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On Nonexpansive and Expansive Semigroup of Order-Preserving Total Mappings in Waist Metric Spaces

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Abstract

In this paper, we introduce nonexpansive and expansive semigroup of order-preserving total mappings ($ON\mathcal{T}_n$) and ($OE\mathcal{T}_n$), respectively, to prove some fixed point theorems in waist metric spaces. We examine the existence of mappings that satisfy the conditions $ON\mathcal{T}_n$ and $OE\mathcal{T}_n$. We also prove that every semigroup of order-preserving total mappings $O\mathcal{T}_n$ has fixed point properties and that the set of fixed points is closed and convex. The present study generalised many previous results on semigroup of order-preserving total mappings $O\mathcal{T}_n$. Efficacy of the results was justified with some practical examples.

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1. Introduction

In the last four decades, semigroup of mappings is one of the areas of application in fixed point theory. In algebra, a semigroup is simply a set S with an associative binary operation. A subset $P \subset S$ is called a subsemigroup of S if it is closed under the binary operation on S.

Let X_n be an ordered finite set in a standard way and let $\alpha : Dom(\alpha) \subseteq X_n \to X_n$ be a self-map. The map α is called a full or total transformation of X_n if $Dom(\alpha) = X_n$. It is said to be partial if $Dom(\alpha) \subseteq X_n$. Otherwise, it is called partial one-to-one or strictly partial. The set of full transformations on X_n, denoted by $\mathcal{T}_n$, forms a semigroup under the composition of mappings called the full transformation semigroup. The set of partial and partial one-to-one transformations on X_n, denoted by $\mathcal{P}_n$ and $\mathcal{I}_n$, respectively, also form a semigroup under the composition of mappings.

The semigroup of order-preserving full transformation of X_n is defined by

$$O\mathcal{T}_n = \{\alpha \in \mathcal{T}_n : x \le y \Rightarrow x\alpha \le y\alpha, \text{ for all } x, y \in X_n\}.$$

Let α be a transformation in $O\mathcal{T}_n$. A point $x^* \in X_n$ is said to be fixed if it coincides with the image α. For $\alpha \in O\mathcal{T}_n$, the fix of α is given by $Fix(\alpha) = \{x \in X_n : x\alpha = x\}$.

Let $OC\mathcal{T}_n$ and $OC^*\mathcal{T}_n$ be subsemigroups of $O\mathcal{T}_n$, then a map-

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ping $\alpha \in OC\mathcal{T}_n$ is called a 'contraction' if

$$|x\alpha - y\alpha| \leq |x - y| \quad \text{for all } x, y \in X_n \tag{1}$$

and a mapping $\alpha \in OC^*\mathcal{T}_n$ is called 'contractive' if

$$|x\alpha - y\alpha| \geq |x - y| \quad \text{for all } x, y \in X_n \tag{2}$$

It is worthy to note there is a difference between 'contraction' and 'contractive' transformations in semigroup and deterministic fixed points. We refer to the following references for standard concepts and terminologies in the semigroup, (see [1, 2, 3, 4, 5]).
If α preserves only distance (with no order), then it satisfies both (1) and (2). This is called an isometry semigroup,

$$|x\alpha - y\alpha| = |x - y| \quad \text{for all } x, y \in X_n. \tag{3}$$

Several fixed point results have been proved on semigroup for the family of isometries under the asymptotic nonexpansive operators [6, 7, 8, 9], Lipschitzian semigroup of mappings [10, 11], commutative semigroup [12] etc. Worthy to mention a few recent studies of fixed points when the parameter set of semigroups is equal to $\{0, 1, 2, 3, \ldots\}$ and $T_n = T^n$ is the n-th iterate of asymptotic pointwise contractions and asymptotic nonexpansive mappings in metric spaces (see [9, 13]). Also, a procedure for constructing and finding the cardinality of order-preserving total transformations with finite fixed points have been considered in [14, 15]. However, we observed through a survey that few or no record of results concerning the existence of the fixed points of a semigroup of order-preserving total mappings. In this respect, existence of semigroup of order-preserving full transformation (which double as a nonlinear operator on X_n) is studied in the present paper. The intuitive notion of semigroup is integrated into a more robust geometric structure to unify some results in the semigroup theory. In concrete, the paper introduces some fixed point theorems for the nonexpansive (and expansive) semigroup of order-preserving mappings to prove some existence of fixed points of the elements of subsemigroups $OC\mathcal{T}_n$ (and $OC^*\mathcal{T}_n$).
We recall Banach's contraction mapping principle [16] which has been used in many areas of applied sciences to study the existence properties of nonlinear operators.

Definition 1. *Let (E, d) be a metric space. A map $T : E \to E$ is called contraction on E if there exists a constant $\lambda \in [0, 1)$ such that for all $x, y \in E$,*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{4}$$

If the condition (4) is weakened, that is $\lambda = 1$, then it reduces to a nonexpansive mapping

$$d(Tx, Ty) \leq d(x, y) \tag{5}$$

Otherwise, it is an expansive mapping. We note that every mapping $T \in OC\mathcal{T}_n$ is a nonexpansive mapping and every mapping $T \in OC^*\mathcal{T}_n$ is an expansive mapping. The inclusion in both cases are strict. For few older results on the family of nonexpansive mappings, see [6, 17, 18, 19, 20, 21].

2. Waist Metric Space

Let $\alpha : Dom(\alpha) \to Im(\alpha)$ be a map in $O\mathcal{T}_n$, where $Dom(\alpha), Im(\alpha) \subset X$. The right waist and left waist of $Dom(\alpha)$ are given, respectively, by

$$w^+(Dom(\alpha)) = \max\{|x| : x \in Dom(\alpha)\}$$

and

$$w^-(Dom(\alpha)) = \min\{|x| : x \in Dom(\alpha)\}.$$

Similarly, the right and left waist of $Im(\alpha)$ are, respectively, given by

$$w^+(\alpha) = \max\{|y| : y \in Im(\alpha)\} \text{ and } w^-(\alpha) = \min\{|y| : y \in Im(\alpha)\}.$$

In view of the above, we introduce a notion of distance function with the left waist $\omega^-(\cdot,\cdot)$ and right waist $\omega^+(\cdot,\cdot)$ terms as follow:

Definition 2. *Let M be a non-empty ordered set and X be a finite subset of M. A function $\omega : X \times X \to X \cup \{0\}$ is called a right and left waist metric if for given transformation α and for each $x, y \in Dom(\alpha) \subseteq X$, the following conditions hold:*

- *W1: $\omega^+(x, y)$ and $\omega^-(x, y)$ are finite and nonnegative integer;*
- *W2: $\omega^+(x, x) = 0$ and $\omega^-(x, x) = 0$;*
- *W3: $\omega^+(x, y) = \omega^+(y, x)$ and $\omega^-(x, y) = \omega^-(y, x)$;*
- *W4: $\omega^+(x, y) + \omega^+(y, z) \geq \omega^+(x, z)$ and $\omega^-(x, y) + \omega^-(y, z) \geq \omega^-(x, z)$ for $x, y, z \in X$.*

The pair $(M, \omega)_\alpha$ is called a waist metric space (WMS). WMS is a weakening form of the canonical metric space and it is classified as pseudometric space.

Example 1. *Let $X = \{1, 2\} \subset M$ be endowed with the waist distance*
$\omega^+_X(x, y) = \max\{|x - y| : x, y \in X\}$ and $\alpha = (1)(2)$, then $\omega^+_X(x, y)$ is a waist metric on X. Similarly for $\omega^-_X(x, y)$.

Example 2. *Let α be a total map on set $X = \{1, 2, 3, 4, 5\} \subset M$ such that $\alpha = (1)(4)(2\,1][3\,4][5\,4] \in O\mathcal{T}_n$, observe that $Im(\alpha) = \{1, 4\}$ and $Dom(\alpha) = \{1, 2, 3, 4, 5\}$. The following are verifiable:*

- *i $w^+(Dom(\alpha)) = 5$ and $w^-(Dom(\alpha)) = 1$.*
- *ii Both $\omega^+_X(x, y)$ and $\omega^-_X(x, y)$ are waist metric on X.*

Remark 1. *If $\alpha \in O\mathcal{T}_n$ for any given set X, then $\omega^-(x, y) = \omega^+(x, y)$. On the other hand, this is not so if α is a partial map $P\mathcal{T}_n$. Since the main focus of this present study is on the mappings in $O\mathcal{T}_n$, we denote $\omega(x, y)$ by a waist metric with no emphasis on left or right waist metric.*

2.1. Completeness of $(M,\omega)_\alpha$

Let $\{x_k\}$ be a sequence in $X \subset (M,\omega)_\alpha$. Since X is a finite set, the convergent of $\{x_k\}$ is vacuously satisfied.
We present the following useful lemmas.

Lemma 1. *A finite set $X \subset M$ is a closed set.*

Proof: Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let $X \cup X' = M$, where $X \cap X' = \emptyset$. For each $x_i \in M$, there is an $\varepsilon-$net such that $x_j \in B(x_i,\varepsilon) \subseteq X'$ for $i \neq j$. Observe that each $B(x_i,\varepsilon)$ is an open ball in M. Let $X' = \cup_{i\in\Delta}\{B(x_i,\varepsilon)\}$, then X' is the union of open balls which itself is an open set in M. Now,

$$M\backslash X' = \cap_{i\in\Delta}\{M\backslash B(x_i,\varepsilon)\} = \{x_1, x_2, \ldots, x_n\}.$$

That is, $X = M\backslash X'$ is a complement of an open set. Thus, X is closed.

Remark 2. *In Lemma 1, observe that for each $y \in X$, $B(x,\varepsilon) \cap X = \{x\}$. This means that no point in X is an accumulation point but every point in X is an isolated point. More so, any metric on a finite space induces a discrete topology (see [22]).*

Definition 3. *Let $(M,\omega)_\alpha$ be a WMS and $X \subset M$. A sequence $x_k \in X$ is said to be a Cauchy sequence in X if for given $\varepsilon-$net, there exist l and k with $l \geq k$ such that $x_l \in B(x_k,\varepsilon)$.*

Lemma 2. *Any convergent sequence in any metric space is a Cauchy sequence.*

Definition 4. *A waist metric space $(M,\omega)_\alpha$ is said to be complete if every Cauchy sequence in M converges to an element in M.*

Theorem 1. *Let $(M,\omega)_\alpha$ be a complete waist metric space and $X \subset M$. The subspace $(X,\omega)_\alpha$ is complete if and only if X is a closed subset of M.*

The proof follows from Lemma 1 and 2. The following concepts are versions of some results in [23, 24].

Definition 5. *Let $(M,\omega)_\alpha$ be a waist metric space. A mapping $u : M\times M\times[0,1] \to M$ is called a convex structure on M if for all $x, y \in M$ and $\lambda \in [0,1]$*

$$\omega(z, u(x,y,\lambda)) \leq \lambda\omega(z,x) + (1-\lambda)\omega(z,y)$$

holds for all $z \in M$. The waist metric space $(M,\omega)_\alpha$ together with a convex structure $u_\lambda = u(x,y,\lambda)$ is called a convex waist metric space.

In Definition 5, a convex waist metric space $(M,\omega,u)_\alpha$ satisfies the following:

$$\omega(u(x,p,\lambda), u(y,p,\lambda)) \leq \lambda\omega(x,y),\ x,y,p \in M,\ \lambda \in [0,1]$$

$$\omega(x,y) = \omega(x,u(x,p,\lambda)) + \omega(u(y,p,\lambda),y),\ x,y \in M,\ \lambda \in [0,1]$$

Definition 6. *A nonempty subset X of a convex waist metric space $(M,\omega,u)_\alpha$ is said to be convex if $u(x,y,\lambda) \in X$ for all $x, y \in X$ and $\lambda \in [0,1]$.*

Definition 7. *A nonempty subset X is said to be p-starshaped, where $p \in X$, provided $u(x,p,\lambda) \in X$ for all $x \in X$ and $\lambda \in [0,1]$, that is, the segment $[p,x] = \{u(x,p,\lambda) : 0 \leq \lambda \leq 1\}$ joining p to x is contained in X for all $x \in X$.*
The set X is said to be starshaped if it is p-starshaped for some $p \in X$.

Clearly, each convex waist metric space is starshaped but not conversely.

Lemma 3. *Let $(M,\omega)_\alpha$ be a waist metric space. Then*

$$\omega^2(z, u_{\frac{1}{2}}) \leq \frac{1}{2}\omega^2(z,x) + \frac{1}{2}\omega^2(z,y) - \frac{1}{4}\omega^2(x,y) \quad (6)$$

for all $x, y, z \in M$.

The proof follows from the parallelogram law.
Inequality (6) is similar to the (CN) inequality of Bruhat and Tits [26].

2.2. Nonexpansive Semigroup of Order-preserving Maps

Let S be a semitopological semigroup and X be a nonempty closed subset of a waist metric space $(M,\omega)_\alpha$. A family $\wp = \{\alpha_s : s \in S\}$ of mappings of X into itself is called a semigroup if it satisfies the following:

- S1: $x\alpha_{st} = x\alpha_s\alpha_t$ for all $s, t \in S$ and $x \in X$;
- S2: for every $x \in X$ the mapping $s \to \alpha_s x$ from S into X is continuous.

The set of all fixed points of semigroup mappings is denoted by $F(\alpha_s) = \{x \in X : x\alpha_s = x \text{ for } s \in S\}$.
More so, any map $\alpha_s \in OC\mathcal{T}_n$ or $\alpha_s \in OC^*\mathcal{T}_n$ possesses the properties as stated in section one. Note that (i) the set of fixed points of order-preserving maps, denoted by $F_o(\alpha_s)$, is a subset of $F(\alpha_s)$. (ii) $F_o(\alpha_s) \subset F_o^n(\alpha_s)$ for all $n > 1$. (iii) If $F_o^n(\alpha_s)$ is singleton for some n, so does $F_o(\alpha_s)$.
Without loss of generality, the notion 'nonexpansive' connotes 'contraction' while 'expansive' connotes 'contractive'. In view of the above, we present some definitions of nonexpansive semigroup of order-preserving mappings in WMS under the property that both $OC\mathcal{T}_n$ and $OC^*\mathcal{T}_n$ have no common maps.

Definition 8. *Let X be a closed subset of $(M,\omega)_\alpha$ and let S be a semitopological semigroup. The map $\alpha_s : X \to X$ is called a nonexpansive semigroup of order-preserving total map $ON\mathcal{T}_n$ if for $x, y \in X$ and $s \in S$,*

$$\omega(x\alpha_s, y\alpha_s) \leq \omega(x,y). \quad (7)$$

Definition 9. *Let X be a closed subset of $(M,\omega)_\alpha$ and let S be a semitopological semigroup. The map $\alpha_s : X \to X$ is called an expansive semigroup of order-preserving total map $OE\mathcal{T}_n$ if for $x, y \in X$ and $s \in S$,*

$$\omega(x\alpha_s, y\alpha_s) > \omega(x,y). \quad (8)$$

3. Main Results

The following lemma is useful in the proof of the main results.

Lemma 4. *If $\wp = \{\alpha_s : s \in S\}$ is a semigroup of continuous mappings of X into itself and $\omega(\alpha_s x, y) \to 0$ as $s \to \infty_{\mathbb{R}}$ for $x, y \in X$, then $y \in F_o(\alpha_s) \subset X$.*

Proof: Let $\varepsilon > 0$ be given. By the continuity of α_t for $t \in S$, there exists $\tau > 0$ such that $\omega(\alpha_t x, \alpha_t y) < \frac{\varepsilon}{2}$ whenever $\omega(x, y) < \tau$ for $x, y \in X$.
Also, since $\omega(\alpha_s x, y) \to 0$ as $s \to \infty_{\mathbb{R}}$, then there exists $u \in S$ such that
$\omega(\alpha_{au} x, y) < \min\{\frac{\varepsilon}{2}, \tau\}$ for each $a \in S$. Thus, $\omega(\alpha_{tau} x, \alpha_t y) < \frac{\varepsilon}{2}$.
Now,

$$\omega(y, \alpha_t y) \le \omega(y, \alpha_{tau} x) + \omega(\alpha_{tau} x, \alpha_t y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it follows that $y \in F_o(\alpha_s)$.

Remark 3. *If $S = \mathbb{N}$, then the hypothesis on α_s would include asymptotic regularity condition. For example, see Theorem 6.7. in [25].*

In the next theorem, we let $A(X, \bar{co}\{\alpha_s x\})$ denote the asymptotic center and $r(x_0, \bar{co}\{\alpha_s x\}$ is the asymptotic radius.

Theorem 2. *Let X be a nonempty closed subset of a convex waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \wp$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself, that is, $\alpha_s \in ON\mathcal{T}_n$ for $s \in S$. If the set $\{\alpha_s x,\ s \in S\}$ is bounded for some $x \in X$ and $y \in A(X, \bar{co}\{\alpha_s x\})$, then $y \in F_o(\alpha_s)$.*

Proof: Let $\{\alpha_s x,\ s \in S\}$ be a bounded net. Define $R := r(y, \bar{co}\{\alpha_s x\})$ for $y \in A(X, \bar{co}\{\alpha_s x\})$ with the property that $\omega(x, y) < R$.
If $R = 0$, then $\limsup \omega(\alpha_s x, y) = 0$ and by Lemma 4, the proof is complete. On the other hand, suppose $R > 0$ and $y \notin F_o(\alpha_s)$, then for given $\varepsilon > 0$ and a subnet $\{s_\beta\}$ in S, we have

$$\omega(\alpha_{s_\beta} y, y) > \varepsilon, \text{ for } s_\beta \in S.$$

Also, since $\omega(\alpha_s x, y) \to 0$ as $s \to \infty_{\mathbb{R}}$, then there exists $\gamma \in S$ such that, for choosing $\nu \ge 0$,

$$\omega(\alpha_\gamma x, y) < R + \nu. \tag{9}$$

Moreover, we have by hypothesis that

$$\begin{aligned}\omega(\alpha_s x, \alpha_s y) &\le \limsup_\beta \omega(\alpha_\beta x, \alpha_\beta y) + \nu \\ &\le \limsup \omega(x, y) + \nu \\ &= R + \nu \end{aligned}\tag{10}$$

By Lemma 3, (9) and (10), we have

$$\begin{aligned}\omega^2(u, \alpha_s x) &\le \frac{1}{2}\omega^2(u, \alpha_s x) + \frac{1}{2}\omega^2(\alpha_s x, \alpha_s y) - \frac{1}{4}\omega^2(y, \alpha_s y) \\ &\le \frac{1}{2}(R+\nu)^2 + \frac{1}{2}(R+\nu)^2 - \frac{1}{4}\varepsilon^2 \le (R-\nu)^2\end{aligned}$$

Thus, $\omega(u, \alpha_s x) < R - \nu$ which implies that

$$r(u, \alpha_s x) < r(y, \bar{co}\{\alpha_s x\}).$$

is a contradiction. Hence, $y \in F_o(\alpha_s)$.
Necessary and sufficient result for Theorem 2 is presented as follow:

Theorem 3. *Let X be a nonempty closed subset of a convex waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \wp$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself, that is, $\alpha_s \in ON\mathcal{T}_n$ for $s \in S$. The set $\{\alpha_s x,\ s \in S\}$ is bounded for some $x \in X$ if and only if $F_o(\alpha_s)$ is nonempty.*

Proof: Assume that $\{\alpha_s x,\ s \in S\}$ is bounded for some $x \in X$, there is a unique element $y \in X$ for which $y \in A(X, \bar{co}\{\alpha_s x\})$. By Theorem 2, $F_o(\alpha_s)$ is nonempty. The converse is obvious.

Remark 4. *If in Theorem 2, the boundedness assumption on $\{\alpha_s x,\ s \in S\}$ is dropped, then another suitable concept is stated in the next theorem.*

Theorem 4. *Let X be a closed subset of a complete waist metric space $(M, \omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s \in \wp$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself, that is, $\alpha_s \in ON\mathcal{T}_n$ for $s \in S$. Then, α_s has at least one fixed point.*

Proof: For $\delta \in (0, 1]$, set $T_\delta = (1 - \delta)\alpha_s$. It follows that T_δ is a δ-contraction on X and by the Banach fixed point theorem, there exists x_δ for $\delta \in (0, 1]$ such that $T_\delta x_\delta = x_\delta$. Now, we show that for $\delta_k \to 0$, the net x_{δ_k} converges to p, where p is a fixed point of α_s. Indeed, for any arbitrary $u \in X$, we have

$$\begin{aligned}\omega^2(x_{\delta_k}, u) &= \omega^2(x_{\delta_k} - p, u - p) \\ &= \omega^2(x_{\delta_k}, p) + \omega^2(p, u) + 2\omega(x_{\delta_k} - p, u - p) \\ &\le \omega^2(x_{\delta_k}, p) + \omega^2(p, u)\end{aligned}$$

By setting $u = \alpha_s p$, we have

$$\limsup \left(\omega^2(x_{\delta_k}, \alpha_s p) - \omega^2(x_{\delta_k}, p)\right) \le \omega^2(p, \alpha_s p) \tag{11}$$

Also, since $T_{\delta_k} x_{\delta_k} = x_{\delta_k}$ and $\delta_k \to 0$ as $k \to \infty$, then

$$\begin{aligned}\lim_{k\to\infty} \omega(x_{\delta_k}, \alpha_s x_{\delta_k}) &= \lim_{k\to\infty} [\omega(x_{\delta_k}, T_{\delta_k} x_{\delta_k}) + \delta_k \omega(0, \alpha_s x_{\delta_k})] \\ &= \lim_{k\to\infty} \omega(x_{\delta_k}, T_{\delta_k} x_{\delta_k}) \to 0\end{aligned}\tag{12}$$

On the other hand, since $\alpha_s \in ON\mathcal{T}_n$, then

$$\omega(\alpha_s p, \alpha_s x_{\delta_k}) \le \omega(p, x_{\delta_k}) \tag{13}$$

We have from (12) and (13) that

$$\omega(x_{\delta_k}, \alpha_s p) \le \omega(x_{\delta_k}, \alpha_s x_{\delta_k}) + \omega(x_{\delta_k}, p)$$

which further implies

$$\limsup[\omega(x_{\delta_k}, \alpha_s p) - \omega(x_{\delta_k}, p)] \le \lim_{k\to\infty} \omega(x_{\delta_k}, \alpha_s x_{\delta_k}) = 0 \tag{14}$$

Also, from (11) and (14), we obtain

$$\limsup\left(\omega^2(x_{\delta_k},\alpha_s p)-\omega^2(x_{\delta_k},p)\right)=\limsup\left(\omega(x_{\delta_k},\alpha_s p)-\omega(x_{\delta_k},p)\right)\times\left(\omega(x_{\delta_k},\alpha_s p)+\omega(x_{\delta_k},p)\right)$$

Thus,

$$\limsup\left(\omega^2(x_{\delta_k},\alpha_s p)-\omega^2(x_{\delta_k},p)\right)=\omega^2(p,\alpha_s p)=0$$

and hence, p is the fixed point of α_s

Theorem 5. *Let X be a closed subset of a complete waist metric space $(M,\omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s\in\wp$ is a mapping satisfying (7), that is, $\alpha_s\in ON\mathcal{T}_n$ for $s\in S$. Then, the set $F_o(\alpha_s)\subset X$ is a nonempty closed convex set.*

Proof: Since α_s satisfies (7), by Theorem 2, α_s has fixed point in X. It is left to show that $F_o(\alpha_s)$ is closed and convex. Firstly, we show that $F_o(\alpha_s)$ is closed. Let $\{x_t\}$ be a net in $F_o(\alpha_s)$ such that $x_t\to x$, then by hypothesis:

$$\omega(\alpha_t x,x)\leq\omega(\alpha_t x,x_t)+\omega(x_t,x)\to 0$$

This implies $\alpha_t x\to x\in F_o(\alpha_s)$. Hence, $F_o(\alpha_s)$ is closed.
Also, let $x,y\in F_o(\alpha_s)$ and $\lambda\in[0,1]$, we have $u_\lambda=u(x,y,\lambda)\in F_o(\alpha_s)$.
Indeed,

$$\omega(\alpha_s u_\lambda,x)=\omega(\alpha_s u_\lambda,\alpha_s x)\leq\omega(u_\lambda,x)$$

Similarly, $\omega(\alpha_s u_\lambda,y)\leq\omega(u_\lambda,y)$. Thus,

$$\omega(x,y)\leq\omega(x,\alpha_s u_\lambda)+\omega(\alpha_s u_\lambda,y)\leq\omega(x,y).$$

This shows that for some a,b with $0\leq a,b\leq 1$, we have
$\omega(x,\alpha_s u_\lambda)=a\omega(x,u_\lambda)$ and $\omega(y,\alpha_s u_\lambda)=b\omega(y,u_\lambda)$
from which it follows that $\alpha_s u_\lambda\in F_o(\alpha_s)$.
By letting $S=\mathbb{N}$ in Theorems 4 and 5, we obtain the existence property of a nonexpansive semigroup of order-preserving total mapping in waist metric spaces as follow:

Corollary 1. *Let X be a closed subset of a convex waist metric space (M,ω). Suppose $\alpha\in\wp=\{\alpha^n:n\in\mathbb{N}\}$ is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself. Then. $F_o(\alpha_s)\neq\emptyset$ if and only if $\{\alpha^n x:n\in\mathbb{N}\}$ is bounded for some $x\in X$. Furthermore, $F_o(\alpha_s)$ is closed and convex.*

Theorem 6. *Let X be a closed subset of a complete waist metric space $(M,\omega)_\alpha$ and let S be a semitopological semigroup. Suppose $\alpha_s\in\wp=\{\alpha_s:s\in S\}$ is a mapping satisfying (8). Then,*

i. the map α_s exists;

ii. $F_o(\alpha_s)$ is nonempty.

Proof:

i. Let $x_1,x_2\in X$ with $x_1<x_2$. Suppose, on contrary, that $\alpha_s\notin OE\mathcal{T}_n$, then α_s is an order-reversing map, that is, $x_1\alpha_s\geq x_2\alpha_s$. By induction,

$$x_k\alpha_s\geq x_{k+1}\alpha_s,\quad\text{for } k=1,2,3,\ldots,n-1$$

Define $x_{k+1}=x_k\alpha_s$. Using condition (8), there gives

$$\begin{aligned}\omega(x_{k+1},x_{k+2})&\geq\omega^+(x_{k+1}\alpha_s,x_{k+2}\alpha_s)\\&>\omega^+(x_{k+1},x_{k+2})\end{aligned}$$

This is a clear contradiction. Hence, $\alpha_s\in OE\mathcal{T}_n$.

ii. Let $\alpha_s\in OE\mathcal{T}_n$. Suppose that $F_o(\alpha_s)$ is empty, that is, there is no fixed β such that $\beta\in F_o(\alpha_s)$, this implies that $\omega^+(\beta,\beta\alpha_s)>\varepsilon$, for $\varepsilon>0$. Let $\omega^+(x_k\alpha_s,\beta)=0$, by condition (8), there results

$$\begin{aligned}\omega(x_k,\beta)&<\omega(x_k\alpha_s,\beta\alpha_s)\\&\leq\omega(x_k\alpha_s,\beta)+\omega(\beta,\beta\alpha_s)\\&=\omega(\beta,\beta\alpha_s)\end{aligned}$$

On the other hand, since α_t is continuous on X for each $t\in S$. Then, for $x_k\in X$ and $\tau>0$, $\omega(x_k,\beta)>\tau$ implies that $\omega(x_k\alpha_t,\beta\alpha_t)>\frac{\varepsilon}{2}$.
Since $\omega^+(x_k\alpha_s,\beta)=0$, then there exists $b\in S$ such that $\omega^+(x_k\alpha_{ab},\beta)>\max\{\frac{\varepsilon}{2},\delta\}$ for each $a\in S$. Intuitively, $\omega^+(x_k\alpha_t\alpha_{ab},\beta\alpha_t)>\frac{\varepsilon}{2}$.
But then,

$$\omega^+(\beta\alpha_t,\beta)\leq\omega^+(\beta\alpha_t,x_k\alpha_{tab})+\omega^+(x_k\alpha_{tab},\beta)$$

and

$$\omega^+(\beta\alpha_t,x_k\alpha_{tab})+\omega^+(x_k\alpha_{tab},\beta)>\varepsilon$$

From the last two inequalities, we obtain $\omega^+(\beta\alpha_t,\beta)=\varepsilon$. This is a contradiction. Therefore, $\omega^+(\beta\alpha_s,\beta)=0$ for each $s\in S$ and $\beta\in F_o(\alpha_s)$.

Remark 5. *The convergence theorems (strong convergence and Δ-convergence) of the map $\alpha_s\in\wp$ satisfying the nonexpansive semigroup of order-preserving total mapping (7) in waist metric spaces can be routinely proved using the next lemma and their left.*

Lemma 5. *Let X be a closed subset of a convex waist metric space (M,ω). Suppose α_s is a nonexpansive semigroup of order-preserving total mapping (7) of X into itself with $F_o(\alpha_s)\neq\emptyset$. Then $\lim_s\omega(\alpha_s x,y)$ exists for each $y\in F_o(\alpha_s)$ and $s\in S$.*

4. Practical Examples

The following three examples are considered to justify the theorems in the main results. The first two examples are from the same family for $n=2$ and $n=3$, respectively, while the third example is given for $n=3$.

Example 3. *Let S be a semigroup and $X=\{1,2\}$. Let $\alpha_s:\{1,2\}\to\{1,2\}$ be the mapping $\alpha_s=(1)(2)$ in $O\mathcal{T}_2$, where $s\in S$, associated to $x\alpha_s=x^2-2x+2$ in $X\subset(M,\omega)_\alpha$.*

The map α_s is a nonexpansive order-preserving total mapping. Hence, it satisfies all hypotheses of Theorem 2 and 4 and the set $F_o(\alpha_s)=\{1,2\}$.

Example 4. *Let α_s be a mapping in $\mathcal{OT}_3$ define by $\alpha : \{1,2,3\} \to \{1,2,3\}$ as $\alpha = (1)[3\ \ 2\ \ 1)$ which is equivalent to the map $x\alpha_s = \frac{x^2-3x}{2} + 2$.*

Here, the map α_s is also a nonexpansive order-preserving total mapping, that is, $\alpha_s \in ON\mathcal{T}_3$. Thus, it satisfies all hypotheses of Theorem 2 and 4. The set $F_o(\alpha_s) = \{1\}$.

Example 5. *Let $\alpha_s : \{1,2,3\} \to \{1,2,3\}$ be given by the mapping $\alpha = (1)[2\ \ 1)(3)$ in $\mathcal{OT}_3$ corresponding to the map $x\alpha_s = x^2 - 3x + 3$.*

In Example 5, all hypotheses of Theorem 6 are satisfied since $\alpha_s \in OE\mathcal{T}_n$ with fixed point $F_o(\alpha_s) = \{1,3\}$.

5. Concluding Remark

This study introduced the nonexpansive and expansive semigroup of order-preserving total mappings to prove some fixed point theorems in complete waist metric spaces. The study also considered some examples (see Examples 3, 4 and 5) on semigroup of order-preserving total mappings to validate the hypotheses of Theorems 4, 5 & 6. Results show that the nonexpansive (and expansive) semigroup of order-preserving mappings compares favorably with the semigroup of mappings $OC\mathcal{T}_n$ (and $OC^*\mathcal{T}_n$). In fact, every mapping $\alpha_s \in \wp$ under the action of subsemigroups $OC\mathcal{T}_n$ and $OC^*\mathcal{T}_n$ is also in $ON\mathcal{T}_n$ and $OE\mathcal{T}_n$, respectively. However, this study only features some elements of subsemigroups of order-preserving full mappings in [27]. Therefore, future studies would be to establish some notions to study other elements of subsemigroups in $\wp$ such as order-preserving partial mappings [1, 28, 15], order decreasing full mappings [29], symmetric inverse semigroups [5], orientation-preserving mappings [30], fence-preserving mappings [31] among others.

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