy of the new method.

1. Introduction

Ordinary differential equations (ODEs) have important applications and are a powerful tool in the study of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. Mathematical models in these vast range of disciplines, describe how quantities change. This leads naturally to the language of ordinary differential equations (ODEs). For instance, Newton's laws in mechanics make it possible to reduce the description of the motion of mass points or solid bodies to solving ordinary differential equations. The computation of radiotechnical circuits or satellite tragetories, studies of the stability of a plane in flight, and explaining the course of chemical reactions are all carried out by studying and solving ordinary differential equations. The most interesting and most important applications of these equations are in the theory of oscillations, physical ship dynamic theory and in automatic control theory. These applied problems in turn produce new formulations of problems in the theory of ordinary differential equations from first, second, third, fourth, fifth, e.t.c, to other higher derivatives in ODEs Conti, Graffi and Sansone[1]. However, the solutions to some of these varieties of higher order ODEs problems do not exist explicitly. Hence, the need to develop numerical methods in the form of implicit linear multistep methods (LMMs) to solve these numerous problems in the form of ODEs. Therefore, in this research paper, we shall consider the general fourth order ordinary differential equation of the form:

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$$y^{(iv)} = f(x, y, y', y'', y'''),$$
$$y(a) = \epsilon_0, y'(a) = \epsilon_1, y''(a) = \epsilon_2, y'''(a) = \epsilon_3 \quad (1)$$

where, $R \times R^m \times R^m \to R^m$ and $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in R$. However, interpolation and collocation techniques have been used overtime for developing implicit linear multistep methods using power series as a basis function. Such authors as Raymond, Skwame and Adiku [2] formulated a four-step one off-grid block method using interpolation and collocation approach for the solution of fourth derivative ordinary differential equations. Two-step hybrid linear multistep block method for solving second, third and fourth order initial value problems of ordinary differential equations directly has also been derived by Abolarin, Kuboye, Adeyefa and Ogunware [3]. Hermite interpolation polynomial as a basis function has extensively been used also for the formulation of implicit block methods. Two point implicit block method of uniform order 6 has been derived for solving fourth-order initial value problems directly by Allogmany, Ismail, Majid and Ibrahim [4] using the aforementioned basis function. In addition, in order to avoid order reduction at solving higher order ODEs, such authors as Kuboye [5], Jena, Mohanty and Mishra [6], Adoghe and Omole [7], Omar and Kuboye [8], Duromola [9], Adesanya, Momoh, Adamu and Tahir [10], Awoyemi, Kayode and Adoghe [11], Awoyemi [12] and Ukpebor, Omole and Adoghe [13] have solved equation (1) directly. Similarly, Omole and Ukpebor [14] and Kuboye, Elusakin and Quadri [15] have both formulated a 4-point and 5-point hybrids block formulae for the solution of system and linear fourth order initial value problems in ordinary differential equations using power series as a basis function via interpolation and collocation techniques with uniform order 4 and 7 respectively. Therefore, we are motivated to improve on Jena et al. [6] who have formulated a 9-point block method of a uniform order 12 for the direct solution of equation (1) by proposing a new special 15-step block numerical method for the direct approximation of (1). Consequently, in section two, we carry out the formulation of 15-step block formula, section three provides the analysis of basic numerical properties, section four considers numerical examples, its implementation and discussion of results and in section five, the conclusion is drawn.

2. Materials and Methods of 15-Step Block Method

In this section, we shall consider the formulation of a 15-step block method for the numerical approximation of equation (1).

2.1. Formulation of 15-Step Block Method

Let us consider the power series approximations of the form:

$$y(x) = \sum_{j=0}^{k+4} a_j x^j \quad (2)$$

which approximates equation (1), so that $a_{j's}$, $j = 0(1)(k+4)$ are parameters to be found and k is the step-number. Thus, the first, second, third and fourth derivatives of equation (2) become:

$$y'(x) = \sum_{j=0}^{k+4} j a_j x^{j-1} \quad (3)$$

$$y''(x) = \sum_{j=0}^{k+4} j(j-1) a_j x^{j-2} \quad (4)$$

$$y'''(x) = \sum_{j=0}^{k+4} j(j-1)(j-2) a_j x^{j-3} \quad (5)$$

$$y^{(iv)}(x) = \sum_{j=0}^{k+4} j(j-1)(j-2)(j-3) a_j x^{j-4} \quad (6)$$

Therefore, equating equation (1) and (6) gives:

$$\sum_{j=0}^{k+4} j(j-1)(j-2)(j-3) a_j x^{j-4} = f(x, y, y', y'', y''') \quad (7)$$

Thus, equation (2), (3), (4) and (5) are then interpolated at $x_{n+i}, i = (k-15)$ and equation (7) is collocated at $x_{n+i}, i = 0(1)k$ to give system of linear equations in the form:

$$CX = D \quad (8)$$

Where,

$$C = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & \cdots & x_n^{k+4} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & \cdots & (k+4)x_n^{k+3} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & \cdots & (k+4)(k+3)x_n^{k+2} \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & \cdots & (k+4)(k+3)(k+2)x_n^{k+1} \\ 0 & 0 & 0 & 0 & 24 & 120x_n & \cdots & (k+4)(k+3)(k+2)(k+1)x_n^k \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+1} & \cdots & (k+4)(k+3)(k+2)(k+1)x_{n+1}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+k} & \cdots & (k+4)(k+3)(k+2)(k+1)x_{n+k}^k \end{bmatrix},$$

$$X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{k+4} \end{bmatrix}, D = \begin{bmatrix} y_n \\ y'_n \\ y''_n \\ y'''_n \\ f_n \\ \vdots \\ f_{n+k} \end{bmatrix}$$

By solving equation (8) with $k = 15$ and $j = 0(1)(k+4)$ (where C is a 20-by-20 matrix), using Gaussian elimination method, $a_{k+4's}$ are obtained and substitution into equation (2) is made to give a continuous linear multistep method(LMM) of the form:

$$y(x) = \alpha_0 y_n + h\alpha_1 y'_n + h^2 \alpha_2 y''_n + h^3 \alpha_3 y'''_n + h^4 \sum_{j=0}^{k} \beta_j f_{n+j} \quad (9)$$

Where, $\xi = x - x_n$

$$\alpha_0 = 1 \quad (10)$$

$$\alpha_1 = \xi \quad (11)$$

$$\alpha_2 = \frac{1}{2}\xi^2 \quad (12)$$

$$\alpha_3 = \frac{1}{6}\xi^3 \quad (13)$$

$$\beta_0 = \frac{1}{24}\xi^4 - \frac{1195757\,\xi^5}{43243200\,h} + \frac{13215487\,\xi^6}{1009008000\,h^2} - \frac{35118025721\,\xi^7}{7628100480000\,h^3} + \frac{2065639\,\xi^8}{1676505600\,h^4} - \frac{277382447\,\xi^9}{1086375628800\,h^5}$$
$$+ \frac{2271089\,\xi^{10}}{54867456000\,h^6} - \frac{54576553\,\xi^{11}}{10346434560000\,h^7} + \frac{4783\,\xi^{12}}{9053130240\,h^8} - \frac{324509\,\xi^{13}}{7846046208000\,h^9} + \frac{109\,\xi^{14}}{43589145600\,h^{10}}$$
$$- \frac{26921\,\xi^{15}}{235381386240000\,h^{11}} + \frac{\xi^{16}}{261534873600\,h^{12}} - \frac{47\,\xi^{17}}{533531142144000\,h^{13}} + \frac{\xi^{18}}{800296713216000\,h^{14}}$$
$$- \frac{\xi^{19}}{1216451004088832000\,h^{15}} \qquad (14)$$

$$\beta_1 = \frac{1}{8}\frac{\xi^5}{h} - \frac{835397\,\xi^6}{8648640\,h^2} + \frac{60461593\,\xi^7}{1412611200\,h^3} - \frac{146723651\,\xi^8}{11176704000\,h^4} + \frac{8548607\,\xi^9}{2874009600\,h^5} - \frac{28162523\,\xi^{10}}{54867456000\,h^6}$$
$$+ \frac{1475953\,\xi^{11}}{21555072000\,h^7} - \frac{7346057\,\xi^{12}}{1034643456000\,h^8} + \frac{5973601\,\xi^{13}}{10461394944000\,h^9} - \frac{24599\,\xi^{14}}{697426329600\,h^{10}} + \frac{223\,\xi^{15}}{135862272000\,h^{11}}$$
$$- \frac{2329\,\xi^{16}}{41845579776000\,h^{12}} + \frac{71\,\xi^{17}}{54721142784000\,h^{13}} - \frac{\xi^{18}}{53801459712000\,h^{14}} + \frac{\xi^{19}}{8109673360588800\,h^{15}} \qquad (15)$$

$$\beta_2 = -\frac{7\,\xi^5}{16\,h} + \frac{1015577\,\xi^6}{2471040\,h^2} - \frac{20848547\,\xi^7}{100900800\,h^3} + \frac{1546697837\,\xi^8}{22353408000\,h^4} - \frac{149466307\,\xi^9}{8941363200\,h^5} + \frac{664545493\,\xi^{10}}{219469824000\,h^6}$$
$$- \frac{289311887\,\xi^{11}}{689762304000\,h^7} + \frac{5795249\,\xi^{12}}{129330432000\,h^8} - \frac{1379251\,\xi^{13}}{373621248000\,h^9} + \frac{40549\,\xi^{14}}{174356582400\,h^{10}} - \frac{5231\,\xi^{15}}{475517952000\,h^{11}}$$
$$+ \frac{989\,\xi^{16}}{2615348736000\,h^{12}} - \frac{61\,\xi^{17}}{6840142848000\,h^{13}} + \frac{59\,\xi^{18}}{457312407552000\,h^{14}} - \frac{\xi^{19}}{1158524765798400\,h^{15}} \qquad (16)$$

$$\beta_3 = \frac{91\,\xi^5}{72\,h} - \frac{1075637\,\xi^6}{855360\,h^2} + \frac{281523559\,\xi^7}{419126400\,h^3} - \frac{23857083361\,\xi^8}{100590336000\,h^4} + \frac{32534861339\,\xi^9}{543187814400\,h^5} - \frac{617568593\,\xi^{10}}{54867456000\,h^6}$$
$$+ \frac{207888451\,\xi^{11}}{129330432000\,h^7} - \frac{5522731\,\xi^{12}}{31352832000\,h^8} + \frac{66465877\,\xi^{13}}{4483454976000\,h^9} - \frac{1989607\,\xi^{14}}{2092278988800\,h^{10}}$$
$$+ \frac{390953\,\xi^{15}}{8559323136000\,h^{11}} - \frac{66571\,\xi^{16}}{41845579776000\,h^{12}} + \frac{6229\,\xi^{17}}{164163428352000\,h^{13}} - \frac{13\,\xi^{18}}{23451918336000\,h^{14}}$$
$$+ \frac{\xi^{19}}{267351869030400\,h^{15}} \qquad (17)$$

$$\beta_4 = -\frac{91\,\xi^5}{32\,h} + \frac{1105667\,\xi^6}{380160\,h^2} - \frac{18107311\,\xi^7}{11289600\,h^3} + \frac{3269363347\,\xi^8}{5588352000\,h^4} - \frac{6129294103\,\xi^9}{40236134400\,h^5} + \frac{3229334839\,\xi^{10}}{109734912000\,h^6}$$
$$- \frac{2970738131\,\xi^{11}}{689762304000\,h^7} + \frac{31183253\,\xi^{12}}{64665216000\,h^8} - \frac{15450997\,\xi^{13}}{373621248000\,h^9} + \frac{235087\,\xi^{14}}{87178291200\,h^{10}}$$
$$- \frac{62483\,\xi^{15}}{475517952000\,h^{11}} + \frac{12121\,\xi^{16}}{2615348736000\,h^{12}} - \frac{139\,\xi^{17}}{1243662336000\,h^{13}} + \frac{29\,\xi^{18}}{17588938752000\,h^{14}}$$
$$- \frac{\xi^{19}}{89117289676800\,h^{15}} \qquad (18)$$

$$\beta_5 = \frac{1001\,\xi^5}{200\,h} - \frac{224737\,\xi^6}{43200\,h^2} + \frac{103207793\,\xi^7}{35280000\,h^3} - \frac{221496517\,\xi^8}{203212800\,h^4} + \frac{176798693\,\xi^9}{609638400\,h^5} - \frac{3138934651\,\xi^{10}}{54867456000\,h^6}$$
$$+ \frac{230071601\,\xi^{11}}{26943840000\,h^7} - \frac{201455717\,\xi^{12}}{206928691200\,h^8} + \frac{126849403\,\xi^{13}}{1494484992000\,h^9} - \frac{3917983\,\xi^{14}}{697426329600\,h^{10}} + \frac{1319249\,\xi^{15}}{4755179520000\,h^{11}}$$
$$- \frac{1507\,\xi^{16}}{152165744640\,h^{12}} + \frac{1201\,\xi^{17}}{4974649344000\,h^{13}} - \frac{23\,\xi^{18}}{6395977728000\,h^{14}} + \frac{\xi^{19}}{40507858944000\,h^{15}} \qquad (19)$$

$$\beta_6 = -\frac{1001\,\xi^5}{144\,h} + \frac{1135697\,\xi^6}{155520\,h^2} - \frac{158534377\,\xi^7}{38102400\,h^3} + \frac{28776650641\,\xi^8}{18289152000\,h^4} - \frac{84007238759\,\xi^9}{197522841600\,h^5}$$
$$+ \frac{2079145531\,\xi^{10}}{24385536000\,h^6} - \frac{26754892001\,\xi^{11}}{2069286912000\,h^7} + \frac{193584983\,\xi^{12}}{129330432000\,h^8} - \frac{21199931\,\xi^{13}}{160123392000\,h^9}$$
$$+ \frac{4645801\,\xi^{14}}{523069747200\,h^{10}} - \frac{1900739\,\xi^{15}}{4279661568000\,h^{11}} + \frac{1271\,\xi^{16}}{79252992000\,h^{12}} - \frac{737\,\xi^{17}}{1865493504000\,h^{13}}$$
$$+ \frac{19\,\xi^{18}}{3197988864000\,h^{14}} - \frac{\xi^{19}}{24304715366400\,h^{15}} \tag{20}$$

$$\beta_7 = \frac{429\,\xi^5}{56\,h} - \frac{1144277\,\xi^6}{141120\,h^2} + \frac{15356659\,\xi^7}{3292800\,h^3} - \frac{603869969\,\xi^8}{338688000\,h^4} + \frac{297351533\,\xi^9}{609638400\,h^5}$$
$$- \frac{1811075923\,\xi^{10}}{18289152000\,h^6} + \frac{218600713\,\xi^{11}}{14370048000\,h^7} - \frac{4306835671\,\xi^{12}}{2414168064000\,h^8} + \frac{557256047\,\xi^{13}}{3487131648000\,h^9}$$
$$- \frac{2522719\,\xi^{14}}{232475443200\,h^{10}} + \frac{174023\,\xi^{15}}{317011968000\,h^{11}} - \frac{25411\,\xi^{16}}{1268047872000\,h^{12}} + \frac{827\,\xi^{17}}{1658216448000\,h^{13}}$$
$$- \frac{113\,\xi^{18}}{14923948032000\,h^{14}} + \frac{\xi^{19}}{18903667507200\,h^{15}} \tag{21}$$

$$\beta_8 = -\frac{429\,\xi^5}{64\,h} + \frac{143839\,\xi^6}{20160\,h^2} - \frac{108870653\,\xi^7}{26342400\,h^3} + \frac{33726391\,\xi^8}{21168000\,h^4} - \frac{178809037\,\xi^9}{406425600\,h^5}$$
$$+ \frac{825125941\,\xi^{10}}{9144576000\,h^6} - \frac{460176289\,\xi^{11}}{32845824000\,h^7} + \frac{2237897\,\xi^{12}}{1347192000\,h^8} - \frac{131166143\,\xi^{13}}{871782912000\,h^9}$$
$$+ \frac{75043\,\xi^{14}}{7264857600\,h^{10}} - \frac{83719\,\xi^{15}}{158505984000\,h^{11}} + \frac{193\,\xi^{16}}{9906624000\,h^{12}} - \frac{29\,\xi^{17}}{59222016000\,h^{13}}$$
$$+ \frac{\xi^{18}}{133249536000\,h^{14}} - \frac{\xi^{19}}{18903667507200\,h^{15}} \tag{22}$$

$$\beta_9 = \frac{1001\,\xi^5}{216\,h} - \frac{128413\,\xi^6}{25920\,h^2} + \frac{109950473\,\xi^7}{38102400\,h^3} - \frac{3413334089\,\xi^8}{3048192000\,h^4} + \frac{15383980811\,\xi^9}{49380710400\,h^5}$$
$$- \frac{3534853769\,\xi^{10}}{54867456000\,h^6} + \frac{652662223\,\xi^{11}}{64665216000\,h^7} - \frac{1249849571\,\xi^{12}}{1034643456000\,h^8} + \frac{495342143\,\xi^{13}}{4483454976000\,h^9}$$
$$- \frac{1780879\,\xi^{14}}{232475443200\,h^{10}} + \frac{3386083\,\xi^{15}}{8559323136000\,h^{11}} - \frac{56057\,\xi^{16}}{3804143616000\,h^{12}} + \frac{5581\,\xi^{17}}{14923948032000\,h^{13}}$$
$$- \frac{37\,\xi^{18}}{6395977728000\,h^{14}} + \frac{\xi^{19}}{24304715366400\,h^{15}} \tag{23}$$

$$\beta_{10} = -\frac{1001\,\xi^5}{400\,h} + \frac{1159721\,\xi^6}{432000\,h^2} - \frac{13852667\,\xi^7}{8820000\,h^3} + \frac{249027661\,\xi^8}{406425600\,h^4} - \frac{1254757531\,\xi^9}{7315660800\,h^5}$$
$$+ \frac{1567631503\,\xi^{10}}{43893964800\,h^6} - \frac{19442163553\,\xi^{11}}{3448811520000\,h^7} + \frac{17594483\,\xi^{12}}{25866086400\,h^8} - \frac{23443657\,\xi^{13}}{373621248000\,h^9}$$
$$+ \frac{765371\,\xi^{14}}{174356582400\,h^{10}} - \frac{543937\,\xi^{15}}{2377589760000\,h^{11}} + \frac{409\,\xi^{16}}{47551795200\,h^{12}} - \frac{137\,\xi^{17}}{6221831168000\,h^{13}}$$
$$+ \frac{11\,\xi^{18}}{3197988864000\,h^{14}} - \frac{\xi^{19}}{40507858944000\,h^{15}} \tag{24}$$

$$\beta_{11} = \frac{91\,\xi^5}{88\,h} - \frac{105727\,\xi^6}{95040\,h^2} + \frac{10139863\,\xi^7}{15523200\,h^3} - \frac{2860697731\,\xi^8}{11176704000\,h^4} + \frac{160975931\,\xi^9}{2235340800\,h^5}$$
$$- \frac{827528203\,\xi^{10}}{54867456000\,h^6} + \frac{103295831\,\xi^{11}}{43110144000\,h^7} - \frac{301248457\,\xi^{12}}{1034643456000\,h^8} + \frac{40445413\,\xi^{13}}{1494484992000\,h^9}$$
$$- \frac{1331119\,\xi^{14}}{697426329600\,h^{10}} + \frac{1049441\,\xi^{15}}{10461394944000\,h^{11}} - \frac{159209\,\xi^{16}}{41845579776000\,h^{12}} + \frac{5381\,\xi^{17}}{54721142784000\,h^{13}}$$
$$- \frac{109\,\xi^{18}}{70355755008000\,h^{14}} + \frac{\xi^{19}}{89117289676800\,h^{15}} \tag{25}$$

$$\beta_{12} = -\frac{91\,\xi^5}{288\,h} + \frac{1165727\,\xi^6}{3421440\,h^2} - \frac{672835853\,\xi^7}{3353011200\,h^3} + \frac{3969230807\,\xi^8}{50295168000\,h^4} - \frac{24226741543\,\xi^9}{1086375628800\,h^5}$$
$$+ \frac{515061979\,\xi^{10}}{109734912000\,h^6} - \frac{1551629011\,\xi^{11}}{2069286912000\,h^7} + \frac{1976179\,\xi^{12}}{21555072000\,h^8} - \frac{9616213\,\xi^{13}}{1120863744000\,h^9}$$
$$+ \frac{14491\,\xi^{14}}{23775897600\,h^{10}} - \frac{138169\,\xi^{15}}{4279661568000\,h^{11}} + \frac{3229\,\xi^{16}}{2615348736000\,h^{12}} - \frac{1321\,\xi^{17}}{41040857088000\,h^{13}}$$
$$+ \frac{\xi^{18}}{1954326528000\,h^{14}} - \frac{\xi^{19}}{267351869030400\,h^{15}} \tag{26}$$

$$\beta_{13} = \frac{7\,\xi^5}{104\,h} - \frac{89849\,\xi^6}{1235520\,h^2} + \frac{8665381\,\xi^7}{201801600\,h^3} - \frac{189340169\,\xi^8}{11176704000\,h^4} + \frac{96662087\,\xi^9}{20118067200\,h^5}$$
$$- \frac{55723553\,\xi^{10}}{54867456000\,h^6} + \frac{1757123\,\xi^{11}}{10777536000\,h^7} - \frac{20736923\,\xi^{12}}{1034643456000\,h^8} + \frac{402847\,\xi^{13}}{213497856000\,h^9}$$
$$- \frac{94109\,\xi^{14}}{697426329600\,h^{10}} + \frac{6847\,\xi^{15}}{951035904000\,h^{11}} - \frac{11611\,\xi^{16}}{41845579776000\,h^{12}} + \frac{5189\,\xi^{17}}{711374856192000\,h^{13}}$$
$$- \frac{107\,\xi^{18}}{914624815104000\,h^{14}} + \frac{\xi^{19}}{1158524765798400\,h^{15}} \tag{27}$$

$$\beta_{14} = -\frac{\xi^5}{112\,h} + \frac{1170017\,\xi^6}{121080960\,h^2} - \frac{8077807\,\xi^7}{1412611200\,h^3} + \frac{50565173\,\xi^8}{22353408000\,h^4} - \frac{17264887\,\xi^9}{26824089600\,h^5}$$
$$+ \frac{29969773\,\xi^{10}}{219469824000\,h^6} - \frac{15185081\,\xi^{11}}{689762304000\,h^7} + \frac{2461967\,\xi^{12}}{905313024000\,h^8} - \frac{673219\,\xi^{13}}{2615348736000\,h^9}$$
$$+ \frac{3229\,\xi^{14}}{174356582400\,h^{10}} - \frac{473\,\xi^{15}}{475517952000\,h^{11}} + \frac{101\,\xi^{16}}{2615348736000\,h^{12}} - \frac{\xi^{17}}{977163264000\,h^{13}}$$
$$+ \frac{53\,\xi^{18}}{3201186852864000\,h^{14}} - \frac{\xi^{19}}{8109673360588800\,h^{15}} \tag{28}$$

$$\beta_{15} = \frac{\xi^5}{1800\,h} - \frac{1171733\,\xi^6}{1945944000\,h^2} + \frac{30946717\,\xi^7}{86682960000\,h^3} - \frac{406841\,\xi^8}{2874009600\,h^4} + \frac{21939781\,\xi^9}{543187814400\,h^5}$$
$$- \frac{3209\,\xi^{10}}{373248000\,h^6} + \frac{899683\,\xi^{11}}{646652160000\,h^7} - \frac{3247\,\xi^{12}}{18811699200\,h^8} + \frac{515261\,\xi^{13}}{31384184832000\,h^9}$$
$$- \frac{71\,\xi^{14}}{59779399680\,h^{10}} + \frac{2747\,\xi^{15}}{42796615680000\,h^{11}} - \frac{\xi^{16}}{398529331200\,h^{12}} + \frac{\xi^{17}}{14923948032000\,h^{13}}$$
$$- \frac{\xi^{18}}{914624815104000\,h^{14}} + \frac{\xi^{19}}{1216451004408832000\,h^{15}} \tag{29}$$

Substituting equation (10) – (29) into equation (9), interpolating and collocating at x_{n+j}, $j = 0, 1; 0, 2; 0, 3; ..., 0, k_¡$ and x_{n+j}, $j = 0(1)k$ gives the solution to equation (1) and is given explicitly below:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + \frac{1}{6}h^3 y'''_n + \frac{27312614539002931}{1161157776629760000}h^4 f_n + \frac{215021456509297}{3547982095257600}h^4 f_{n+1}$$
$$-\frac{2405950254864953}{13516122267648000}h^4 f_{n+2} + \frac{228535928736331}{465478700544000}h^4 f_{n+3} - \frac{43830916431996773}{40548366802944000}h^4 f_{n+4}$$
$$+\frac{1983716565322187}{1055947052160000}h^4 f_{n+5} - \frac{188870621369290423}{72987060245299200}h^4 f_{n+6} + \frac{16765512493838707}{5913303492096000}h^4 f_{n+7}$$
$$-\frac{77901096585757121}{31537618624512000}h^4 f_{n+8} + \frac{388205122274959 19}{22808456326656000}h^4 f_{n+9} - \frac{26549037149067547}{28963119144960000}h^4 f_{n+10}$$
$$+\frac{63857023621987}{168951528345600}h^4 f_{n+11} - \frac{42069828441973351}{364935301226496000}h^4 f_{n+12} + \frac{31069509811453}{1267136462592000}h^4 f_{n+13}$$
$$-\frac{307336703482477}{94612855873536000}h^4 f_{n+14} + \frac{14651877357209}{72572361039360000}h^4 f_{n+15} \quad (30)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + \frac{4}{3}h^3 y'''_n + \frac{776157610312243}{3118343638410000}h^4 f_n + \frac{12110365177663}{11549420883000}h^4 f_{n+1}$$
$$-\frac{320666666657}{123743795175}h^4 f_{n+2} + \frac{313723318722181}{44547766263000}h^4 f_{n+3} - \frac{7257112564561}{471404934000}h^4 f_{n+4}$$
$$+\frac{659444520473071}{24748759035000}h^4 f_{n+5} - \frac{815371875214873}{22273883131500}h^4 f_{n+6} + \frac{30824521197751}{769961392200}h^4 f_{n+7}$$
$$-\frac{804553592927429}{23098841766000}h^4 f_{n+8} + \frac{1068040546872449}{44547766263000}h^4 f_{n+9} - \frac{26608028437433}{2062396586250}h^4 f_{n+10}$$
$$+\frac{537006152783}{101015343000}h^4 f_{n+11} - \frac{2625137049893}{1619918773200}h^4 f_{n+12} + \frac{568425874883}{1649917269000}h^4 f_{n+13}$$
$$-\frac{790392890503}{17324131324500}h^4 f_{n+14} + \frac{4419708470941}{1559171819205000}h^4 f_{n+15} \quad (31)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + \frac{9}{2}h^3 y'''_n + \frac{602530652428169}{648922193920000}h^4 f_n + \frac{19434314988567}{4055763712000}h^4 f_{n+1}$$
$$-\frac{189942041947311}{18540634112000}h^4 f_{n+2} + \frac{63328896091}{2228441600}h^4 f_{n+3} - \frac{1152715866154893}{18540634112000}h^4 f_{n+4}$$
$$+\frac{311770415766357}{2896974080000}h^4 f_{n+5} - \frac{2741097040974839}{18540634112000}h^4 f_{n+6} + \frac{1311392773004949}{8111527424000}h^4 f_{n+7}$$
$$-\frac{3650761245166821}{25956887756800}h^4 f_{n+8} + \frac{56087920654093}{579394816000}h^4 f_{n+9} - \frac{4828796962028649}{92703170560000}h^4 f_{n+10}$$
$$+\frac{12434976965421}{579394816000}h^4 f_{n+11} - \frac{121294082904151}{18540634112000}h^4 f_{n+12} + \frac{14650769187}{10534451200}h^4 f_{n+13}$$
$$-\frac{23901979612131}{129784438784000}h^4 f_{n+14} + \frac{464062820041}{40557637120000}h^4 f_{n+15}$$

$$(32)$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + \frac{32}{3}h^3 y'''_n + \frac{451207826917664}{194896477400625}h^4 f_n + \frac{57491991503264}{4331032831125}h^4 f_{n+1}$$
$$-\frac{5241628601168}{206239658625}h^4 f_{n+2} + \frac{58358490710752}{795495826125}h^4 f_{n+3} - \frac{1525291110688}{9518753475}h^4 f_{n+4}$$
$$+\frac{286110321363872}{1031198293125}h^4 f_{n+5} - \frac{303253476312368}{795495826125}h^4 f_{n+6} + \frac{601876165585312}{1443677610375}h^4 f_{n+7}$$
$$-\frac{174546287711456}{481225870125}h^4 f_{n+8} + \frac{278040058459552}{1113694156575}h^4 f_{n+9} - \frac{59370049162288}{441942125625}h^4 f_{n+10}$$
$$+\frac{11415845225504}{206239658625}h^4 f_{n+11} - \frac{93955520375296}{5568470782875}h^4 f_{n+12} + \frac{317044076768}{88388425125}h^4 f_{n+13}$$
$$-\frac{137148361136}{288735522075}h^4 f_{n+14} + \frac{5751617237536}{194896477400625}h^4 f_{n+15} \quad (33)$$

$$y_{n+5} = y_n + 5\,hy'_n + \frac{25}{2}\,h^2 y''_n + \frac{125}{6}\,h^3 y'''_n + \frac{95264979269192125}{20436376868683776}\,h^4 f_n$$
$$+ \frac{168412149190625}{5913303492096}\,h^4 f_{n+1} - \frac{16289893064271875}{324386934423552}\,h^4 f_{n+2} + \frac{2509888032209375}{16587968237568}\,h^4 f_{n+3}$$
$$- \frac{35503567768034375}{108128978141184}\,h^4 f_{n+4} + \frac{80665948618375}{141777506304}\,h^4 f_{n+5} - \frac{2282582909693346875}{2919482409811968}\,h^4 f_{n+6}$$
$$+ \frac{13484181814896875}{15768809312256}\,h^4 f_{n+7} - \frac{563136843495371875}{756902846988288}\,h^4 f_{n+8} + \frac{46722464671271875}{91233825306624}\,h^4 f_{n+9}$$
$$- \frac{29797815959624875}{108128978141184}\,h^4 f_{n+10} + \frac{2302123472646875}{20274183401472}\,h^4 f_{n+11} - \frac{101052862907884375}{2919482409811968}\,h^4 f_{n+12}$$
$$+ \frac{3825296084375}{519850856448}\,h^4 f_{n+13} - \frac{2212679175659375}{2270708540964864}\,h^4 f_{n+14} + \frac{7029876481375}{116115777662976}\,h^4 f_{n+15} \quad (34)$$

$$y_{n+6} = y_n + 6\,hy'_n + 18\,h^2 y''_n + 36\,h^3 y'''_n + \frac{1302374156467}{158428270000}\,h^4 f_n + \frac{166010903703}{3168565400}\,h^4 f_{n+1}$$
$$- \frac{98189047503}{1131630500}\,h^4 f_{n+2} + \frac{615303609259}{2263261000}\,h^4 f_{n+3} - \frac{2646014872329}{4526522000}\,h^4 f_{n+4}$$
$$+ \frac{11487075212241}{11316305000}\,h^4 f_{n+5} - \frac{1103785049}{791350}\,h^4 f_{n+6} + \frac{24170242181949}{15842827000}\,h^4 f_{n+7}$$
$$- \frac{42060841828767}{31685654000}\,h^4 f_{n+8} + \frac{2068042917719}{2263261000}\,h^4 f_{n+9} - \frac{2782160441883}{5658152500}\,h^4 f_{n+10}$$
$$+ \frac{91711309509}{452652200}\,h^4 f_{n+11} - \frac{279567458927}{4526522000}\,h^4 f_{n+12}$$
$$+ \frac{29717015319}{2263261000}\,h^4 f_{n+13} - \frac{3443405418}{1980353375}\,h^4 f_{n+14} + \frac{8557581379}{79214135000}\,h^4 f_{n+15} \quad (35)$$

$$y_{n+7} = y_n + 7\,hy'_n + \frac{49}{2}\,h^2 y''_n + \frac{343}{6}\,h^3 y'''_n + \frac{493224413054950241}{37238296043520000}\,h^4 f_n$$
$$+ \frac{102260630103289}{1175451264000}\,h^4 f_{n+1} - \frac{7568023997612911}{55167845990400}\,h^4 f_{n+2} + \frac{206970176772057433}{4654787700544000}\,h^4 f_{n+3}$$
$$- \frac{783475292841925019\,h^4 f_{n+4}}{827517689856000}\,h^4 f_{n+4} + \frac{8886369763068371}{5387484960000}\,h^4 f_{n+5}$$
$$- \frac{16869125122720872113}{7447659208704000}\,h^4 f_{n+6} + \frac{29871704176307}{12055910400}\,h^4 f_{n+7} - \frac{1982337311331162081}{91946409984000}\,h^4 f_{n+8}$$
$$+ \frac{43176101394439517}{29092418784000}\,h^4 f_{n+9} - \frac{330447014263845703}{4137588449280000}\,h^4 f_{n+10} + \frac{5673457014546587}{17239951872000}\,h^4 f_{n+11}$$
$$- \frac{149427441213124277}{1489531841740800}\,h^4 f_{n+12} + \frac{275759583050483}{12929963904000}\,h^4 f_{n+13} - \frac{59927250442381}{21218402304000}\,h^4 f_{n+14}$$
$$+ \frac{290057854303}{1652978340000}\,h^4 f_{n+15} \quad (36)$$

$$y_{n+8} = y_n + 8\,hy'_n + 32\,h^2 y''_n + \frac{256}{3}\,h^3 y'''_n + \frac{3895558043809792}{194896477400625}\,h^4 f_n$$
$$+ \frac{193850692468736}{1443677610375}\,h^4 f_{n+1} - \frac{125956008558592}{618718975875}\,h^4 f_{n+2} + \frac{108065421131776}{159099165225}\,h^4 f_{n+3}$$
$$- \frac{295522157750272}{206239658625}\,h^4 f_{n+4} + \frac{7751983865397248}{3093594879375}\,h^4 f_{n+5} - \frac{19137382555205632}{5568470782875}\,h^4 f_{n+6}$$
$$+ \frac{1809910071427072}{481225870125}\,h^4 f_{n+7} - \frac{6607665740288}{2019129525}\,h^4 f_{n+8} + \frac{12544381621239808}{5568470782875}\,h^4 f_{n+9}$$
$$- \frac{25512731459584}{21044863125}\,h^4 f_{n+10} + \frac{309076971618304}{618718975875}\,h^4 f_{n+11} - \frac{847973274318848}{5568470782875}\,h^4 f_{n+12}$$
$$+ \frac{1335378968576}{41247931725}\,h^4 f_{n+13} - \frac{18568423555072}{4331032831125}\,h^4 f_{n+14} + \frac{51915343659008}{194896477400625}\,h^4 f_{n+15} \quad (37)$$

$$y_{n+9} = y_n + 9\,hy'_n + \frac{81}{2}\,h^2 y''_n + \frac{243}{2}\,h^3 y'''_n + \frac{18625133626628643}{648922193920000}\,h^4 f_n$$

$$+ \frac{1591587721573497}{8111527424000}\,h^4 f_{n+1} - \frac{5340901247722377}{18540634112000}\,h^4 f_{n+2} + \frac{1141699342205313}{1158789632000}\,h^4 f_{n+3}$$

$$- \frac{1091696657937021}{529732403200}\,h^4 f_{n+4} + \frac{10481199823834029}{2896974080000}\,h^4 f_{n+5} - \frac{91850956406711577}{18540634112000}\,h^4 f_{n+6}$$

$$+ \frac{44009977111307523}{8111527424000}\,h^4 f_{n+7} - \frac{612567442391345667}{129784438784000}\,h^4 f_{n+8} + \frac{5266011862773}{1620684800}\,h^4 f_{n+9}$$

$$- \frac{162099315257981103}{92703170560000}\,h^4 f_{n+10} + \frac{119277892927593}{165541376000}\,h^4 f_{n+11} - \frac{4072438241603397}{18540634112000}\,h^4 f_{n+12}$$

$$+ \frac{211374969363}{4526522000}\,h^4 f_{n+13} - \frac{14592591850059}{2359717068800}\,h^4 f_{n+14} + \frac{15583153071177}{40557637120000}\,h^4 f_{n+15} \quad (38)$$

$$y_{n+10} = y_n + 10\,hy'_n + 50\,h^2 y''_n + \frac{500}{3}\,h^3 y'''_n + \frac{197774498840375}{4989349821456}\,h^4 f_n$$

$$+ \frac{76168854565625}{277186101192}\,h^4 f_{n+1} - \frac{1296060059375}{3299834538}\,h^4 f_{n+2} + \frac{44460274821875}{32398375464}\,h^4 f_{n+3}$$

$$- \frac{20508142934375}{7199638992}\,h^4 f_{n+4} + \frac{66257006246125}{13199338152}\,h^4 f_{n+5} - \frac{1222568114959375}{178191065052}\,h^4 f_{n+6}$$

$$+ \frac{694980202403125}{92395367064}\,h^4 f_{n+7} - \frac{402852494434375}{61596911376}\,h^4 f_{n+8} + \frac{1604910232878125}{356382130104}\,h^4 f_{n+9}$$

$$- \frac{83883071750}{34613649}\,h^4 f_{n+10} + \frac{13181028190625}{13199338152}\,h^4 f_{n+11} - \frac{216981310746875}{712764260208}\,h^4 f_{n+12}$$

$$+ \frac{2562778803125}{39598014456}\,h^4 f_{n+13} - \frac{395952809375}{46197683532}\,h^4 f_{n+14} + \frac{1328461659125}{2494674910728}\,h^4 f_{n+15} \quad (39)$$

$$y_{n+11} = y_n + 11\,hy'_n + \frac{121}{2}\,h^2 y''_n + \frac{1331}{6}\,h^3 y'''_n + \frac{61603729660715825561}{11611157776629760000}\,h^4 f_n$$

$$+ \frac{19997710920159577}{53757304473600}\,h^4 f_{n+1} - \frac{19160068546902768599}{3686215163904000}\,h^4 f_{n+2}$$

$$+ \frac{39142139354400203}{21158122752000}\,h^4 f_{n+3} - \frac{4686113275972795271}{1228738387968000}\,h^4 f_{n+4}$$

$$+ \frac{3884821540842611237}{575971119360000}\,h^4 f_{n+5} - \frac{8722148090842645789}{947883899289600}\,h^4 f_{n+6}$$

$$+ \frac{1809824212823561759}{179191014912000}\,h^4 f_{n+7} - \frac{7547732019302950763}{8601168715776000}\,h^4 f_{n+8}$$

$$+ \frac{6267565444696350157}{1036748014848000}\,h^4 f_{n+9} - \frac{2854635393056124029}{877670277120000}\,h^4 f_{n+10}$$

$$+ \frac{1187685734092033}{886109414400}\,h^4 f_{n+11} - \frac{1355561827910396151}{33175936475136000}\,h^4 f_{n+12}$$

$$+ \frac{68072706364399}{783634176000}\,h^4 f_{n+13} - \frac{22834016763745847}{1984885088256000}\,h^4 f_{n+14}$$

$$+ \frac{51871823638164079}{72572361039360000}\,h^4 f_{n+15} \quad (40)$$

$$y_{n+12} = y_n + 12\,hy'_n + 72\,h^2 y''_n + 288\,h^3 y'''_n + \frac{62288947936}{900160625}\,h^4 f_n + \frac{970010779104}{1980353375}\,h^4 f_{n+1}$$

$$- \frac{37981121616}{56581525}\,h^4 f_{n+2} + \frac{686872050784}{282907625}\,h^4 f_{n+3} - \frac{1407334462848}{282907625}\,h^4 f_{n+4}$$

$$+ \frac{12486203584416}{1414538125}\,h^4 f_{n+5} - \frac{3400616961584}{282907625}\,h^4 f_{n+6} + \frac{5232260492064}{396070675}\,h^4 f_{n+7}$$

$$- \frac{22708918277856}{1980353375}\,h^4 f_{n+8} + \frac{2236670113376}{282907625}\,h^4 f_{n+9} - \frac{6013168148016}{1414538125}\,h^4 f_{n+10}$$

$$+ \frac{495777441312}{282907625}\,h^4 f_{n+11} - \frac{2324971232}{4352425}\,h^4 f_{n+12} + \frac{32128912224}{282907625}\,h^4 f_{n+13}$$

$$- \frac{29783960496}{1980353375}\,h^4 f_{n+14} + \frac{841151264}{900160625}\,h^4 f_{n+15} \quad (41)$$

$$y_{n+13} = y_n + 13\,hy'_n + \frac{169}{2}\,h^2 y''_n + \frac{2197}{6}\,h^3 y'''_n + \frac{86779718953850369369}{9825181118686720000}\,h^4 f_n$$

$$+ \frac{430007608906153603}{682304249088000}\,h^4 f_{n+1} - \frac{883011992382639353}{1039701712896000}\,h^4 f_{n+2}$$

$$+ \frac{1093285607607322649}{350899328102400}\,h^4 f_{n+3} - \frac{19802369214482134397}{3119105138688000}\,h^4 f_{n+4}$$

$$+ \frac{3671517181341961757}{324906785280000}\,h^4 f_{n+5} - \frac{431204822493929017859}{28071946248192000}\,h^4 f_{n+6}$$

$$+ \frac{7688899400227183177}{454869499392000}\,h^4 f_{n+7} - \frac{7111838608315340341}{485194132684800}\,h^4 f_{n+8}$$

$$+ \frac{4437011288736058799}{438624160128000}\,h^4 f_{n+9} - \frac{84741976626999074461}{15595525693440000}\,h^4 f_{n+10}$$

$$+ \frac{145708237149829601}{64981357056000}\,h^4 f_{n+11} - \frac{1743000544456694861}{2551995113472000}\,h^4 f_{n+12}$$

$$+ \frac{5662729165476331}{38988814233600}\,h^4 f_{n+13} - \frac{139985003179042333}{7277911990272000}\,h^4 f_{n+14}$$

$$+ \frac{73385695962908641}{61407382417920000}\,h^4 f_{n+15} \quad (42)$$

$$y_{n+14} = y_n + 14\,hy'_n + 98\,h^2 y''_n + \frac{1372}{3}\,h^3 y'''_n + \frac{1006280506843003}{9091380870000}\,h^4 f_n$$

$$+ \frac{26776412350933}{33671781000}\,h^4 f_{n+1} - \frac{53335962667519}{50507671500}\,h^4 f_{n+2} + \frac{3566435834783353}{909138087000}\,h^4 f_{n+3}$$

$$- \frac{107133265333577}{13468712400}\,h^4 f_{n+4} + \frac{7171100242899187}{5050076715000}\,h^4 f_{n+5} - \frac{2189555205249664}{113642260875}\,h^4 f_{n+6}$$

$$+ \frac{238278143310809}{11223927000}\,h^4 f_{n+7} - \frac{1238370377602421}{67343562000}\,h^4 f_{n+8} + \frac{2310124038552169}{181827617400}\,h^4 f_{n+9}$$

$$- \frac{573731688450277}{84179452500}\,h^4 f_{n+10} + \frac{284604385256219}{101015343000}\,h^4 f_{n+11} - \frac{1556780681533049}{1818276174000}\,h^4 f_{n+12}$$

$$+ \frac{558564964069}{3061071000}\,h^4 f_{n+13} - \frac{121946014477}{5050767150}\,h^4 f_{n+14} + \frac{6819244114711}{4545690435000}\,h^4 f_{n+15} \quad (43)$$

$$y_{n+15} = y_n + 15\,hy'_n + \frac{225}{2}\,h^2 y''_n + \frac{1125}{2}\,h^3 y'''_n + \frac{141760259867125}{1038275510272}\,h^4 f_n$$
$$+ \frac{16008433396875}{16223054848}\,h^4 f_{n+1} - \frac{191857931184375}{148325072896}\,h^4 f_{n+2} + \frac{6435135446875}{1324331008}\,h^4 f_{n+3}$$
$$- \frac{1454924082628125}{148325072896}\,h^4 f_{n+4} + \frac{1271375605125}{72424352}\,h^4 f_{n+5} - \frac{3527747563259375}{148325072896}\,h^4 f_{n+6}$$
$$+ \frac{212840309896875}{8111527424}\,h^4 f_{n+7} - \frac{23572142846465625}{1038275510272}\,h^4 f_{n+8} + \frac{1137431715625}{72424352}\,h^4 f_{n+9}$$
$$- \frac{178243328212875}{21189296128}\,h^4 f_{n+10} + \frac{32313630965625}{9270317056}\,h^4 f_{n+11} - \frac{156469002934375}{148325072896}\,h^4 f_{n+12}$$
$$+ \frac{525121903125}{2317579264}\,h^4 f_{n+13} - \frac{30805101646875}{1038275510272}\,h^4 f_{n+14} + \frac{60144050875}{32446109696}\,h^4 f_{n+15} \quad (44)$$

Thus, equation (30) – (44) is the new method named (NS4O15M). Therefore, we take the first derivative of equation (9) and evaluate at x_{n+j}, $j = 0(1)k$ to have:

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}\,h^2 y'''_n + \frac{2646946687904237}{32011868528640000}\,h^3 f_n$$
$$+ \frac{9331210633373}{35568742809600}\,h^3 f_{n+1} - \frac{525466618744679}{711374856192000}\,h^3 f_{n+2} + \frac{11892772256669}{5884534656000}\,h^3 f_{n+3}$$
$$- \frac{50067223151677}{11291664384000}\,h^3 f_{n+4} + \frac{6838723295804269}{889218570240000}\,h^3 f_{n+5} - \frac{13547754274847623}{1280474741145600}\,h^3 f_{n+6}$$
$$+ \frac{6130273402717}{529296768000}\,h^3 f_{n+7} - \frac{2391130153285807}{237124952064000}\,h^3 f_{n+8} + \frac{11115802946972267}{1600593426432000}\,h^3 f_{n+9}$$
$$- \frac{13298258493701081}{3556874280960000}\,h^3 f_{n+10} + \frac{195767508553}{127031224320}\,h^3 f_{n+11} - \frac{3008598640512121}{6402373705728000}\,h^3 f_{n+12}$$
$$+ \frac{5923791011249}{59281238016000}\,h^3 f_{n+13} - \frac{855970682491}{64670441472000}\,h^3 f_{n+14} + \frac{822811814369}{1000370891520000}\,h^3 f_{n+15} \quad (45)$$

$$y'_{n+2} = y'_n + 2\,hy''_n + 2\,h^2 y'''_n + \frac{6414418122631}{15630795180000}\,h^3 f_n + \frac{176725029073}{86837751000}\,h^3 f_{n+1}$$
$$- \frac{653662511}{141775920}\,h^3 f_{n+2} + \frac{9816667434881}{781539759000}\,h^3 f_{n+3} - \frac{866478322811}{31577364000}\,h^3 f_{n+4}$$
$$+ \frac{404117238667}{8513505000}\,h^3 f_{n+5} - \frac{101892006563273}{1563079518000}\,h^3 f_{n+6} + \frac{412585117387}{5789183400}\,h^3 f_{n+7}$$
$$- \frac{7177626750949}{115783668000}\,h^3 f_{n+8} + \frac{680466388363}{15949791000}\,h^3 f_{n+9} - \frac{19933118316169}{868377510000}\,h^3 f_{n+10}$$
$$+ \frac{821247749563}{86837751000}\,h^3 f_{n+11} - \frac{1802310446791}{625231807200}\,h^3 f_{n+12} + \frac{53212387759}{86837751000}\,h^3 f_{n+13}$$
$$- \frac{1565842507}{19297278000}\,h^3 f_{n+14} + \frac{19699416731}{3907698795000}\,h^3 f_{n+15} \quad (46)$$

$$y'_{n+3} = y'_n + 3\,hy''_n + \frac{9}{2}\,h^2 y'''_n + \frac{690720887139}{697016320000}\,h^3 f_n + \frac{32059975167}{5544448000}\,h^3 f_{n+1}$$
$$- \frac{981108537597}{88711168000}\,h^3 f_{n+2} + \frac{16959465003}{536166400}\,h^3 f_{n+3} - \frac{67574744254179}{975822848000}\,h^3 f_{n+4}$$
$$+ \frac{746266958199}{6223360000}\,h^3 f_{n+5} - \frac{160777027856499}{975822848000}\,h^3 f_{n+6} + \frac{2585730381813}{14350336000}\,h^3 f_{n+7}$$
$$- \frac{30594531344121}{195164569600}\,h^3 f_{n+8} + \frac{6580669587693}{60988928000}\,h^3 f_{n+9} - \frac{40468694141943}{697016320000}\,h^3 f_{n+10}$$
$$+ \frac{5836036102983}{243955712000}\,h^3 f_{n+11} - \frac{1016547199431}{139403264000}\,h^3 f_{n+12} + \frac{18909103959}{12197785600}\,h^3 f_{n+13}$$
$$- \frac{200320234209}{975822848000}\,h^3 f_{n+14} + \frac{15557090403}{1219778560000}\,h^3 f_{n+15} \quad (47)$$

$$y'_{n+4} = y'_n + 4\,hy''_n + 8\,h^2 y'''_n + \frac{891141341312}{488462349375}\,h^3 f_n + \frac{5951251456}{516891375}\,h^3 f_{n+1}$$

$$-\frac{395392552}{20138625}\,h^3 f_{n+2} + \frac{5846193073216}{97692469875}\,h^3 f_{n+3} - \frac{443144332}{3408075}\,h^3 f_{n+4}$$

$$+\frac{12223798950016}{54273594375}\,h^3 f_{n+5} - \frac{30236186795768}{97692469875}\,h^3 f_{n+6} + \frac{1224829520192}{3618239625}\,h^3 f_{n+7}$$

$$-\frac{1065684738736}{3618239625}\,h^3 f_{n+8} + \frac{6657405184}{32837805}\,h^3 f_{n+9} - \frac{657887719304}{6030399375}\,h^3 f_{n+10}$$

$$+\frac{9958015424}{221524875}\,h^3 f_{n+11} - \frac{1338659514532}{97692469875}\,h^3 f_{n+12} + \frac{31620742784}{10854718875}\,h^3 f_{n+13}$$

$$-\frac{837478216}{2170943775}\,h^3 f_{n+14} + \frac{900559936}{37574026875}\,h^3 f_{n+15} \quad (48)$$

$$y'_{n+5} = y'_n + 5\,hy''_n + \frac{25}{2}\,h^2 y'''_n + \frac{149078112485525}{51218989645824}\,h^3 f_n + \frac{27357607800625}{1422749712384}\,f_{n+1}$$

$$-\frac{57524077045625}{1896999616512}\,h^3 f_{n+2} + \frac{9768839123125}{100037089152}\,h^3 f_{n+3} - \frac{108220654263125}{517363531776}\,h^3 f_{n+4}$$

$$+\frac{516319144825}{1421328384}\,h^3 f_{n+5} - \frac{3653176292400625}{7316998520832}\,h^3 f_{n+6} + \frac{32375202664375}{59281238016}\,h^3 f_{n+7}$$

$$-\frac{901457665911875}{1896999616512}\,h^3 f_{n+8} + \frac{4188606865466875}{12804747411456}\,h^3 f_{n+9} - \frac{1001792026947025}{5690998849536}\,h^3 f_{n+10}$$

$$+\frac{252938010625}{3487131648}\,h^3 f_{n+11} - \frac{14708026245625}{665181683712}\,h^3 f_{n+12} + \frac{955425548125}{203249958912}\,h^3 f_{n+13}$$

$$-\frac{3542687313125}{5690998849536}\,h^3 f_{n+14} + \frac{1934553625}{50018544576}\,h^3 f_{n+15} \quad (49)$$

$$y'_{n+6} = y'_n + 6\,hy''_n + 18\,h^2 y'''_n + \frac{10124252319}{2382380000}\,h^3 f_n + \frac{689167593}{23823800}\,h^3 f_{n+1}$$

$$-\frac{10275078321}{238238000}\,h^3 f_{n+2} + \frac{2468567643}{17017000}\,h^3 f_{n+3} - \frac{145912136193}{476476000}\,h^3 f_{n+4}$$

$$+\frac{318410446077}{595595000}\,h^3 f_{n+5} - \frac{34946139}{47600}\,h^3 f_{n+6} + \frac{95665826601}{119119000}\,h^3 f_{n+7}$$

$$-\frac{47568246489}{68068000}\,h^3 f_{n+8} + \frac{57304332759}{119119000}\,h^3 f_{n+9} - \frac{6293482551}{24310000}\,h^3 f_{n+10}$$

$$+\frac{2541447099}{23823800}\,h^3 f_{n+11} - \frac{15494777187}{476476000}\,h^3 f_{n+12} + \frac{6920469}{1001000}\,h^3 f_{n+13}$$

$$-\frac{218119509}{238238000}\,h^3 f_{n+14} + \frac{33879999}{595595000}\,h^3 f_{n+15} \quad (50)$$

$$y'_{n+7} = y'_n + 7\,hy''_n + \frac{49}{2}\,h^2 y'''_n + \frac{346932898878727}{59391221760000}\,h^3 f_n$$

$$+\frac{36848690199839}{907365888000}\,h^3 f_{n+1} - \frac{168555897445747}{2903570841600}\,h^3 f_{n+2} + \frac{6601074966881017}{32665171968000}\,h^3 f_{n+3}$$

$$-\frac{555890398779737}{1319804928000}\,h^3 f_{n+4} + \frac{3356052363748651}{4536829440000}\,h^3 f_{n+5} - \frac{132440457899414987}{130660687872000}\,h^3 f_{n+6}$$

$$+\frac{625723050379}{564019200}\,h^3 f_{n+7} - \frac{4671757797600943}{4839284736000}\,h^3 f_{n+8} + \frac{5427096160617551}{8166292992000}\,h^3 f_{n+9}$$

$$-\frac{25961087688408481}{72589271040000}\,h^3 f_{n+10} + \frac{534888408252611}{3629463552000}\,h^3 f_{n+11} - \frac{234803718886513}{5226427514880}\,h^3 f_{n+12}$$

$$+\frac{787863604577}{82487808000}\,h^3 f_{n+13} - \frac{18363232338017}{14517854208000}\,h^3 f_{n+14} + \frac{755030443391}{9607403520000}\,h^3 f_{n+15} \quad (51)$$

$$y'_{n+8} = y'_n + 8\,hy''_n + 32\,h^2 y'''_n + \frac{16988198848}{2210236875}\,h^3 f_n + \frac{53560565248}{986792625}\,h^3 f_{n+1}$$

$$-\frac{815067367168}{10854718875}\,h^3 f_{n+2} + \frac{1050018913792}{3907698795}\,h^3 f_{n+3} - \frac{859150752128}{1550674125}\,h^3 f_{n+4}$$

$$+\frac{53114085282304}{54273594375}\,h^3 f_{n+5} - \frac{130629987954944}{97692469875}\,h^3 f_{n+6} + \frac{757517948416}{516891375}\,h^3 f_{n+7}$$

$$-\frac{2150503744}{1686825}\,h^3 f_{n+8} + \frac{85740073246208}{97692469875}\,h^3 f_{n+9} - \frac{25634459993344}{54273594375}\,h^3 f_{n+10}$$

$$+\frac{301807773184}{1550674125}\,h^3 f_{n+11} - \frac{5796336118912}{97692469875}\,h^3 f_{n+12} + \frac{27384514048}{2170943775}\,h^3 f_{n+13}$$

$$-\frac{18132713216}{10854718875}\,h^3 f_{n+14} + \frac{50697789952}{488462349375}\,h^3 f_{n+15} \qquad (52)$$

$$y'_{n+9} = y'_n + 9\,hy''_n + \frac{81}{2}\,h^2 y'''_n + \frac{47735374876317}{4879114240000}\,h^3 f_n$$

$$+\frac{1312259291757}{18765824000}\,h^3 f_{n+1} - \frac{10036420749}{106496000}\,h^3 f_{n+2} + \frac{10518533830593}{30494464000}\,h^3 f_{n+3}$$

$$-\frac{137554269574383}{195164569600}\,h^3 f_{n+4} + \frac{1526337505582101}{1219778560000}\,h^3 f_{n+5} - \frac{1663567405083603}{975822848000}\,h^3 f_{n+6}$$

$$+\frac{57085662261447}{30494464000}\,h^3 f_{n+7} - \frac{1587493105268517}{975822848000}\,h^3 f_{n+8} + \frac{7800788673}{6963200}\,h^3 f_{n+9}$$

$$-\frac{2941612400903409}{4879114240000}\,h^3 f_{n+10} + \frac{473504266647}{1905904000}\,h^3 f_{n+11} - \frac{73906005035817}{975822848000}\,h^3 f_{n+12}$$

$$+\frac{3928134597579}{243955712000}\,h^3 f_{n+13} - \frac{416165679981}{195164569600}\,h^3 f_{n+14} + \frac{57389067}{433160000}\,h^3 f_{n+15} \qquad (53)$$

$$y'_{n+10} = y'_n + 10\,hy''_n + 50\,h^2 y'''_n + \frac{43351425625}{3572753184}\,h^3 f_n + \frac{60830018125}{694702008}\,h^3 f_{n+1}$$

$$-\frac{12345544375}{106877232}\,h^3 f_{n+2} + \frac{384753416875}{893188296}\,h^3 f_{n+3} - \frac{47592424375}{54486432}\,h^3 f_{n+4}$$

$$+\frac{22085668025}{14177592}\,h^3 f_{n+5} - \frac{26465861018125}{12504636144}\,h^3 f_{n+6} + \frac{539270756875}{231567336}\,h^3 f_{n+7}$$

$$-\frac{1871692800625}{926269344}\,h^3 f_{n+8} + \frac{8709301139375}{6252318072}\,h^3 f_{n+9} - \frac{1040166025}{1388016}\,h^3 f_{n+10}$$

$$+\frac{214532249375}{694702008}\,h^3 f_{n+11} - \frac{336344910625}{3572753184}\,h^3 f_{n+12} + \frac{421341875}{21051576}\,h^3 f_{n+13}$$

$$-\frac{283286875}{106877232}\,h^3 f_{n+14} + \frac{1029672275}{6252318072}\,h^3 f_{n+15} \qquad (54)$$

$$y'_{n+11} = y'_n + 11\,hy''_n + \frac{121}{2}\,h^2 y'''_n + \frac{42887155137009007}{2910169866240000}\,h^3 f_n$$

$$+\frac{12377607151943}{115482931200}\,h^3 f_{n+1} - \frac{183316518915541}{1319804928000}\,h^3 f_{n+2} + \frac{76566704845774583}{145508493312000}\,h^3 f_{n+3}$$

$$-\frac{1399043605047649}{1319804928000}\,h^3 f_{n+4} + \frac{4262048618111119}{2245501440000}\,h^3 f_{n+5} - \frac{3522716671290113}{1369491701760}\,h^3 f_{n+6}$$

$$+\frac{15283394277950909}{5389203456000}\,h^3 f_{n+7} - \frac{52954438598926117}{21556813824000}\,h^3 f_{n+8} + \frac{8814989815349839}{5196731904000}\,h^3 f_{n+9}$$

$$-\frac{294676003269851291}{323352207360000}\,h^3 f_{n+10} + \frac{1907340759533}{5076172800}\,h^3 f_{n+11} - \frac{66668404334979491}{582033973248000}\,h^3 f_{n+12}$$

$$+\frac{98429973965983}{4041902592000}\,h^3 f_{n+13} - \frac{69521484528617}{21556813824000}\,h^3 f_{n+14} + \frac{11214181843469}{55964805120000}\,h^3 f_{n+15} \qquad (55)$$

$$y'_{n+12} = y'_n + 12\,hy''_n + 72\,h^2 y'''_n + \frac{1309780608}{74449375}\,h^3 f_n + \frac{1917572544}{14889875}\,h^3 f_{n+1}$$

$$-\frac{489571704}{2977975}\,h^3 f_{n+2} + \frac{9399103872}{14889875}\,h^3 f_{n+3} - \frac{18828258012}{14889875}\,h^3 f_{n+4} + \frac{24164791488}{10635625}\,h^3 f_{n+5}$$

$$-\frac{6534821544}{2127125}\,h^3 f_{n+6} + \frac{10105364736}{2977975}\,h^3 f_{n+7} - \frac{2570011632}{875875}\,h^3 f_{n+8}$$

$$+\frac{30227141184}{14889875}\,h^3 f_{n+9} - \frac{81008278056}{74449375}\,h^3 f_{n+10} + \frac{6696008064}{14889875}\,h^3 f_{n+11}$$

$$-\frac{4481292}{32725}\,h^3 f_{n+12} + \frac{61932096}{2127125}\,h^3 f_{n+13} - \frac{5219352}{1353625}\,h^3 f_{n+14} + \frac{17836032}{74449375}\,h^3 f_{n+15} \quad (56)$$

$$y'_{n+13} = y'_n + 13\,hy''_n + \frac{169}{2}\,h^2 y'''_n + \frac{50976734501772521}{2462451425280000}\,h^3 f_n$$

$$+\frac{231607232521561}{1520031744000}\,h^3 f_{n+1} - \frac{10507261454113931}{54721142784000}\,h^3 f_{n+2} + \frac{327984650433739}{439723468800}\,h^3 f_{n+3}$$

$$-\frac{7396667135771797}{4974649344000}\,h^3 f_{n+4} + \frac{183307899755534437}{68401428480000}\,h^3 f_{n+5} - \frac{254414562335962769}{70355755008000}\,h^3 f_{n+6}$$

$$+\frac{570182964741793}{142502976000}\,h^3 f_{n+7} - \frac{1800041137343969}{521153740800}\,h^3 f_{n+8} + \frac{294815320789342379}{123122571264000}\,h^3 f_{n+9}$$

$$-\frac{140128717971047}{109486080000}\,h^3 f_{n+10} + \frac{454258655878469}{855017856000}\,h^3 f_{n+11} - \frac{79225841723025829}{492490285056000}\,h^3 f_{n+12}$$

$$+\frac{2682311572421}{78173061120}\,h^3 f_{n+13} - \frac{248596418804357}{54721142784000}\,h^3 f_{n+14} + \frac{21720378795557}{76951607040000}\,h^3 f_{n+15} \quad (57)$$

$$y'_{n+14} = y'_n + 14\,hy''_n + 98\,h^2 y'''_n + \frac{7676045327671}{318995820000}\,h^3 f_n + \frac{28667764727}{161109000}\,h^3 f_{n+1}$$

$$-\frac{23816981209}{107406000}\,h^3 f_{n+2} + \frac{13878519596837}{15949791000}\,h^3 f_{n+3} - \frac{489732836761}{283551840}\,h^3 f_{n+4}$$

$$+\frac{27659304286957}{8860995000}\,h^3 f_{n+5} - \frac{134110052796167}{31899582000}\,h^3 f_{n+6} + \frac{2752397206979}{590733000}\,h^3 f_{n+7}$$

$$-\frac{9488406575333}{2362932000}\,h^3 f_{n+8} + \frac{8897984343731}{3189958200}\,h^3 f_{n+9} - \frac{26342918970319}{17721990000}\,h^3 f_{n+10}$$

$$+\frac{122149393583}{196911000}\,h^3 f_{n+11} - \frac{696997533883}{3752892000}\,h^3 f_{n+12} + \frac{71589567763}{1772199000}\,h^3 f_{n+13}$$

$$-\frac{3740727473}{708879600}\,h^3 f_{n+14} + \frac{26185092311}{79748955000}\,h^3 f_{n+15} \quad (58)$$

$$y'_{n+15} = y'_n + 15\,hy''_n + \frac{225}{2}\,h^2 y'''_n + \frac{216067472925}{7806582784}\,h^3 f_n + \frac{100262210625}{487911424}\,h^3 f_{n+1}$$

$$-\frac{179965535625}{709689344}\,h^3 f_{n+2} + \frac{1959327448125}{1951645696}\,h^3 f_{n+3} - \frac{2213858930625}{1115226112}\,h^3 f_{n+4}$$

$$+\frac{1754723396925}{487911424}\,h^3 f_{n+5} - \frac{37748275074375}{7806582784}\,h^3 f_{n+6} + \frac{1496243626875}{278806528}\,h^3 f_{n+7}$$

$$-\frac{36049237025625}{7806582784}\,h^3 f_{n+8} + \frac{1568033338125}{487911424}\,h^3 f_{n+9} - \frac{13327761065625}{7806582784}\,h^3 f_{n+10}$$

$$+\frac{199792130625}{278806528}\,h^3 f_{n+11} - \frac{1651019848125}{7806582784}\,h^3 f_{n+12} + \frac{23381533125}{487911424}\,h^3 f_{n+13}$$

$$-\frac{2568718125}{459210752}\,h^3 f_{n+14} + \frac{752826975}{1951645696}\,h^3 f_{n+15} \quad (59)$$

Similarly, the second derivative of equation (9) is found and then evaluated at x_{n+j}, $j = 0(1)k$ to give:

$$y''_{n+1} = y''_n + hy'''_n + \frac{3206819325906617}{16005934264320000} h^2 f_n$$
$$+ \frac{111956703448001}{133382785536000} h^2 f_{n+1} - \frac{780902869927541}{355687428096000} h^2 f_{n+2} + \frac{9502268460740177}{1600593426432000} h^2 f_{n+3}$$
$$- \frac{1975509756311887}{152437469184000} h^2 f_{n+4} + \frac{9965996481056899}{444609285120000} h^2 f_{n+5} - \frac{98538747853228853}{3201186852864000} h^2 f_{n+6}$$
$$+ \frac{5985224981757391}{177843714048000} h^2 f_{n+7} - \frac{3470876036880139}{118562476032000} h^2 f_{n+8} + \frac{251938134205319}{12504636144000} h^2 f_{n+9}$$
$$- \frac{57837526128808793}{5335311421440000} h^2 f_{n+10} + \frac{113475037891729}{25406244864000} h^2 f_{n+11} - \frac{4358164674336769}{3201186852864000} h^2 f_{n+12}$$
$$+ \frac{77205249783577}{266765571072000} h^2 f_{n+13} - \frac{13631425781543}{355687428096000} h^2 f_{n+14} + \frac{19055094997949}{8002967132160000} h^2 f_{n+15} \quad (60)$$

$$y''_{n+2} = y''_n + 2hy'''_n + \frac{7100763759181}{15630795180000} h^2 f_n$$
$$+ \frac{79686372907}{28945917000} h^2 f_{n+1} - \frac{198060940481}{37216179000} h^2 f_{n+2} + \frac{11595391593263}{781539759000} h^2 f_{n+3}$$
$$- \frac{3762629808143}{115783668000} h^2 f_{n+4} + \frac{73285372020391}{1302566265000} h^2 f_{n+5} - \frac{120811992265001}{1563079518000} h^2 f_{n+6}$$
$$+ \frac{815470210927}{9648639000} h^2 f_{n+7} - \frac{25538181924601}{347351004000} h^2 f_{n+8} + \frac{807085379107}{15949791000} h^2 f_{n+9}$$
$$- \frac{492563604418}{18091198125} h^2 f_{n+10} + \frac{2922359996183}{260513253000} h^2 f_{n+11} - \frac{10689180138419}{3126159036000} h^2 f_{n+12}$$
$$+ \frac{21039751529}{28945917000} h^2 f_{n+13} - \frac{4559016221}{47366046000} h^2 f_{n+14} + \frac{23367229601}{3907698795000} h^2 f_{n+15} \quad (61)$$

$$y''_{n+3} = y''_n + 3hy'''_n + \frac{1724745170011}{2439557120000} h^2 f_n + \frac{1156246932303}{243955712000} h^2 f_{n+1}$$
$$- \frac{3669218612307}{487911424000} h^2 f_{n+2} + \frac{15741501473}{670208000} h^2 f_{n+3} - \frac{25023934662201}{487911424000} h^2 f_{n+4}$$
$$+ \frac{15482577601863}{174254080000} h^2 f_{n+5} - \frac{59583820501187}{487911424000} h^2 f_{n+6} + \frac{16294185200871}{121977856000} h^2 f_{n+7}$$
$$- \frac{56712794491323}{487911424000} h^2 f_{n+8} + \frac{19519814663081}{243955712000} h^2 f_{n+9} - \frac{15006216511971}{348508160000} h^2 f_{n+10}$$
$$+ \frac{8453922669}{476476000} h^2 f_{n+11} - \frac{376990411261}{69701632000} h^2 f_{n+12} + \frac{25501122657}{22177792000} h^2 f_{n+13}$$
$$- \frac{74295242361}{487911424000} h^2 f_{n+14} + \frac{5770019551}{609889280000} h^2 f_{n+15} \quad (62)$$

$$y''_{n+4} = y''_n + 4hy'''_n + \frac{468835748086}{488462349375} h^2 f_n + \frac{31276065416}{4652022375} h^2 f_{n+1}$$
$$- \frac{237066268}{24613875} h^2 f_{n+2} + \frac{3223897961912}{97692469875} h^2 f_{n+3} - \frac{25089869444}{357847875} h^2 f_{n+4}$$
$$+ \frac{2201175147736}{18091198125} h^2 f_{n+5} - \frac{16340498219092}{97692469875} h^2 f_{n+6} + \frac{1986215722936}{10854718875} h^2 f_{n+7}$$
$$- \frac{21338011906}{134008875} h^2 f_{n+8} + \frac{1529765976712}{13956067125} h^2 f_{n+9} - \frac{9604572197884}{162820783125} h^2 f_{n+10}$$
$$+ \frac{4188081368}{172297125} h^2 f_{n+11} - \frac{723896144432}{97692469875} h^2 f_{n+12} + \frac{51299745704}{32564156625} h^2 f_{n+13}$$
$$- \frac{754844036}{3618239625} h^2 f_{n+14} + \frac{6331431944}{488462349375} h^2 f_{n+15} \quad (63)$$

$$y''_{n+5} = y''_n + 5\,hy'''_n + \frac{31054788444685}{25609494822912}\,h^2 f_n$$
$$+ \frac{6194003019125}{711374856192}\,h^2 f_{n+1} - \frac{100287550861375}{8536498274304}\,h^2 f_{n+2} + \frac{545635775073875}{12804747411456}\,h^2 f_{n+3}$$
$$- \frac{22793304846625}{258681765888}\,h^2 f_{n+4} + \frac{164744717905}{1065996288}\,h^2 f_{n+5} - \frac{776841970197875}{3658499260416}\,h^2 f_{n+6}$$
$$+ \frac{110166567234875}{474249904128}\,h^2 f_{n+7} - \frac{575215088287625}{2845499424768}\,h^2 f_{n+8} + \frac{890981273520125}{6402373705728}\,h^2 f_{n+9}$$
$$- \frac{213109965786455}{2845499424768}\,h^2 f_{n+10} + \frac{131726502560375}{4268249137152}\,h^2 f_{n+11} - \frac{3129099276125}{332590841856}\,h^2 f_{n+12}$$
$$+ \frac{3176113625}{1587890304}\,h^2 f_{n+13} - \frac{2261231861125}{8536498274304}\,h^2 f_{n+14} + \frac{210742341295}{12804747411456}\,h^2 f_{n+15} \quad (64)$$

$$y''_{n+6} = y''_n + 6\,hy'''_n + \frac{3491237903}{2382380000}\,h^2 f_n + \frac{1273496769}{119119000}\,h^2 f_{n+1}$$
$$- \frac{3302866863}{238238000}\,h^2 f_{n+2} + \frac{888545417}{17017000}\,h^2 f_{n+3} - \frac{50509176723}{476476000}\,h^2 f_{n+4} + \frac{112104615069}{595595000}\,h^2 f_{n+5}$$
$$- \frac{30620993}{119000}\,h^2 f_{n+6} + \frac{33546751263}{119119000}\,h^2 f_{n+7} - \frac{116779083507}{476476000}\,h^2 f_{n+8}$$
$$+ \frac{20098920059}{119119000}\,h^2 f_{n+9} - \frac{15452545149}{170170000}\,h^2 f_{n+10} + \frac{4457401749}{119119000}\,h^2 f_{n+11}$$
$$- \frac{5435403101}{476476000}\,h^2 f_{n+12} + \frac{41270877}{17017000}\,h^2 f_{n+13} - \frac{4782384}{14889875}\,h^2 f_{n+14} + \frac{11885633}{595595000}\,h^2 f_{n+15} \quad (65)$$

$$y''_{n+7} = y''_n + 7\,hy'''_n + \frac{561270290962031}{326651719680000}\,h^2 f_n + \frac{138008915043671}{10888390656000}\,h^2 f_{n+1}$$
$$- \frac{115995161213911}{7258927104000}\,h^2 f_{n+2} + \frac{1009684768894753}{16332585984000}\,h^2 f_{n+3} - \frac{2698346531507767}{21776781312000}\,h^2 f_{n+4}$$
$$+ \frac{4028977853553919}{18147317760000}\,h^2 f_{n+5} - \frac{19697678682126767}{65330343936000}\,h^2 f_{n+6} + \frac{2100668420753}{6345216000}\,h^2 f_{n+7}$$
$$- \frac{696945974650877}{2419642368000}\,h^2 f_{n+8} + \frac{6477310161851851}{32665171968000}\,h^2 f_{n+9} - \frac{11619838717354259}{108883906560000}\,h^2 f_{n+10}$$
$$+ \frac{79805888878681}{1814731776000}\,h^2 f_{n+11} - \frac{875850118609039}{65330343936000}\,h^2 f_{n+12} + \frac{2387301668339}{837568512000}\,h^2 f_{n+13}$$
$$- \frac{2740021526389}{7258927104000}\,h^2 f_{n+14} + \frac{3740727473}{159497910000}\,h^2 f_{n+15} \quad (66)$$

$$y''_{n+8} = y''_n + 8\,hy'''_n + \frac{962791153472}{488462349375}\,h^2 f_n + \frac{159116760448}{10854718875}\,h^2 f_{n+1}$$
$$- \frac{589264474048}{32564156625}\,h^2 f_{n+2} + \frac{6977664005248}{97692469875}\,h^2 f_{n+3} - \frac{219901261024}{1550674125}\,h^2 f_{n+4}$$
$$- \frac{33760546806464}{97692469875}\,h^2 f_{n+6} + \frac{1379590216064}{3618239625}\,h^2 f_{n+7} - \frac{25115068768}{75907125}\,h^2 f_{n+8}$$
$$+ \frac{22258735804544}{97692469875}\,h^2 f_{n+9} - \frac{6655353722816}{54273594375}\,h^2 f_{n+10} + \frac{235080913792}{4652022375}\,h^2 f_{n+11}$$
$$- \frac{1504988356384}{97692469875}\,h^2 f_{n+12} + \frac{35552129152}{10854718875}\,h^2 f_{n+13} - \frac{14124842944}{32564156625}\,h^2 f_{n+14}$$
$$+ \frac{13164251008}{488462349375}\,h^2 f_{n+15} \quad (67)$$

$$y''_{n+9} = y''_n + 9\,hy'''_n + \frac{5425276143393}{2439557120000}\,h^2 f_n + \frac{507508999443}{30494464000}\,h^2 f_{n+1} - \frac{1408763359683}{69701632000}\,h^2 f_{n+2}$$
$$+ \frac{1797053294583}{22177792000}\,h^2 f_{n+3} - \frac{77925040957971}{487911424000}\,h^2 f_{n+4} + \frac{176608903026609}{609889280000}\,h^2 f_{n+5}$$
$$- \frac{190121626816461}{487911424000}\,h^2 f_{n+6} + \frac{105298772316291}{243955712000}\,h^2 f_{n+7} - \frac{181947002167377}{487911424000}\,h^2 f_{n+8}$$
$$+ \frac{15689842209}{60928000}\,h^2 f_{n+9} - \frac{337995172999251}{2439557120000}\,h^2 f_{n+10} + \frac{13927989639699}{243955712000}\,h^2 f_{n+11}$$
$$- \frac{8492014219401}{487911424000}\,h^2 f_{n+12} + \frac{451361269269}{121977856000}\,h^2 f_{n+13} - \frac{18392351427}{37531648000}\,h^2 f_{n+14}$$
$$+ \frac{37139996961}{1219778560000}\,h^2 f_{n+15} \quad (68)$$

$$y''_{n+10} = y''_n + 10\,hy'''_n + \frac{61940195105}{25009272288}\,h^2 f_n + \frac{38819735375}{2084106024}\,h^2 f_{n+1}$$
$$- \frac{7755391625}{347351004}\,h^2 f_{n+2} + \frac{11564784875}{127598328}\,h^2 f_{n+3} - \frac{1480662560375}{8336424096}\,h^2 f_{n+4}$$
$$+ \frac{32091122155}{99243144}\,h^2 f_{n+5} - \frac{5423884076375}{12504636144}\,h^2 f_{n+6} + \frac{334822805875}{694702008}\,h^2 f_{n+7}$$
$$- \frac{384255384875}{926269344}\,h^2 f_{n+8} + \frac{1800509413375}{6252318072}\,h^2 f_{n+9} - \frac{321370705}{2082024}\,h^2 f_{n+10}$$
$$+ \frac{44211042625}{694702008}\,h^2 f_{n+11} - \frac{69319839125}{3572753184}\,h^2 f_{n+12} + \frac{8597247125}{2084106024}\,h^2 f_{n+13}$$
$$- \frac{759050375}{1389404016}\,h^2 f_{n+14} + \frac{212230705}{6252318072}\,h^2 f_{n+15} \quad (69)$$

$$y''_{n+11} = y''_n + 11\,hy'''_n + \frac{3971658023273897}{1455084933120000}\,h^2 f_n$$
$$+ \frac{2266813707059}{109983744000}\,h^2 f_{n+1} - \frac{48389901241627}{1979707392000}\,h^2 f_{n+2} + \frac{1823189496513649}{18188561664000}\,h^2 f_{n+3}$$
$$- \frac{100348334174341}{513257472000}\,h^2 f_{n+4} + \frac{86610066741761629}{242514155520000}\,h^2 f_{n+5} - \frac{139054076635086833}{291016986624000}\,h^2 f_{n+6}$$
$$+ \frac{53121794226743}{99800064000}\,h^2 f_{n+7} - \frac{14769476593075457}{32335220736000}\,h^2 f_{n+8} + \frac{6621237757011361}{20786927616000}\,h^2 f_{n+9}$$
$$- \frac{9127089742965557}{53892034560000}\,h^2 f_{n+10} + \frac{9366906277091}{133249536000}\,h^2 f_{n+11} - \frac{6230159494371769}{291016986624000}\,h^2 f_{n+12}$$
$$+ \frac{8175398913517}{1796401152000}\,h^2 f_{n+13} - \frac{58463252748769}{97005662208000}\,h^2 f_{n+14} + \frac{13621603946467}{363771233280000}\,h^2 f_{n+15} \quad (70)$$

$$y''_{n+12} = y''_n + 12\,hy'''_n + \frac{222031178}{74449375}\,h^2 f_n + \frac{336427848}{14889875}\,h^2 f_{n+1}$$
$$- \frac{395464572}{14889875}\,h^2 f_{n+2} + \frac{1635581336}{14889875}\,h^2 f_{n+3} - \frac{3177823608}{14889875}\,h^2 f_{n+4}$$
$$+ \frac{4157834328}{10635625}\,h^2 f_{n+5} - \frac{1110232588}{2127125}\,h^2 f_{n+6} + \frac{8675883864}{14889875}\,h^2 f_{n+7}$$
$$- \frac{7426541862}{14889875}\,h^2 f_{n+8} + \frac{5198849816}{14889875}\,h^2 f_{n+9} - \frac{13721678676}{74449375}\,h^2 f_{n+10}$$
$$+ \frac{1158088968}{14889875}\,h^2 f_{n+11} - \frac{3815012}{163625}\,h^2 f_{n+12} + \frac{10574568}{2127125}\,h^2 f_{n+13}$$
$$- \frac{9806508}{14889875}\,h^2 f_{n+14} + \frac{3046952}{74449375}\,h^2 f_{n+15} \quad (71)$$

$$y''_{n+13} = y''_n + 13\,hy'''_n + \frac{3983186767192217}{1231225712640000}\,h^2 f_n$$

$$+ \frac{504351825845881}{20520428544000}\,h^2 f_{n+1} - \frac{261504344043007}{9120190464000}\,h^2 f_{n+2} + \frac{2100826446847817}{17588938752000}\,h^2 f_{n+3}$$

$$- \frac{18984767728152889}{82081714176000}\,h^2 f_{n+4} + \frac{1210240131830197}{2850059520000}\,h^2 f_{n+5} - \frac{199061994113000267}{35177877504000}\,h^2 f_{n+6}$$

$$+ \frac{8656591000332301}{13680285696000}\,h^2 f_{n+7} - \frac{1642958954761241}{3040063488000}\,h^2 f_{n+8} + \frac{23360194872458951}{61561285632000}\,h^2 f_{n+9}$$

$$- \frac{11668896830043479}{58629795840000}\,h^2 f_{n+10} + \frac{388626555599243}{4560095232000}\,h^2 f_{n+11} - \frac{5998735795932529}{246245142528000}\,h^2 f_{n+12}$$

$$+ \frac{8030528674181}{1465744896000}\,h^2 f_{n+13} - \frac{594992073871}{829108224000}\,h^2 f_{n+14} + \frac{27410337567119}{615612856320000}\,h^2 f_{n+15} \quad (72)$$

$$y''_{n+14} = y''_n + 14\,hy'''_n + \frac{1112609593123}{318995820000}\,h^2 f_n + \frac{47075937433}{1772199000}\,h^2 f_{n+1}$$

$$- \frac{327517459171}{10633194000}\,h^2 f_{n+2} + \frac{2059176926639}{15949791000}\,h^2 f_{n+3} - \frac{1767851783719}{7088796000}\,h^2 f_{n+4}$$

$$+ \frac{12198755857183}{26582985000}\,h^2 f_{n+5} - \frac{4871439361817}{7974895500}\,h^2 f_{n+6} + \frac{404310610109}{590733000}\,h^2 f_{n+7}$$

$$- \frac{4139166288943}{7088796000}\,h^2 f_{n+8} + \frac{6562564086043}{15949791000}\,h^2 f_{n+9} - \frac{3810288983977}{17721990000}\,h^2 f_{n+10}$$

$$+ \frac{496658084279}{5316597000}\,h^2 f_{n+11} - \frac{1636772074721}{63799164000}\,h^2 f_{n+12} + \frac{1113970361}{161109000}\,h^2 f_{n+13}$$

$$- \frac{3740727473}{5316597000}\,h^2 f_{n+14} + \frac{3740727473}{79748955000}\,h^2 f_{n+15} \quad (73)$$

$$y''_{n+15} = y''_n + 15\,hy'''_n + \frac{14605252055}{3903291392}\,h^2 f_n + \frac{55679491125}{1951645696}\,h^2 f_{n+1}$$

$$- \frac{127956427875}{3903291392}\,h^2 f_{n+2} + \frac{134732597875}{975822848}\,h^2 f_{n+3} - \frac{147877626375}{557613056}\,h^2 f_{n+4}$$

$$+ \frac{951630836355}{1951645696}\,h^2 f_{n+5} - \frac{2519631003875}{3903291392}\,h^2 f_{n+6} + \frac{175914493875}{243955712}\,h^2 f_{n+7}$$

$$- \frac{2381638669875}{3903291392}\,h^2 f_{n+8} + \frac{834744780875}{1951645696}\,h^2 f_{n+9} - \frac{856960688385}{3903291392}\,h^2 f_{n+10}$$

$$+ \frac{13247717625}{139403264}\,h^2 f_{n+11} - \frac{93624917875}{3903291392}\,h^2 f_{n+12} + \frac{14479300125}{1951645696}\,h^2 f_{n+13}$$

$$+ \frac{1501410375}{3903291392}\,h^2 f_{n+14} + \frac{50188465}{487911424}\,h^2 f_{n+15} \quad (74)$$

Also, we take the third derivative of equation (9) and evaluate at x_{n+j}, $j = 0(1)k$ to give:

$$y'''_{n+1} = y'''_n + \frac{25221445}{98402304}\,hf_n + \frac{105145058757073}{62768369664000}\,hf_{n+1}$$

$$- \frac{20999287611259}{5706215424000}\,hf_{n+2} + \frac{612744541065337}{62768369664000}\,hf_{n+3} - \frac{189568380436867}{8966909952000}\,hf_{n+4}$$

$$+ \frac{2285168598349733}{62768369664000}\,hf_{n+5} - \frac{3129453071993581}{62768369664000}\,hf_{n+6} + \frac{1138313909617631}{20922789888000}\,hf_{n+7}$$

$$- \frac{988788576755233}{20922789888000}\,hf_{n+8} + \frac{679781959848881}{20922789888000}\,hf_{n+9} - \frac{1096355235402331}{62768369664000}\,hf_{n+10}$$

$$+ \frac{64486158419069}{8966909952000}\,hf_{n+11} - \frac{137515713789319}{62768369664000}\,hf_{n+12} + \frac{29219384284087}{62768369664000}\,hf_{n+13}$$

$$- \frac{3867689367599}{62768369664000}\,hf_{n+14} + \frac{2639651053}{689762304000}\,hf_{n+15} \quad (75)$$

$$y'''_{n+2} = y'''_n + \frac{631693279}{2501928000} hf_n + \frac{15268129873}{7662154500} hf_{n+1} - \frac{302093934503}{122594472000} hf_{n+2}$$
$$+ \frac{31510558777}{3831077250} hf_{n+3} - \frac{2248853768341}{122594472000} hf_{n+4} + \frac{245046332159}{7662154500} hf_{n+5}$$
$$- \frac{1802000069137}{40864824000} hf_{n+6} + \frac{30858062633}{638512875} hf_{n+7} - \frac{1720640706481}{40864824000} hf_{n+8}$$
$$+ \frac{222285459167}{7662154500} hf_{n+9} - \frac{1915228492981}{122594472000} hf_{n+10} + \frac{24675139849}{3831077250} hf_{n+11}$$
$$- \frac{240803930183}{122594472000} hf_{n+12} + \frac{3200718769}{7662154500} hf_{n+13} - \frac{6783778481}{122594472000} hf_{n+14}$$
$$+ \frac{3012}{875875} hf_{n+15} \quad (76)$$

$$y'''_{n+3} = y'''_n + \frac{1674820979}{6623232000} hf_n + \frac{56902877331}{28700672000} hf_{n+1}$$
$$- \frac{60273315579}{28700672000} hf_{n+2} + \frac{794168588449}{86102016000} hf_{n+3} - \frac{550323797343}{28700672000} hf_{n+4}$$
$$+ \frac{135645163833}{4100096000} hf_{n+5} - \frac{1303414182247}{28700672000} hf_{n+6} + \frac{203566719249}{4100096000} hf_{n+7}$$
$$- \frac{1239549466761}{28700672000} hf_{n+8} + \frac{2559362190283}{86102016000} hf_{n+9} - \frac{459042547569}{28700672000} hf_{n+10}$$
$$+ \frac{189136929441}{28700672000} hf_{n+11} - \frac{3529763983}{1757184000} hf_{n+12} + \frac{12256162053}{28700672000} hf_{n+13}$$
$$- \frac{1622998893}{28700672000} hf_{n+14} + \frac{9166839}{2609152000} hf_{n+15} \quad (77)$$

$$y'''_{n+4} = y'''_n + \frac{1328507}{5255250} hf_n + \frac{3800835068}{1915538625} hf_{n+1} - \frac{579872212}{273648375} hf_{n+2}$$
$$+ \frac{18445487108}{1915538625} hf_{n+3} - \frac{70153091887}{3831077250} hf_{n+4} + \frac{62404505692}{1915538625} hf_{n+5}$$
$$- \frac{86024128376}{1915538625} hf_{n+6} + \frac{31404619756}{638512875} hf_{n+7} - \frac{54690363709}{1277025750} hf_{n+8}$$
$$+ \frac{2690307652}{91216125} hf_{n+9} - \frac{30412523756}{1915538625} hf_{n+10} + \frac{12534712108}{1915538625} hf_{n+11}$$
$$- \frac{7643539093}{3831077250} + \frac{812610068}{1915538625} hf_{n+13}$$
$$- \frac{107625136}{1915538625} hf_{n+14} + \frac{53500}{15324309} hf_{n+15} \quad (78)$$

$$y'''_{n+5} = y'''_n + \frac{887775845}{3511517184} hf_n + \frac{10946710975}{5518098432} hf_{n+1}$$
$$- \frac{81687827825}{38626689024} hf_{n+2} + \frac{4819615460525}{502146957312} hf_{n+3} - \frac{8971075692025}{502146957312} hf_{n+4}$$
$$+ \frac{16743731110985}{502146957312} hf_{n+5} - \frac{1081685073325}{23911759872} hf_{n+6} + \frac{8274717147275}{167382319104} hf_{n+7}$$
$$- \frac{7198523492725}{167382319104} hf_{n+8} + \frac{14864967700175}{502146957312} hf_{n+9} - \frac{7999362395575}{502146957312} hf_{n+10}$$
$$+ \frac{3296252627975}{502146957312} hf_{n+11} - \frac{1004846299475}{502146957312} hf_{n+12} + \frac{30518628325}{71735279616} hf_{n+13}$$
$$- \frac{28291221275}{502146957312} hf_{n+14} + \frac{803745}{229605376} hf_{n+15} \quad (79)$$

$$y'''_{n+6} = y'''_n + \frac{340115}{1345344} hf_n + \frac{6950787}{3503500} hf_{n+1} - \frac{118634421}{56056000} hf_{n+2}$$
$$+ \frac{2404277}{250250} hf_{n+3} - \frac{1004179551}{56056000} hf_{n+4} + \frac{118528077}{3503500} hf_{n+5}$$
$$- \frac{7491091147}{168168000} hf_{n+6} + \frac{43106373}{875875} hf_{n+7} - \frac{2403399321}{56056000} hf_{n+8}$$
$$+ \frac{310373351}{10510500} hf_{n+9} - \frac{127297257}{8008000} hf_{n+10} + \frac{11476851}{1751750} hf_{n+11}$$
$$- \frac{10179391}{5096000} hf_{n+12} + \frac{135273}{318500} hf_{n+13} - \frac{450453}{8008000} hf_{n+14} + \frac{1312}{375375} hf_{n+15} \quad (80)$$

$$y'''_{n+7} = y'''_n + \frac{148080429}{585728000} hf_n + \frac{2541298850209}{1280987136000} hf_{n+1}$$
$$- \frac{2709977083289}{1280987136000} hf_{n+2} + \frac{1118294324987}{116453376000} hf_{n+3} - \frac{22922743552933}{1280987136000} hf_{n+4}$$
$$+ \frac{43245378870197}{1280987136000} hf_{n+5} - \frac{56382862132093}{1280987136000} hf_{n+6} + \frac{21283573174031}{426995712000} hf_{n+7}$$
$$- \frac{18372161728753}{426995712000} hf_{n+8} + \frac{12638994737537}{426995712000} hf_{n+9} - \frac{20401231357003}{1280987136000} hf_{n+10}$$
$$+ \frac{8406116820059}{1280987136000} hf_{n+11} - \frac{232954185989}{116453376000} hf_{n+12} + \frac{544780276327}{1280987136000} hf_{n+13}$$
$$- \frac{72145233503}{1280987136000} hf_{n+14} + \frac{4482518383}{1280987136000} hf_{n+15} \quad (81)$$

$$y'''_{n+8} = y'''_n + \frac{69181108}{273648375} hf_n + \frac{3800295904}{1915538625} hf_{n+1} - \frac{368501776}{174139875} hf_{n+2}$$
$$+ \frac{18400755872}{1915538625} hf_{n+3} - \frac{4900116016}{273648375} hf_{n+4} + \frac{64738663904}{1915538625} hf_{n+5}$$
$$- \frac{28171120816}{638512875} hf_{n+6} + \frac{32195047904}{638512875} hf_{n+7} - \frac{27104604508}{638512875} hf_{n+8}$$
$$+ \frac{56498981024}{1915538625} hf_{n+9} - \frac{30436015696}{1915538625} hf_{n+10} + \frac{1792450592}{273648375} hf_{n+11}$$
$$- \frac{3825896576}{1915538625} hf_{n+12} + \frac{813518752}{1915538625} hf_{n+13} - \frac{107747312}{1915538625} hf_{n+14} + \frac{1312}{375375} hf_{n+15} \quad (82)$$

$$y'''_{n+9} = y'''_n + \frac{148080429}{585728000} hf_n + \frac{56938145427}{28700672000} hf_{n+1}$$
$$- \frac{60718067451}{28700672000} hf_{n+2} + \frac{275617326987}{28700672000} hf_{n+3} - \frac{513628536159}{28700672000} hf_{n+4}$$
$$+ \frac{969126008463}{28700672000} hf_{n+5} - \frac{1264262164839}{28700672000} hf_{n+6} + \frac{1442792036727}{28700672000} hf_{n+7}$$
$$- \frac{1200258046089}{28700672000} hf_{n+8} + \frac{861743554713}{28700672000} hf_{n+9} - \frac{458089367409}{28700672000} hf_{n+10}$$
$$+ \frac{188548120737}{28700672000} hf_{n+11} - \frac{57454723701}{28700672000} hf_{n+12} + \frac{12212476677}{28700672000} hf_{n+13}$$
$$- \frac{1617125229}{28700672000} hf_{n+14} + \frac{803745}{229605376} hf_{n+15} \quad (83)$$

$$y'''_{n+10} = y'''_n + \frac{340115}{1345344} hf_n + \frac{121610725}{61297236} hf_{n+1} - \frac{2075551475}{980755776} hf_{n+2}$$
$$+ \frac{294433325}{30648618} hf_{n+3} - \frac{17564076025}{980755776} hf_{n+4} + \frac{295984445}{8756748} hf_{n+5}$$
$$- \frac{43273790575}{980755776} hf_{n+6} + \frac{36779675}{729729} hf_{n+7} - \frac{13743878725}{326918592} hf_{n+8}$$
$$+ \frac{627611225}{20432412} hf_{n+9} - \frac{15176096665}{980755776} hf_{n+10} + \frac{18169175}{2786238} hf_{n+11}$$
$$- \frac{39878275}{20015424} hf_{n+12} + \frac{25987525}{61297236} hf_{n+13} - \frac{55091525}{980755776} hf_{n+14} + \frac{53500}{15324309} hf_{n+15} \quad (84)$$

$$y'''_{n+11} = y'''_n + \frac{887775845}{3511517184} hf_n + \frac{11319967290643}{5706215424000} hf_{n+1}$$
$$- \frac{132626474849}{62705664000} hf_{n+2} + \frac{54784127878667}{5706215424000} hf_{n+3} - \frac{102073339405279}{5706215424000} hf_{n+4}$$
$$+ \frac{192568525522703}{5706215424000} hf_{n+5} - \frac{83717504794957}{1902071808000} hf_{n+6} + \frac{95525302183421}{1902071808000} hf_{n+7}$$
$$- \frac{79485015923203}{1902071808000} hf_{n+8} + \frac{24771909147359}{815173632000} hf_{n+9} - \frac{83924967291121}{5706215424000} hf_{n+10}$$
$$+ \frac{39756270033953}{5706215424000} hf_{n+11} - \frac{11548005399029}{5706215424000} hf_{n+12} + \frac{2443388534917}{5706215424000} hf_{n+13}$$
$$- \frac{322980342509}{5706215424000} hf_{n+14} + \frac{9166839}{2609152000} hf_{n+15} \quad (85)$$

$$y'''_{n+12} = y'''_n + \frac{1328507}{5255250} hf_n + \frac{248268}{125125} hf_{n+1} - \frac{12972}{6125} hf_{n+2}$$
$$+ \frac{25263044}{2627625} hf_{n+3} - \frac{31416549}{1751750} hf_{n+4} + \frac{29661108}{875875} hf_{n+5} - \frac{5536136}{125125} hf_{n+6}$$
$$+ \frac{44305452}{875875} hf_{n+7} - \frac{74018277}{1751750} hf_{n+8} + \frac{81177812}{2627625} hf_{n+9} - \frac{13324692}{875875} hf_{n+10}$$
$$+ \frac{6858276}{875875} hf_{n+11} - \frac{8502661}{5255250} hf_{n+12} + \frac{51204}{125125} hf_{n+13} - \frac{48192}{875875} hf_{n+14} + \frac{3012}{875875} hf_{n+15} \quad (86)$$

$$y'''_{n+13} = y'''_n + \frac{1674820979}{6623232000} hf_n + \frac{9574328794069}{4828336128000} hf_{n+1} - \frac{10180906367021}{4828336128000} hf_{n+2}$$
$$+ \frac{6600937081531}{689762304000} hf_{n+3} - \frac{85872829106857}{4828336128000} hf_{n+4} + \frac{161714860824569}{4828336128000} hf_{n+5}$$
$$- \frac{210181855059553}{4828336128000} hf_{n+6} + \frac{79669188209819}{1609445376000} hf_{n+7} - \frac{65878891642213}{1609445376000} hf_{n+8}$$
$$+ \frac{47601806972213}{1609445376000} hf_{n+9} - \frac{9733223634433}{689762304000} hf_{n+10} + \frac{33795680175959}{4828336128000} hf_{n+11}$$
$$- \frac{2990410218211}{4828336128000} hf_{n+12} + \frac{3733892296147}{4828336128000} hf_{n+13} - \frac{44874067901}{689762304000} hf_{n+14}$$
$$+ \frac{2639651053}{689762304000} hf_{n+15} \quad (87)$$

$$y'''_{n+14} = y'''_n + \frac{631693279}{2501928000}\,hf_n + \frac{311056753}{156370500}\,hf_{n+1} - \frac{5395044599}{2501928000}\,hf_{n+2}$$
$$+ \frac{765940609}{78185250}\,hf_{n+3} - \frac{46375653541}{2501928000}\,hf_{n+4} + \frac{5525678207}{156370500}\,hf_{n+5}$$
$$- \frac{39205297537}{833976000}\,hf_{n+6} + \frac{712193069}{13030875}\,hf_{n+7} - \frac{39205297537}{833976000}\,hf_{n+8}$$
$$+ \frac{5525678207}{156370500}\,hf_{n+9} - \frac{46375653541}{2501928000}\,hf_{n+10}$$
$$+ \frac{765940609}{78185250}\,hf_{n+11} - \frac{5395044599}{2501928000}\,hf_{n+12} + \frac{311056753}{156370500}\,hf_{n+13} + \frac{631693279}{2501928000}\,hf_{n+14} \quad (88)$$

$$y'''_{n+15} = y'''_n + \frac{25221445}{98402304}\,hf_n + \frac{442589775}{229605376}\,hf_{n+1} - \frac{388226775}{229605376}\,hf_{n+2} + \frac{1746295975}{229605376}\,hf_{n+3}$$
$$- \frac{372104925}{32800768}\,hf_{n+4} + \frac{4103141115}{229605376}\,hf_{n+5} - \frac{10001664025}{688816128}\,hf_{n+6} + \frac{1698012675}{229605376}\,hf_{n+7}$$
$$+ \frac{1698012675}{229605376}\,hf_{n+8} - \frac{10001664025}{688816128}\,hf_{n+9} + \frac{4103141115}{229605376}\,hf_{n+10} - \frac{372104925}{32800768}\,hf_{n+11}$$
$$+ \frac{1746295975}{229605376}\,hf_{n+12} - \frac{388226775}{229605376}\,hf_{n+13} + \frac{442589775}{229605376}\,hf_{n+14} + \frac{25221445}{98402304}\,hf_{n+15} \quad (89)$$

3. Analysis of the Basic Properties

Theorem 3.1 (Source: Lambert [21]).

No zero-stable linear multistep method of step number k can have order exceeding $k+1$ when k is odd, or $k+2$ when k is even.
Remark: For a proof of this theorem, the reader is referred to [18].

3.1. Order of the Block Method

We define the linear operator associated with the new method, (NS4O15M) as:

$$L(y(x); h) = A_0 Y_m - A_1 Y_{m-1} - h A_2 Y'_{m-1} - h^2 B_0 Y''_{m-1} - h^3 B_1 Y'''_{m-1} - h^4 (C_0 F_m + C_1 F_{m-1}) \quad (90)$$

where,

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+k} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \end{bmatrix}, Y'_{m-1} = \begin{bmatrix} y'_{n-k+1} \\ y'_{n-k+2} \\ \vdots \\ y'_n \end{bmatrix},$$

$$Y''_{m-1} = \begin{bmatrix} y''_{n-k+1} \\ y''_{n-k+2} \\ \vdots \\ y''_n \end{bmatrix}, Y'''_{m-1} = \begin{bmatrix} y'''_{n-k+1} \\ y'''_{n-k+2} \\ \vdots \\ y'''_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+k} \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-k+1} \\ f_{n-k+2} \\ \vdots \\ f_n \end{bmatrix}$$

Taylor series expansion of equation (90) gives:

$$L(y(x); h) = E_0 y(x) + h E_1 y'(x) + h_2^2 E_2 y''(x) + \ldots + h_p^{(p)} E_p y^{(p)}(x) + h_{p+1}^{(p+1)} E_{p+1} y^{(p+1)}(x) + \ldots \quad (91)$$

The new method, (NS4O15M) and its linear operator equation (90) are said to be of order p if and only if $E_0 = E_1 = E_2 = \ldots = E_p = E_{p+1} = E_{p+2} = 0, E_{p+3} = 0, E_{p+4} \neq 0$ in equation (91). Thus, the new method, (NS4O15M) is of a uniform order 16 with the error constants:

$$E_{p+4} =$$

$$\left[\begin{array}{ccccc} \frac{977656794693207}{524009663299584000}, & \frac{4331373510263}{1663116607152000}, & \frac{10908847155123}{1038275510272000}, & \frac{27085200034}{999469114875}, & \frac{432858835650625}{7785286426165248}, \\ \frac{25150957551}{253485232000}, & \frac{3201049858326341}{19860424556544000}, & \frac{289018658816}{1181190772125}, & \frac{366465164205771}{1038275510272000}, & \frac{6508856531875}{13304932857216}, \\ \frac{31281013654043197}{47637242118144000}, & \frac{242877078}{282907625}, & \frac{575350492228736471}{524009663299584000}, & \frac{514075874669}{372979728000}, & \frac{14142409239375}{8306204082176} \end{array} \right]^T$$

Hence, our new derived method (NS4O15M) is inline with theorem 3.1.

3.2. Consistency

A numerical method, according to Lambert [20], [21], is said to be consistent if its order, that is, $p \geq 1$. Therefore, the new method (NS4O15M), whose order is *16*, is certainly consistent.

3.3. Zero Stability

The new method (NS4O15M) is said to be zero stable if no roots of the first characteristic polynomial, $|r_s| > 1, \forall s = 1, 2, \ldots, N$ and that one of its roots is simple. That is,

$$\rho(r) = \left| (rA_0 - A_1) \right| \tag{92}$$

Where,

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
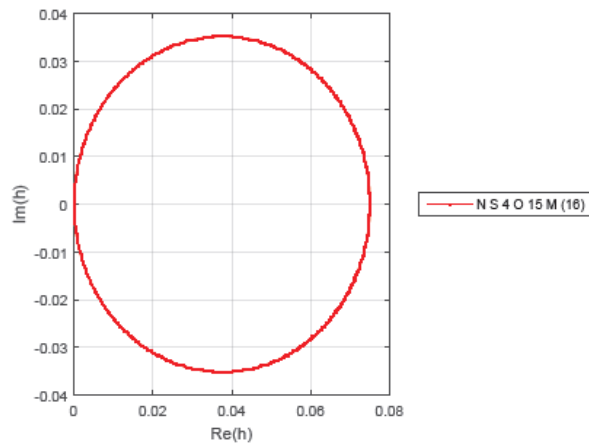

Figure 1. Absolute Stability Region of NS4O15M(16)

So that equation (92) is evaluated and solved for r to give:
$r = 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.$ Thus, the new method (NS4O15M) is zero stable.

3.4. Convergency

According to Lambert [21], a numerical method is convergent if it is both zero stable and consistent. Hence, the new method (NS4O15M) is certainly convergent inline with (3.2) and (3.3)

3.5. Absolute Stability

Definition 3.5.1 (Source: Lambert [20]).

The LMM (NS4O15M) is said to be A-Stable if the region of absolute stability includes the entire left half of the z-plane (that is $z \in (-\infty, 0)$). Therefore, the absolute stability region for the new method (NS4O15M) is plotted in the spirit Lambert [20], [21] and Okuonghae and Ikhile [22] using MATLAB. Hence, the stability polynomial is given by:

$$R(t; \overline{h}) = \frac{141760259867125}{1038275510272} t^{14}(\overline{h})^4 + \frac{1125}{2} t^{14}(\overline{h})^3$$
$$+ \frac{225}{2} t^{14}(\overline{h})^2$$
$$+ 15 t^{14}(\overline{h}) + (t - 1) t^{14} \tag{93}$$

Where, $\overline{h} = \lambda h$

Since A-stability is a severe property that is desired by all numerical methods, by Definition (3.5.1) the new method (NS4O15M) is A-Stable. Therefore, it is certain that Dahlquist barrier no longer posses restrictions on the use of step sizes in the method. Also, it is clear from Figure 1 that the new method has small unstable region and a wider region of absolute stability which proved its convergence with approximate solutions having little or no deviation from the true solution as a result of small change in input data Lambert [20]. Hence, the new method is numerically stable and it is investigated in section four below.

4. Numerical Examples

The new method (NS4O15M) employed Taylor series approach to incorporate all the initial values for the solution of (1). The following numerical examples are used to test the accuracy of the new method and comparison are made with some selected numerical methods in recent literature. All numerical iterations are carried out on MATLAB Software environment.

Problem 4.1 (Source: Jena et al. [6]).

$$y^{(iv)} = e^z \left(z^4 + 14 z^3 + 49 z^2 + 32 z - 12\right), \ h = 0.1,$$
$$y(0) = 0, \ y'(0) = 0, \ y''(0) = 2, \ y'''(0) = -6,$$

Exact Solution: $y(z) = z^2 (1 - z)^2 e^z$,

The numerical results for Problem 4.1 is as shown in Table 1

Problem 4.2 (Source: Ukpebor et al. [13]).

Consider the special fourth order below
$$u^{(iv)} = x, \ u(0) = 0, \ u'(0) = 1, \ u''(0) = 0, \ u'''(0) = 0, \ h = 0.1$$

Exact Solution: $u(0) = \frac{x^5}{120} + x$

The numerical results for Problem 4.2 is as shown in Table **2**

Problem 4.3 (Source: Ukpebor et al. [13]).

Consider the linear differential equation of fourth order:
$$u^{(iv)} + u'' = 0,$$
$$u(0) = 0, \ u'(0) = \frac{-1.1}{72-50\pi}, \ u''(0) = \frac{1}{144-100\pi}, \ u'''(0) = \frac{1.2}{144-100\pi}, \ h = 0.01,$$

Exact Solution: $u(x) = \frac{1-x-\cos(x)-1.2\sin(x)}{144-100\pi}$

The numerical results for Problem 4.3 is as shown in Table 3

Problem 4.4.

To confirm the application of the new method NS4O15M(16), we solve a physical problem from ship dynamics. As stated by Familua and Omole [17], when a sinusoidal wave of frequency Ω passes long a ship or offshore structure, the resultant fluid actions vary with time t. Therefore, consider the fourth-order problem as:
$$y^{(iv)} + 3y'' + y(2 + \varepsilon \cos(\Omega t)) = 0, \ t > 0$$

Is subjected to the following initial conditions:
$$y(0) = 1, \ y'(0) = 0, \ y''(0) = 0, \ y'''(0) = 0, \ h = \frac{1}{320}$$

where $\varepsilon = 0$, for the existence of the theoretical solution: $y(t) = 2\cos(t) - \cos(t\sqrt{2})$.

Problem 4.5 (Source: Allogmany et al., [4]).

We shall consider a nonlinear initial value problem of the form:
$$y^{iv} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x + x^2), 0 \le x \le 1$$
$$y(0) = 1, \ y'(0) = 1, \ y''(0) = 3, \ y'''(0) = 1$$

Exact Solution: $y(x) = x^2 + e^x$

We also compare the results for this problem that has been solved by [19]. The results are shown in Table 5. To further illustrate the

Table 1. Comparison of Absolute Maximum Error for Problem 4.1 when h=0.1

| x	Error in ODE45	Error in Jena et al. [6]	Error in NS4O15M
0.1	1.0326 e-07	1.5370 e-14	1.7347 e-18
0.2	1.6792 e-07	8.2021 e-14	1.6653 e-16
0.3	2.5340 e-07	3.6666 e-13	4.3715 e-16
0.4	3.6477 e-07	6.3424 e-13	8.3267 e-16
0.5	5.0816 e-07	6.7024 e-13	5.2736 e-16
0.6	6.9093 e-07	5.2608 e-13	3.7748 e-15
0.7	9.2191 e-07	3.3906 e-13	4.6629 e-15
0.8	1.2116 e-06	1.9011 e-13	5.1209 e-15
0.9	1.5727 e-06	9.6152 e-14	1.3854 e-14
1.0	4.1649 e-06	4.4983 e-14	1.9791 e-14
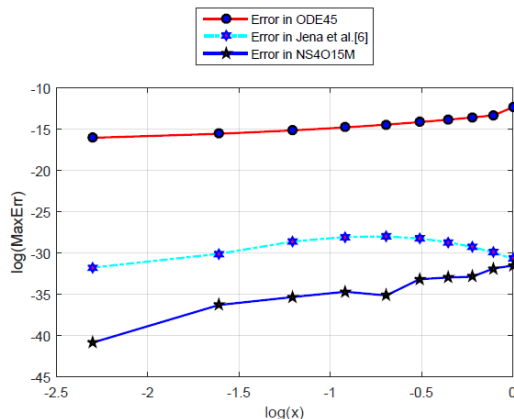
Figure 2. Efficiency Curves for Test Problem 4.1

accuracy of (NS4O15M) with comparison to other recent existing methods such as Jena et al. [6], Ukpebor et al. [13] and Familua et al. [17], we have also presented the efficiency curves for Problem 4.1, 4.3 and 4.4 and are given below:

5. Results and Discussions

This research paper has considered five special fourth order ODEs problems in recent literature. Problems 4.1 and 4.2 are solved using a step size, h=0.1, Problem 4.3 has been solved using a step-size, h=0.01 while Problem 4.4 and 4.5 have been solved using a step-

Table 2. Comparison of Absolute Maximum Error for Problem 4.2 when h=0.1

x	Error in Omar et al. [8]	Error in Ukpebor et al. [13]	Error in NS4O15M
0.1	1.002087 e-12	0.00 e-00	0.00 e+00
0.2	0.000000 e+00	0.00 e-00	0.00 e+00
0.3	0.000000 e+00	0.00 e-00	0.00 e+00
0.4	0.000000 e+00	0.00 e-00	0.00 e+00
0.5	1.002087 e-12	0.00 e-00	0.00 e+00
0.6	2.755907 e-12	0.00 e-00	0.00 e+00
0.7	3.507306 e-12	0.00 e-00	0.00 e+00
0.8	3.507306 e-12	2.00 e-18	0.00 e+00
0.9	4.175569 e-12	2.00 e-18	1.11 e-16
1.0	4.175569 e-12	1.00 e-17	0.00 e+00

Table 3. Comparison of Absolute Maximum Error for Problem 4.3 when h=0.01

x	Error in Adesanya et al. [10]	Error in Ukpebor et al. [13]	Error in NS4O15M
0.01	8.5052 e-19	8.0000 e-20	5.4210 e-20
0.02	1.3010 e-18	9.1500 e-19	5.4210 e-20
0.03	4.7704 e-18	3.4150 e-18	2.7105 e-19
0.04	1.7347 e-17	8.2220 e-18	1.0842 e-19
0.05	4.3368 e-17	1.5965 e-17	3.2526 e-19
0.06	9.5409 e-17	2.7404 e-17	3.2526 e-19
0.07	1.8127 e-16	3.4329 e-17	3.2526 e-19
0.08	3.1571 e-16	3.2551 e-17	4.3368 e-19
0.09	5.1868 e-16	6.5927 e-17	2.1684 e-19
0.10	8.0491 e-16	1.1919 e-16	6.5052 e-19

Table 4. Comparison of Absolute Maximum Error for Problem 4.4 when $h = \frac{1}{320}$ (Application problem arising from ship dynamics)

x	Error in block method, p=7, in [17]	Error in PC-method, p=7, [17]	Error in method NS4O15M(16)
0.003125	6.685763 e-13	5.685763 e-10	0.000000e+00
0.006250	1.458489 e-11	1.767654 e-10	1.110223e-16
0.009375	1.082968 e-10	5.909878 e-09	0.000000e+00
0.001250	3.917803 e-10	5.767654 e-09	2.220446e-16
0.015625	1.025145 e-09	1.100202 e-08	7.771561e-16
0.018750	2.217319 e-09	6.898767 e-08	2.886580e-15
0.021875	4.226068 e-09	4.636354 e-08	8.548717e-15
0.025000	7.358019 e-09	5.787654 e-07	2.187139e-14
0.028125	1.196868 e-08	2.245763 e-07	4.951595e-14
0.031250	1.846249 e-08	2.846249 e-07	1.035838e-13

Table 5. Comparison of Absolute Maximum Error for Problem 4.5 when $h = \frac{1}{320}$

x	IHB8 [19]	I2PBDO6 [4]	NS4O15M(16)
0.003125	4.261834 e-14	0.0000000 e+00	2.220446 e-16
0.006250	7.314820 e-13	3.9099252 e-16	0.000000 e+00
0.009375	1.625321 e-12	7.2441945 e-16	2.220446 e-16
0.001250	6.568972 e-12	9.9936929 e-16	0.000000 e+00
0.015625	2.849147 e-12	1.2947886 e-15	0.000000 e+00
0.018750	4.231457 e-12	0.0000000 e+00	2.220446 e-16
0.021875	1.246852 e-11	1.5163355 e-15	6.661338 e-16
0.025000	5.146142 e-11	2.8632323 e-15	1.332268 e-15
0.028125	3.765423 e-11	4.5622733 e-15	2.886580 e-15

size of $h = \frac{1}{320}$ Jena et al. [6] has solved Problem 4.1 using 9-step block method with uniform order *12* in comparison with ODE45 and Awoyemi et al. [11]. Similarly, Ukpebor et al. [13] derived a 7-step block formula for solving fourth order ODEs with a uniform order *4* and results compared with Duromola [9]. However, results for our test problems are shown in Tables 1, 2 and 3 respectively. Also, Familua et al. [17] formulated a five point mono hybrid block method for the solution of nth order ordinary differential equations and results compared in Table 4 with direct block and predictor-corrector block methods and also compared with our new method. Results from Tables 4, 5 and Figure 4, showed that the new method is considerably efficient at solving application problems as ship dynamics and nonlinear IVPs. It is certain from the tables that the new derived method (NS4O15M) gave better accuracy than Jena et al. [6], ODE45 Omar et al. [8], Adesanya et al. [10], Awoyemi et al. [11], Ukpebor et al. [13] and Familua et al. [17] respectively. Generally, the new method, (NS4O15M) showed improved approximations as $h \rightarrow 0$ on some of the test examples considered. Further analysis using the efficiency
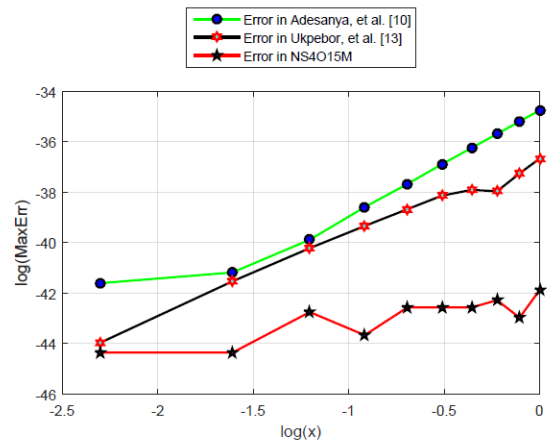

Figure 3. Efficiency Curves for Test Problem 4.3

curves in Figures 2, 3 and 4 showed that the new method has small scale errors unlike other existing methods compared.
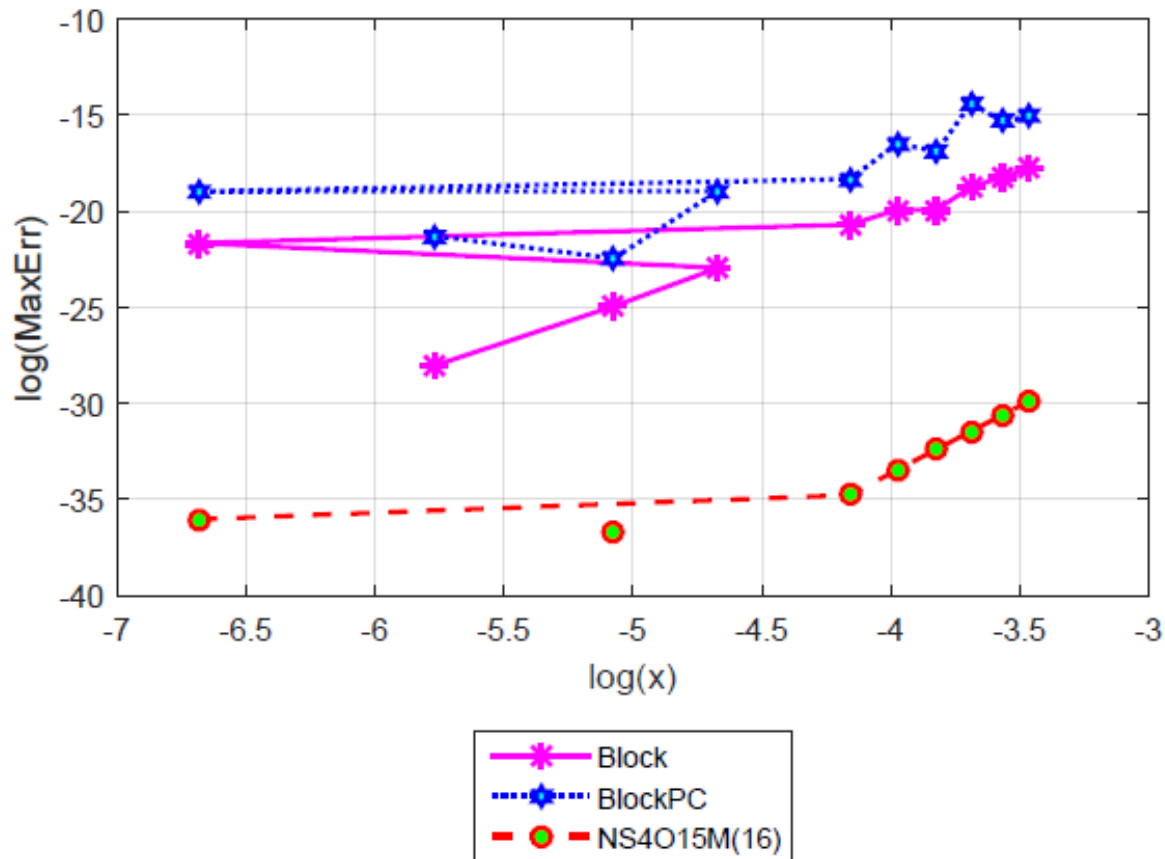
Figure 4. Efficiency Curves for Test Problem 4.4

6. Conclusion

In this research paper, a new block method that approximates 15-step simultaneously for the solution of (1) has been formulated. The new method is derived through interpolation and collocation techniques using power series as a basis function. The numerical properties of the method have been investigated and proven to be zero stable, consistent and A-stable. Test results showed that the new method is capable of solving general fourth order ODEs including physical problems from ship dynamics and nonlinear IVPs with better accuracy than other existing methods considered in this research paper. Therefore, the new method 15-step with uniform higher order *16*, proved better accuracy over 7-step with uniform order *4* and 9-step with uniform higher order *12*, among others. Hence, the new method (NS4O15M) should be considered as a viable alternative for solving general fourth order ODEs.

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