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On Lemniscate of Bernoulli of q-Janowski type

Afis Saliu[a,b,*], Semiu Oladipupo Oladejo[b]

[a] *Department of Mathematics, University of the Gambia, MDI Road, Kanifing P.O. Box 3530, Serrekunda, The Gambia*
[b] *Department of Mathematics, Gombe State University P.M.B 127, Tudun Wada, Gombe, Gombe State, Nigeria*

Abstract

In this article, we introduce the q-analogue of functions characterized by the lemniscate of Bernoulli in the right-half plane and define the class $\mathbb{L}_q^*(A,B)$. Furthermore, we study the geometric properties of this class, which include coefficient inequalities, subordination factor sequence property, radii results and Fekete-Szegö problems. Some deductions of our results show relevant connections between this present work and the existing ones in many literature. It is worthy of note that some of our results are sharp.

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1. Introduction and Preliminaries

The theory of univalent functions had its history from the Riemann mapping theorem [1]. This theorem aroused the interest of many researchers to start working in this field. For example, Bieberbach [1] proved that for every univalent function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U := \{z \in \mathbb{C} : |z| < 1\}) \,, \tag{1}$$

$$|a_2| \leq 2$$

and conjectured that, in general,

$$|a_n| \leq n \,, \quad n \geq 2.$$

This conjecture stood for more than fifty years until it was finally settled by De Branges [2] in 1985. Geometric Function Theory (GFT) has witnessed many developements and introduction of new subfamilies of univalent functions within these intervening years. In particular, Ma and Minda [3] gave a comprehensive classification of classes S^* and C of starlike and convex functions [1], respectively. For this purpose, they considered the class $\mathbb{A}$ of normalized analytic functions (1) and a univalent function φ (which maps U onto domains symmetric with the real axis and starlike with respect to 1) normalized such that $\varphi(0) = 1$ and $\varphi'(0) > 0$. Thus, the classes are respectively defined as follows:

$$S^*(\varphi) = \left\{ f \in \mathbb{A} : \frac{zf'(z)}{f(z)} \prec \varphi \right\}$$

and

$$C(\varphi) = \left\{ f \in \mathbb{A} : zf' \in S^*(\varphi) \right\} \,.$$

In particular, if we chose $\varphi = (1+z)/(1-z)$, S^* and C respectively become the usual classes S^* and C of starlike and convex

*Corresponding author tel. no: +2348066221155, +22941154
Email address: asaliu@utg.edu.gm, saliugsu@gmail.com (Afis Saliu)

functions. Sokół and Stankiewicz [4] introduced the class $\mathbb{L}^*$ by setting $\varphi(z) = \sqrt{1+z}$. This class consists of those functions that map U onto the lemniscate of Bernoulli. They obtained radius of convexity, growth and distortion results for this class. Also, early coefficients of functions in $\mathbb{L}^*$ were derived in [5]. Furthermore, Ali et al. [6] and Sokół [7] determined various radii results associated with this class. Moreover, Raza and Malik [8] proved the third Hankel determinant associated with $\mathbb{L}^*$. For other related works, see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] for details.

One of the most recent approaches in GFT is the concept of calculus without the notion of limits known as q-calculus, or quantum calculus [21, 22]. This idea was first discussed in the theory of univalent functions by Ismail et al. [23]. They introduced the class S_q^* of q-starlike functions, which have led to many new results and techniques in quantum calculus associated with GFT. Recently, Khan et al. [24] gave a q-analogue of $\sqrt{1+z}$ and introduced the class $\mathbb{L}_q^*$. Moreover, they proved few coefficient bounds and obtained the third Hankel determinant for the class.

Motivated by these works and in particular, [24, 8] we introduce a new class $\mathbb{L}_q^*(A,B)$ and establish many coefficient inequalities, subordination factor sequence, radii results and Fekete-Szegö bounds for this class.

2. Definitions and Lemmas

Let $\mathcal{W}$ be the class of analytic functions

$$w(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in U \tag{2}$$

such that $w(0) = 0$ and $|w(z)| < 1$. These functions are known as Schwarz functions. If $f(z)$ and $g(z)$ are analytic functions in U, then $f(z)$ is subordinate to $g(z)$ (written as $f(z) \prec g(z)$) if there exists $w(z) \in \mathcal{W}$ such that $f(z) = g(w(z)),\ z \in U$. In addition, if g is univalent in U, then $f(0) = g(0)$ and $f(U) \subset g(U)$. An analytic function

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{3}$$

is a function with positive real part. The class of all such functions is denoted by $\mathbb{P}$. More generally, for $-1 \le B < A \le 1$, the class $\mathbb{P}(A,B)$ consists of functions $p(z)$ of the form (3) satisfying the subordination condition

$$p(z) \prec \frac{1+Az}{1+Bz}, \quad z \in U.$$

On the other hand, $p \in \mathbb{P}(A,B)$ is called a Janowski function [25] if and only if

$$p(z) = \frac{(1+A)h(z) + (1-A)}{(1+B)h(z) + (1-B)}, \quad -1 \le B < A \le 1,$$

where $h \in \mathbb{P}$.

Definition 2.1 (Subordinating Factor Sequence)**.** *[26] A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ of the form* (1) *is analytic, univalent and convex in U, we have the subornation given by*

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in U,\ a_1 := 1).$$

Definition 2.2. *[27] Let $q \in (0,1)$. Then the q-number $[n]_q$ is given as*

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \sum_{\iota=0}^{n-1} q^\iota = 1 + q + q^2 + \cdots + q^{n-1}, & n \in \mathbb{N}, \\ n, & \text{as } q \to 1^-. \end{cases} \tag{4}$$

and the q-derivative of a complex valued function $f(z)$ in U is given by

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & z \neq 0 \\ f'(0), & z = 0, \\ f'(z), & \text{as } q \to 1^-. \end{cases} \tag{5}$$

From the above explanation, it is easy to see that for $f(z)$ given by (1)*,*

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{6}$$

Using the idea of q-calculus, Khan et al. [24] extended the work of Raza and Malik [8] by presenting the following class:

Definition 2.3. *[24] A function of the form* (1) *belongs to the class $\mathbb{L}_q^*$ if and only if*

$$\left| \left(\frac{zD_q f(z)}{f(z)} \right)^2 - \frac{1}{1-q} \right| < \frac{1}{1-q}, \tag{7}$$

or equivalently,

$$\left(\frac{zD_q f(z)}{f(z)} \right)^2 \prec \frac{2(1+z)}{2+(1-q)z}, \quad z \in U,\ q \in (0,1). \tag{8}$$

Inspired by the work of Khan et al. [24] and Raza and Malik [8], and in view of the Janowski functions, we introduce the following class.

Definition 2.4. *A function of the form* (1) *belongs to the class $\mathbb{L}_q^*(A,B)$ if and only if*

$$\left| \frac{(B-1)\left(\frac{zD_q f(z)}{f(z)}\right)^2 - (A-1)}{(B+1)\left(\frac{zD_q f(z)}{f(z)}\right)^2 - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}, \tag{9}$$

or equivalently,

$$\left(\frac{zD_q f(z)}{f(z)} \right)^2 \prec \frac{(AO_1 + O_3)z + 4}{(BO_1 + O_3)z + 4}, \quad zU,\ q \in (0,1), \tag{10}$$

where

$$O_1 = 1 + q, \quad O_3 = 3 - q, \quad -1 \le B < A \le 1. \tag{11}$$

Remark 2.5. *For $A = 1$ and $B = -1$, we are back to the class in Definition 2.3. In addition to that, if $q \to 1^-$, Definition 2.4 reduces to the one given in [4, 7].*

To prove our main findings, we need the following results

Lemma 2.6. *[28] If $w \in \mathcal{W}$ is of the form (2), then for a real number σ,*

$$|w_2 - \sigma w_1^2| \leq \begin{cases} -\sigma, & \text{for } \sigma \leq -1, \\ 1, & \text{for } -1 \leq \sigma \leq 1, \\ \sigma & \text{for } \sigma \geq 1. \end{cases}$$

When $\sigma < -1$ or $\sigma > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < \sigma < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $\sigma = -1$ if and only if $w(z) = \frac{z(x+z)}{1+xz}$ $(0 \leq x \leq 1)$ or one of its rotations while for $\sigma = 1$, equality holds if and only if $w(z) = -\frac{z(x+z)}{1+xz}$ $(0 \leq x \leq 1)$ or one of its rotations.

Also, the sharp upper bound above can be improved as follows when $-1 < \sigma < 1$:

$$|w_2 - \sigma w_1^2| + (1+\sigma)|w_1|^2 \leq 1 \ \ (-1 < \sigma \leq 0)$$

and

$$|w_2 - \sigma w_1^2| + (1-\sigma)|w_1|^2 \leq 1 \ \ (0 < \sigma < 1).$$

Lemma 2.7. *[26] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0, \quad (zU).$$

In the next section, we present our main findings.

3. Main Results

Theorem 3.1. *Let $f \in \mathbb{A}$ and suppose*

$$\sum_{n=3}^{\infty}\left\{\sum_{j=1}^{n-1}\left[2\left([j]_q[n-j]_q - 1\right)\right.\right. \tag{12}$$

$$\left.\left. + \left|(B+1)[j]_q[n-j]_q - (A+1)\right|\right]\left|a_j a_{n-j}\right|\right\} < \left|B - A\right|. \tag{13}$$

Then $f \in \mathbb{L}_q^*(A,B)$

Proof. Suppose condition (12) is satisfied. Then it suffices to prove (9). Now,

$$\begin{aligned}
&\left|\frac{(B-1)\left(\frac{zD_qf(z)}{f(z)}\right)^2 - (A-1)}{(B+1)\left(\frac{zD_qf(z)}{f(z)}\right)^2 - (A+1)} - \frac{1}{1-q}\right| \\
\leq &\left|\frac{(B-1)\left(\frac{zD_qf(z)}{f(z)}\right)^2 - (A-1)}{(B+1)\left(\frac{zD_qf(z)}{f(z)}\right)^2 - (A+1)} - 1\right| + \frac{q}{1-q} \\
=2 &\left|\frac{\left(zD_qf(z)\right)^2 - (f(z))^2}{(B+1)\left(zD_qf(z)\right)^2 - (A+1)(f(z))^2}\right| + \frac{q}{1+q} \\
=2 &\left|\frac{\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} a_j a_{n-j}[j]_q[n-j]_q\right)z^n - \sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} a_j a_{n-j}\right)z^n}{(1+B)\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} a_j a_{n-j}[j]_q[n-j]_q\right)z^n - (A+1)\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} a_j a_{n-j}\right)z^n}\right| + \frac{q}{1-q} \\
=2 &\left|\frac{\sum_{n=3}^{\infty}\left[\sum_{j=1}^{n-1} a_j a_{n-j}\left([j]_q[n-j]_q - 1\right)\right]z^n}{(B-A)z^2 + \sum_{n=3}^{\infty}\left\{\sum_{j=1}^{n-1} a_j a_{n-j}\left[(B+1)[j]_q[n-j]_q - (A+1)\right]\right\}z^n}\right| \\
\leq &\frac{2\sum_{n=3}^{\infty}\left[\sum_{j=1}^{n-1}\left|a_j a_{n-j}\right|\left([j]_q[n-j]_q - 1\right)\right]}{|B-A| + \sum_{n=3}^{\infty}\left[\sum_{j=1}^{n-1}\left|a_j a_{n-j}\right|\left|(B+1)[j]_q[n-j]_q - (A+1)\right|\right]} + \frac{q}{1-q}.
\end{aligned}$$

This last expression is bounded by $\frac{1}{1-q}$ provided (12) is satisfied. Hence, we have the required result. We observe that Theorem 3.1 implies the following result. □

Corollary 3.2. *Let $f \in \mathbb{A}$. Then $f \in \mathbb{L}_q^*(A,B)$ if*

$$\sum_{n=2}^{\infty}\left[2q[n-1]_q+\left|(B+1)[n]_q-(A+1)\right|\right]|a_n| < |B-A|. \quad (14)$$

Proof. Let

$$\Phi_{j,n} = \left[2\left([j]_q[n-j]_q-1\right)+\left|(B+1)[j]_q[n-j]_q - (A+1)\right|\right]\left|a_j a_{n-j}\right| < |B-A|$$

in (12). Then

$$\sum_{n=3}^{\infty}\left(\Phi_{1,n}+\Phi_{2,n}+\Phi_{3,n}+\cdots+\Phi_{n-1,n}\right) < |B-A|,$$

which implies that

$$\sum_{n=3}^{\infty}\Phi_{n-1,n} < |B-A|.$$

That is

$$\sum_{n=3}^{\infty}\left[2\left([n-1]_q-1\right)+\left|(B+1)[n-1]_q-(A+1)\right|\right]|a_{n-1}| < |B-A|$$

or

$$\sum_{n=2}^{\infty}\left[2q[n-1]_q+\left|(B+1)[n]_q-(A+1)\right|\right]|a_n| < |B-A|$$

□

For $A=1, B=-1$ in Corollary **??**corollary1, we are led to the following assertion.

Corollary 3.3. *Let $f \in \mathbb{A}$. Then $f \in \mathbb{L}_q^*(1,-1)$ if*

$$\sum_{n=2}^{\infty}[n]_q|a_n| < 1.$$

As $q \to 1^-$ in Corollary **??**corollary2, we have

Corollary 3.4. *Let $f \in \mathbb{A}$. Then $f \in \mathbb{L}^*$ if*

$$\sum_{n=2}^{\infty} n|a_n| < 1.$$

By Corollary 3.2, the class $L_q^*(A,B)$ is assumed to be a subclass of $\mathbb{L}_q^*(A,B)$ consisting of $f(z)$ of the form (14) and satisfying the condition (14). Therefore, by the coefficient inequality for the class $L_q^*(A,B)$, we have the following results.

Theorem 3.5. *Let $-1 \le B_1 \le B_2 < A_1 \le A_2 \le 1$. If $f \in \mathbb{A}$, then*
$L_q^*(A_2,B_2) \subset L_q^*(A_1,B_1)$.

Proof. By the hypothesis of the theorem, we have that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[2q[n-1]_q+\left|(B_1+1)[n-1]_q-(A_1+1)\right|\right]|a_n| \\
&\le \sum_{n=2}^{\infty}\left[2q[n-1]_q+\left|(B_2+1)[n-1]_q-(A_2+1)\right|\right]|a_n| \\
&\le |B_2-A_2| \\
&\le |B_1-A_1|.
\end{aligned}$$

Thus, $f \in L_q^*(A_1,B_1)$. □

Theorem 3.6. *Let $f_j \in L_q^*(A,B)$ and be of the form*

$$f_j(z)=\sum_{n=1}^{\infty}a_{n,j}z^n, \quad j=1,2,\ldots m,\ a_{1,j}=1.$$

Then $G(z) \in L_q^*(A,B)$, where

$$G(z)=\sum_{j=1}^{m}|c_j|f_j(z) \quad \text{with} \quad \sum_{j=1}^{m}|c_j|=1.$$

Proof. In view of (14), we have

$$\sum_{n=2}^{\infty}\left[\frac{2q[n-1]_q+\left|(B+1)[n]_q-(A+1)\right|}{|B-A|}\right]|a_{n,j}| < 1.$$

Therefore,

$$\begin{aligned}
G(z) &= \sum_{j=1}^{m}|c_j|f_j(z) \\
&= \sum_{j=1}^{m}|c_j|\left(z+\sum_{n=2}^{\infty}a_{n,j}z^n\right) \\
&= \sum_{j=1}^{m}\sum_{n=1}^{\infty}|c_j|a_{n,j}z^n \\
&= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{m}|c_j|a_{n,j}\right)z^n.
\end{aligned}$$

But

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[\frac{2q[n-1]_q+\left|(B+1)[n]_q-(A+1)\right|}{|B-A|}\right]\left|\sum_{j=1}^{m}|c_j|a_{n,j}\right| \\
&\le \sum_{j=1}^{m}\left\{\sum_{n=2}^{\infty}\left[\frac{2q[n-1]_q+\left|(B+1)[n]_q-(A+1)\right|}{|B-A|}\right]|a_{n,j}|\right\}|c_j| \\
&< \sum_{j=1}^{m}|c_j| \\
&= 1.
\end{aligned}$$

Hence, $G(z) \in L_q^*(A,B)$. □

Theorem 3.7. *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. If $f, g \in L_q^*(A, B)$, then their weighted mean*

$$F_j(z) = \frac{(1-j)f(z) + (1+j)g(z)}{2} \tag{15}$$

also belongs to $L_q^*(A, B)$.

Proof. From (15), we have

$$F_j(z) = z + \sum_{n=2}^{\infty} \left[\frac{(1-j)a_n + (1+j)b_n}{2} \right] z^n.$$

To show that $F_j(z) \in L_q^*(A, B)$, we need to show that

$$\sum_{n=2}^{\infty} \Big[2q[n-1]_q + \Big|(B+1)[n]_q - (A+1)\Big| \Big] \left| \frac{(1-j)a_n + (1+j)b_n}{2} \right| < |B-A|.$$

For this, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \Big[2q[n-1]_q + \Big|(B+1)[n]_q - (A+1)\Big| \Big] \left| \frac{(1-j)a_n + (1+j)b_n}{2} \right| \\
&\leq \left(\frac{1-j}{2}\right) \sum_{n=2}^{\infty} \Big[2q[n-1]_q + \Big|(B+1)[n]_q - (A+1)\Big| \Big] |a_n| \\
&+ \left(\frac{1+j}{2}\right) \sum_{n=2}^{\infty} \Big[2q[n-1]_q + \Big|(B+1)[n]_q - (A+1)\Big| \Big] |b_n| \\
&< |B-A|.
\end{aligned}$$

□

Theorem 3.8. *Suppose $f_j(z) \in L_q^*(A, B)$. Then the arithmetic mean $F(z)$ of $f_j(z)$ given by*

$$F(z) = \frac{1}{n} \sum_{j=1}^{n} f_j(z)$$

is also in $L_q^*(A, B)$.

Proof. We have

$$\begin{aligned}
F(z) &= \frac{1}{n} \sum_{j=1}^{n} f_j(z) \\
&= \frac{1}{n} \sum_{j=1}^{n} \left(z + \sum_{k=2}^{\infty} a_{k,j} z^k \right) \\
&= z + \sum_{k=2}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} a_{k,j} \right) z^k.
\end{aligned}$$

Since $f_j(z) \in L_q^*(A, B)$, then

$$\begin{aligned}
&\sum_{k=2}^{\infty} \Big[2q[k-1]_q + \Big|(B+1)[k]_q - (A+1)\Big| \Big] \left| \frac{1}{n} \sum_{j=1}^{n} a_{k,j} \right| \\
&\leq \frac{1}{n} \sum_{j=1}^{n} \left\{ \sum_{k=2}^{\infty} \Big[2q[k-1]_q + \Big|(B+1)[k]_q - (A+1)\Big| \Big] \right\} |a_{k,j}| \\
&< \frac{1}{n} \sum_{j=1}^{n} |B-A| \\
&= |B-A|.
\end{aligned}$$

Thus, $F \in L_q^*(A, B)$. □

In the next theorem, we provide a sharp subordination result involving class $L_q^*(A, B)$.

Theorem 3.9. *Let $f \in L_q^*(A, B)$. Then for every convex function $g(z)$ in U, we have*

$$\frac{2q + |(1+B)q + B - A|}{2\left[2q + |B-A| + |(1+B)q + B - A|\right]} (f * g)(z) \prec g(z) \tag{16}$$

and

$$\operatorname{Re} f(z) > -\frac{\left[2q + |B-A| + |(1+B)q + B - A|\right]}{2q + |(1+B)q + B - A|}. \tag{17}$$

The constant

$$\frac{2q + |(1+B)q + B - A|}{2\left[2q + |B-A| + |(1+B)q + B - A|\right]}$$

cannot be replaced by any larger one.

Proof. Let $f \in L_q^*(A, B)$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be convex in U. Then

$$\begin{aligned}
&\frac{2q + |(1+B)q + B - A|}{2\left[2q + |B-A| + |(1+B)q + B - A|\right]} (f * g)(z) \\
&= \frac{2q + |(1+B)q + B - A|}{2\left[2q + |B-A| + |(1+B)q + B - A|\right]} \left(z + \sum_{n=2}^{\infty} a_n b_n z^n \right).
\end{aligned}$$

Therefore, by Definition 2.1, the assertion of our theorem holds if the sequence

$$\left\{ \frac{2q + |(1+B)q + B - A|}{2\left[2q + |B-A| + |(1+B)q + B - A|\right]} a_n \right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 2.7, this will hold true if and only if

$$\operatorname{Re} \left[1 + \sum_{n=1}^{\infty} \frac{2q + |(1+B)q + B - A|}{\left[2q + |B-A| + |(1+B)q + B - A|\right]} a_n z^n \right] > 0 \quad (z \in U).$$

Since

$$2q[n-1]_q + \Big|(B+1)[n]_q - (A+1)\Big|$$

is an increasing function of n $(n \geq 2)$, we have

$$\text{Re}\left[1+2\sum_{n=1}^{\infty}\frac{2q+|(1+B)q+B-A|}{2\left[2q+|B-A|+|(1+B)q+B-A|\right]}a_nz^n\right]$$
$$=\text{Re}\left[1+\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}z\right.$$
$$\left.+\sum_{n=2}^{\infty}\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}a_nz^n\right]$$
$$\geq 1-\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}r$$
$$-\sum_{n=2}^{\infty}\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}|a_n|\,r^n$$

$$>1-\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}$$
$$-\sum_{n=2}^{\infty}\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}|a_n|$$
$$\geq 1-\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}$$
$$-\frac{1}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}\sum_{n=2}^{\infty}\left[2q[n-1]_q\right.$$
$$\left.+\left|(B+1)[n]_q-(A+1)\right|\right]|a_n|$$
$$>1-\frac{2q+|(1+B)q+B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}$$
$$-\frac{|B-A|}{\left[2q+|B-A|+|(1+B)q+B-A|\right]}$$
$$=0,$$

where we have used condition (12). This proves (16). Furthermore, condition (17) is achieved by setting

$$g_0(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n$$

To prove the sharpness of the constant $\frac{2q+|B-A|}{4(q+|B-A|)}$, we consider the function given by

$$f(z)=z-\frac{|B-A|}{2q+|(1+B)q+B-A|}z^2.$$

Evidently, $f\in L_q^*(A,B)$ and from (16), we have

$$\frac{2q+|(1+B)q+B-A|}{2\left[2q+|B-A|+|(1+B)q+B-A|\right]}f(z)\prec\frac{z}{1-z}\quad(zU).$$

It is observed that

$$\min\left\{\text{Re}\left(\frac{2q+|(1+B)q+B-A|}{2\left[2q+|B-A|+|(1+B)q+B-A|\right]}f(z)\right)\right\}=-\frac{1}{2}.$$

This completes the proof. □

For $A=1, B=-1$ in Theorem 3.9, we have the following result.

Corollary 3.10. *Let $f\in L_q^*\subset\mathbb{L}_q^*$. then for every convex function $g(z)$ in U, we have*

$$\frac{1+q}{2(q+2)}(f*g)(z)\prec g(z)$$

and

$$\text{Re}\, f(z)>-\frac{2+q}{1+q}.$$

Moreover, the constant $\frac{1+q}{2(2+q)}$ cannot be a larger one.

As $q\to 1^-$ in Corollary 3.10, we obtain the following assertion.

Corollary 3.11. *Let $f\in L^*\subset\mathbb{L}^*$. then for every convex function $g(z)$ in U, we have*

$$\frac{1}{3}(f*g)(z)\prec g(z)$$

and

$$\text{Re}\, f(z)>-\frac{3}{2}.$$

The constant factor $\frac{1}{3}$ cannot be replaced by a larger one.

In the following theorems, we present some radii results associated with the class $\mathbb{L}_q^*(A,B)$.

Theorem 3.12. *$\mathbb{L}_q^*(A,B)\subset\mathbb{L}_q^*$ in the disc $|z|<R_q(A,B)$, $-1\leq B<A\leq 1$, where*

$$R_q(A,B)=\min\left\{\frac{4}{2(A-B)+\left[2(1+B)+(1-q)(1-A)\right]},\ 1\right\}.\quad(18)$$

Proof. To determine $\mathbb{L}_q^*$ radius, we need to find R such that $0<R<1$ and

$$H(zR):=\frac{(AO_1+O_3)\,z+4}{(BO_1+O_3)\,z+4}\prec\frac{2(1+z)}{2+(1-q)z}:=Q(z)\quad(z\in U),$$

where O_1 and O_3 are given by (11). This is equivalent to

$$|Q^{-1}(H(Rz))|\leq 1.$$

It is easy to see that

$$|Q^{-1}(H(Rz))|=\left|\frac{2(A-B)Rz}{4+\left[(2B-(1-q)A)+O_3\right]Rz}\right|$$
$$\leq\frac{2(A-B)R}{4-\left[2(1+B)+(1-q)(1-A)\right]R}.$$

The last expression is bounded by 1 if

$$2(A-B)R\leq 4-\left[2(1+B)+(1-q)(1-A)\right]R.$$

This completes the proof. □

Theorem 3.13. *Let $f\in S_q^*$. Then $f\in\mathbb{L}_q^*$ in the disc $|z|<R_q$, where*

$$R_q=\frac{1}{3+\sqrt{11-2q+|2q-1|}}.\quad(19)$$

Proof. Since $f \in S_q^*$, then

$$\frac{zD_qf(z)}{f(z)} \prec \frac{1+z}{1-qz} \quad (z \in U, \text{ see [29, 30, 31]}),$$

and by subordination property, we have

$$\left(\frac{zD_qf(z)}{f(z)}\right)^2 \prec \left(\frac{1+z}{1-qz}\right)^2 := S(z) \quad (z \in U).$$

We need to determine the smallest positive radius R such that

$$S(Rz) \prec Q(z) \quad (z \in U),$$

which is equivalent to showing that

$$\left|Q^{-1}(S(Rz))\right| \leq 1 \quad (z \in U).$$

Now, we can obviously see that

$$\left|Q^{-1}(S(Rz))\right| = \left|\frac{2\left(2R+(1-q)R^2z^2\right)}{1-2Rz-(2q-1)R^2z^2}\right| \leq \frac{2\left(2(1-q)+R^2\right)}{1-2R-|2q-1|R^2}.$$

This expression is bounded by 1 if

$$(2(1-q)+|2q-1|)\,R^2+6R-1 \leq 0.$$

Let $T(R) = (2(1-q)+|2q-1|)\,R^2+6R-1$. Then $T(0)T(1) < 0$. Therefore, there exists $R \in (0,1)$ such that $T(R) = 0$. Hence, we have the result. □

As $q \to 1^-$ in Theorem 3.13, we get the following result.

Corollary 3.14. *Let $f \in S^*$. Then $f \in \mathbb{L}^*$ in the disc $|z| < \frac{1}{3+\sqrt{10}}$.*

In the next theorem, using Lemma 2.6, we present the Fekete-Szegö inequality for the class $\mathbb{L}_q^*(A,B)$.

Theorem 3.15. *Let $f \in \mathbb{L}_q^*$. Then for a real number μ,*

$$|a_3-\mu a_2^2| \leq \left(\frac{A-B}{8q}\right)\begin{cases} -\frac{qV(q)+2\mu(A-B)(1+q)^2}{16q}, \\ \quad \text{for } \mu \leq -\frac{q(16+V(q))}{2(A-B)(1+q)^2} := \rho_1, \\ 1, \quad \text{for } \rho_1 \leq \mu \leq \frac{q(16-V(q))}{2(A-B)(1+q)^2} := \rho_2, \\ \frac{qV(q)+2\mu(A-B)(1+q)^2}{16q}, \quad \text{for } \mu \geq \rho_2,. \end{cases}$$

It is asserted also that

$$|a_3-\mu a_2^2| + \left[\mu + \frac{q(16+V(q))}{2(A-B)(1+q)^2}\right]|a_2|^2 \leq \frac{A-B}{8q},$$
$$-\frac{q(16+V(q))}{2(A-B)(1+q)^2} < \mu \leq -\frac{qV(q)}{2(A-B)(1+q)^2}$$

and

$$|a_3-\mu a_2^2| - \left[\mu - \frac{q(16-V(q))}{2(A-B)(1+q)^2}\right]|a_2|^2 \leq \frac{A-B}{8q},$$
$$-\frac{qV(q)}{2(A-B)(1+q)^2} < \mu \leq \frac{q(16-V(q))}{2(A-B)(1+q)^2}$$

where

$$V(q) = (A+3B-4)q^2-(A-5B-12)q-2(A-B). \quad (20)$$

These bounds are sharp.

Proof. Let $f \in \mathbb{L}_q^*(A,B)$. Then

$$\frac{zD_qf(z)}{f(z)} = \sqrt{\frac{1+A_1w(z)}{1+B_1w(z)}},$$

where

$$w(z) = \sum_{n=1}^{\infty} w_nz^n \in \mathcal{W}, \quad A_1 = \frac{AO_1+O_3}{4}, \quad \text{and} \quad B_1 = \frac{BO_1+O_3}{4}.$$

Therefore,

$$\begin{aligned} &1+qa_2z+q\left([2]_qa_3-a_2^2\right)z^2+\ldots \\ &= 1+\frac{w_1O_1(A-B)}{8}z-\frac{O_1(A-B)}{8}\left[\frac{1}{16}\Big(AO_1+4O_3 \right.\\ &\left.\quad +3BO_1\Big)w_1^2-w_2\right]z^2+\ldots. \end{aligned}$$

On comparing coefficients, we have

$$a_2 = \frac{(A-B)O_1w_1}{8q} \quad \text{and} \quad a_3 = \frac{A-B}{8q}\left(w_2-\frac{V(q)}{16}w_1^2\right),$$

so that

$$a_3-\mu a_2^2 = \frac{A-B}{8q}(w_2-\sigma w_1^2),$$

where

$$\sigma = \frac{qV(q)+2\mu(A-B)O_1^2}{16q}.$$

Then we have the required result by applying Lemma 2.6. These inequalities are sharp for the functions

$$\begin{cases} \bar{\lambda}f_1(z\lambda), & \text{for } \mu \in (-\infty,\rho_1)\cup(\rho_2,\infty), \\ \bar{\lambda}f_2(z\lambda), & \text{for } \rho_1 \leq \mu \leq \rho_2, \\ \bar{\lambda}f_3(z\lambda), & \text{for } \mu = \rho_1, \\ \bar{\lambda}f_4(z\lambda), & \text{for } \mu = \rho_2, \end{cases}$$

where $|\lambda| = 1$ and

$$\frac{zD_qf_1(z)}{f_1(z)} = \sqrt{\frac{1+A_1z}{1+B_1z}}$$
$$\frac{zD_qf_2(z)}{f_2(z)} = \sqrt{\frac{1+A_1z^2}{1+B_1z^2}}$$
$$\frac{zD_qf_3(z)}{f_3(z)} = \sqrt{\frac{1+A_1w_x(z)}{1+B_1w_x(z)}}$$
$$\frac{zD_qf_4(z)}{f_4(z)} = \sqrt{\frac{1-A_1w_x(z)}{1-B_1w_x(z)}}$$

with

$$w_x(z) = \frac{z(x+z)}{1+xz}, \quad 0 \leq x \leq 1.$$

□

Remark 3.16. *For $A = 1, B = -1$, Theorem 3.15 reduces to the Fekete-Szegö inequalities presented by Khan [24]. Further, as $q \to 1^-$ we are led to the result of Raza and Malik [8].*

4. Conclusion

In this work, we introduced the class $\mathbb{L}_q^*(A, B)$ and obtained Coefficient inequalities, subordination factor sequence property radii results and Fekete-Szegö inequality for it. Overall, we presented many consequences of our investigation. These results are fascinated essentially by their particular cases and consequences. Also, to have more new hypotheses under present assessments, new extension and applications can be investigated with some positive and novel results in different fields of science, particularly, in GFT. For more details about the applications, one may go through [32, 33].

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