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OPTIMIZATION OF ENERGY RECEPTION IN SYSTEMS POWERED BY PERIODICAL MOMENT

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Transmission can play an important role in mechanical systems of energy transfer. A theoretical analysis of a special type transmission is presented in this paper. The analysis is supported by computer simulation results obtained for the systems with a transmission of this type.

Key words: transmission, energy reception, periodical moment

1. Introduction

The transmission can play a very important role in mechanical energy transfer systems. By choosing a transmission with certain structures and parameters, it is possible to achieve optimal conditions for the reception of energy from engine; the achievement of such conditions may increase the power output. The problem is relatively simple when energy generated due to work of a constant force (or a constant moment). In this case, a constant ratio gear can offer conditions for optimal energy reception. The problem becouses more complicated in the case when energy is generated due to work of a periodical variable force, of which depends on the position of loaded element. In such a case, it can be profitable to use a mechanical transmission and one should define the structure of transmission mechanism, as well as specify the values of parameters affecting the transmission ratio.

The biomechanical drive (e.g., a man-powered aircraft, bicykle) is an example of the drive, where the moment depends on an angle of rotation (cf Ernst (1994)). In these examples, application of a transmission other then the conventional one may increase the power output from an engine, and in result, the machine motion may stabilize at a higher value of the average velocity (cf Wójcik (1992)), or some other optimum value.

2. Formulation of the problem

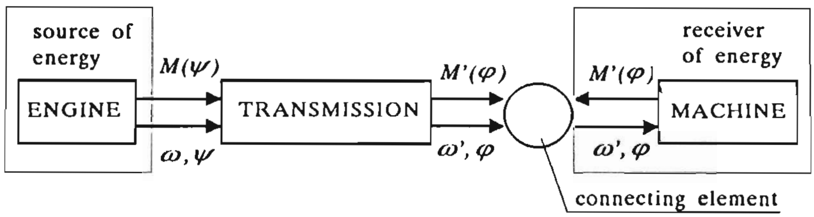


Fig. 1.

In the case being analyzed, the active moment (the moment of motor acting on the drive shaft) is a periodical function of the drive shaft angle of rotation (Fig.1). The work done by this moment during one drive shaft revolution is

$$L_{(2\pi)} = \int_0^{2\pi} M(\psi)\, d\psi \tag{2.1}$$

where

ψ – angle of rotation of the drive shaft

M – active moment.

The work done by any given moment (for a single revolution) is constant, an independent of time. However, the average power supplied by an engine at stable motion is

$$N_{avg} = \frac{1}{T} L_{(2\pi)} \tag{2.2}$$

where T is the time of one drive shaft revolution (period).

Angular velocity of the drive shaft is variable. Using $\widetilde{\omega}$ to represent the average velocity of stable engine motion, the following dependence can be defined

$$T = \frac{2\pi}{\widetilde{\omega}} \tag{2.3}$$

therefore, the average power is

$$N_{avg} = \frac{\widetilde{\omega}}{2\pi} L_{(2\pi)} \tag{2.4}$$

Increase in the average velocity causes the power output increase, which is possible due to a proper configuration of transmission, or adequate ratio change (without the average ratio change). The transmission ratio i, i.e. the relation of the driven shaft velocity $\dot{\varphi} = \omega'$ to that of the drive shaft $\dot{\psi} = \omega$, is not constant, but is a function of the drive shaft rotation angle and transmission parameters λ_ν

$$i(\psi, \lambda_\nu) = \frac{\dot{\varphi}}{\dot{\psi}} \qquad \nu = 1, ..., \mu \tag{2.5}$$

where μ is the number of transmission parameters.

The velocity of driven shaft ω' can also be variable. Since variable active moment and variable transmission ratio are periodic (the periodic variable passive moment can appear). The system motion is stabilized at the average velocity for which the average power supplied (from the motor) is equal to the average power absorbed (by the machine). Stable system motion occurs when the passive moment (or the active moment) is dependent on the velocity, which is always true in real systems.

Presenting the driven shaft velocity as a sum of constant component (average velocity) and variable components, respectively, we have

$$\omega'(t) = \tilde{\omega}' + \Delta\omega'(t) \tag{2.6}$$

where

$\tilde{\omega}'$ – driven shaft average velocity

$\Delta\omega'(t)$ – deviation from the average velocity (variable component)

and the passive moment in a power-series form (for the sake of simplicity, the function arguments ω' and $\Delta\omega'$, are omitted)

$$M'(\varphi, \omega') = M'(\varphi) + c_1\omega' + c_2\omega'^2 + ... \tag{2.7}$$

where the series coefficients, respectively, are

$$c_1 = \frac{\partial M'(\varphi, \omega')}{\partial \omega'}\Big|_{\omega'=0} \qquad c_2 = \frac{\partial^2 M'(\varphi, \omega')}{\partial \omega'^2}\Big|_{\omega'=0} \tag{2.8}$$

the average power on the driven shaft (received from the transmission) is defined by the relation

$$N'_{avg} = \frac{1}{T'}\int\limits_0^{T'} \Big[M'(\varphi) + c_1(\widetilde{\omega}' + \Delta\omega') + c_2(\widetilde{\omega}' + \Delta\omega')^2 + ...\Big](\widetilde{\omega}' + \Delta\omega')\, dt \tag{2.9}$$

where T' is the period of function in the stabilized motion.

The average power on the driven shaft Eq (2.9) can be shown as the sum of two components

$$N'_{avg} = \Big(N'_{avg}\Big)_{\widetilde{\omega}'} + \Big(N'_{avg}\Big)_{\Delta\omega'} \tag{2.10}$$

were the first one includes the components dependent only on the average velocity

$$\Big(N'_{avg}\Big)_{\widetilde{\omega}'} = \frac{1}{T'}\int\limits_0^{T'} \Big[M'(\varphi) + c_1\widetilde{\omega}' + c_2\widetilde{\omega}'^3 + ...\Big]\widetilde{\omega}'\, dt \tag{2.11}$$

while the second one includes components dependent on avariable component of the driven shaft velocity $\Delta\omega'$

$$\begin{aligned}\Big(N'_{avg}\Big)_{\Delta\omega'} &= \frac{1}{T'}\int\limits_0^{T'} M'(\varphi)\Delta\omega'\, dt + \frac{1}{T'}\int\limits_0^{T'} \Big[c_1(\Delta\omega')^2 + \\ &+ 2c_1\widetilde{\omega}'\Delta\omega' + c_2(\Delta\omega')^3 + 3c_2\widetilde{\omega}'^2\Delta\omega' + 3c_2\widetilde{\omega}'(\Delta\omega')^2 + ...\Big]\, dt\end{aligned} \tag{2.12}$$

The average power on the driven shaft (2.9) depends on the average velocity of the driven shaft and its variable component $\Delta\omega'$; therefore, the average power is also dependent on the transmission way. By a proper choice of a periodical variable transmission ratio (i.e., by modifying the transmission) one can affect that part of the average power on the driven shaft (2.12), which depends on the variable component of velocity. In this way, one can change the average velocity of the stabilized motion of the system. Therefore, the system motion is stabilized at the average velocity dependent on a periodical variable transmission ratio (i.e., dependent on the transmission parameters), provided the condition of the equality of average power supplied and received from the system is satisfied

$$N'_{avg} = N_{avg} \tag{2.13}$$

The transmission is modified by changing the values of its parameters λ_ν, while the average transmission ratio remains unchanged. The average transmission ratio $\tilde{i}$ is independent of the parameters λ_ν and can be written as follows

$$i = \frac{\tilde{\omega}'}{\tilde{\omega}} = \frac{1}{2\pi} \int\limits_0^{2\pi} i(\psi, \lambda_\nu)\, d\psi \tag{2.14}$$

Eq (2.14) allows also the comparison between the systems with variable transmission ratio and constant transmission ratio, respectively to be made. This proves the thesis that the increase in power received from the motor can result from application of a transmission with a variable instantaneous transmission ratio, while the driven shaft angle of rotation, corresponding to the revolution of the drive shaft through 360° (average transmission ratio), is the same as in a system with a constant transmission ratio.

According to Eq (2.14), the average velocity of the drive shaft is also a function of the transmission parameters

$$\tilde{\omega}'(\lambda_\nu) = \frac{1}{T} \int\limits_0^{T} \omega(t, \lambda_\nu)\, dt \tag{2.15}$$

Proper choice the transmission parameters λ_ν ensures higher values of the average velocity to be obtained. The necessary condition for reaching the maximum possible average velocity is

$$\frac{\partial \tilde{\omega}'(\lambda_\nu)}{\partial \lambda_\nu} = 0 \qquad\qquad \nu = 1, ..., \mu \tag{2.16}$$

Then also, by virtue of Eq (2.4) the average power received achieves the maximum

$$N_{avg} = N_{avg_{(max)}} \tag{2.17}$$

The above represents optimization of energy reception in the system. The optimal values for the transmission parameters can be calculated using Eq (2.16) or using numerical methods of optimization. It should be noted however, that it can be done only when the active moment is a periodical function and the system can operate at a variable drive shaft velocity.

3. Equation of the system motion

In the diagram of the system shown in Fig.1 there was used an equivalent moment of inertia concentrated at the driven shaft. Kinetic energy of the system in terms of the drive shaft rotation angle ψ (generalized coordinate) can be written as follows

$$E = \frac{1}{2}J\Big[\dot{\psi}, i(\psi, \lambda_\nu)\Big]^2 \tag{3.1}$$

while the generalized force assumes the form

$$Q_\psi = M(\psi) - M'i(\psi, \lambda_\nu) \tag{3.2}$$

then the equation for motion, after some simple transformations reads

$$J\ddot{\psi} + J\frac{1}{i(\psi, \lambda_\nu)}\frac{\partial i(\psi, \lambda_\nu)}{\partial \psi}\dot{\psi}^2 = \frac{1}{[i(\psi, \lambda_\nu)]^2}M(\psi) - \frac{1}{i(\psi, \lambda_\nu)}M' \tag{3.3}$$

In a singular case, the resisting moment acting on the driven shaft, Eq (2.7), contains a constant component and a component dependent on velocity

$$M' = M_c + c\dot{\varphi} \tag{3.4}$$

then, after taking Eq (2.4) into account, Eq (2.11) can be rewritten as

$$J\ddot{\psi} + J\frac{1}{i(\psi, \lambda_\nu)}\frac{\partial i(\psi, \lambda_\nu)}{\partial \psi}\dot{\psi}^2 = \frac{1}{[i(\psi, \lambda_\nu)]^2}M(\psi) - \frac{1}{i(\psi, \lambda_\nu)}M_c - c\dot{\psi} \tag{3.5}$$

Solution to the equation in a generalized form is impossible. It is possible, however, to arrive at a numerical solution for some functions representing the transmission ratio and driving moment. Optimization by selecting the transmission parameters can also be carried out then. An analytical solution is possible only for selected functions describing the transmission ratio, after further simplification of Eq (3.4). Basing on the simplified equations, the optimization conditions can be presented in analytical form as well.

4. Kinematics of chain transmission

Realization of the variable ratio is rather easy in the case of chain transmission. One or both gears are replaced with an element of different shape (non-circular). The considerations presented here in after apply to the transmission in which only the gear on the drive shaft has been replaced with an element of different shape. It was assumed that there were no deformable elements in the system and that the active segment of the chain was in rectilinear. Fig.2 shows a diagram of the transmission.

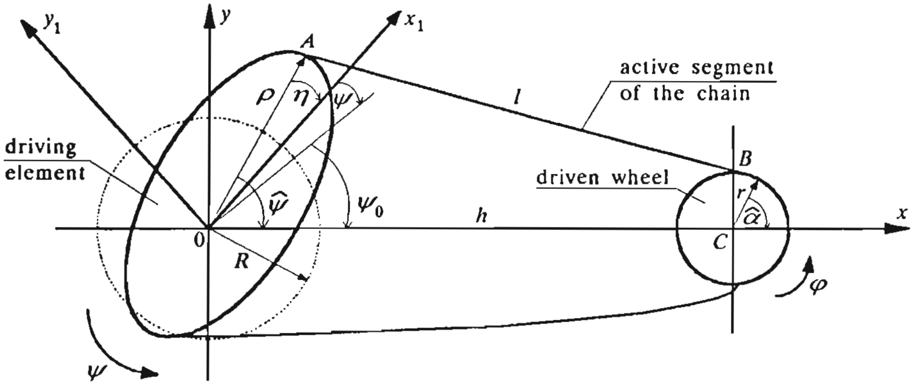


Fig. 2.

Arrows were assigned to segments OA, AB, BC, CO. Taking into consideration that the polygon formed of these vectors is a closed figure

$$\vec{OA} + \vec{AB} + \vec{BC} + \vec{CO} = \vec{0} \tag{4.1}$$

after projecting vectors on the $0xy$ system axes, the following equations are obtained

$$\begin{aligned} &\rho\cos\widehat{\psi} + l\sin\widehat{\alpha} - r\cos\widehat{\alpha} - h = 0 \\ &\rho\sin\widehat{\psi} - l\cos\widehat{\alpha} - r\sin\widehat{\alpha} = 0 \end{aligned} \tag{4.2}$$

The active part of the chain remains tangent to the element on the drive shaft and to the driven gear, from which the following relationship results

$$\frac{\frac{d\rho}{d\eta}\sin\widehat{\psi} + \rho\cos\widehat{\psi}}{\frac{d\rho}{d\eta}\cos\widehat{\psi} - \rho\sin\widehat{\psi}} = -\frac{\cos\widehat{\alpha}}{\sin\widehat{\alpha}} \tag{4.3}$$

Taking into account that the chain length is constant, after describing the shape of driving element in terms of polar coordinates (ρ, η), next equation is obtained

$$-\int\limits_0^{\eta} \sqrt{\Big(\frac{d\rho}{d\eta}\Big)^2 + \rho^2}\, d\eta + l + r\Delta\alpha - r\varphi = l_0 \tag{4.4}$$

where (see Fig.2)

$\vec{\rho}$ – vector connecting the drive shaft axis with the point of contact of the chain with the driving element

$\widehat{\psi}$ – angle representing position of the vector $\vec{\rho}$ in the $0xy$ system (system at rest)

η – angle representing position of the vector $\vec{\rho}$ in the $0x_1y_1$ system (system in motion)

$\vec{r}$ – vector connecting the driven shaft axis with the point of contact of the chain with the driven gear

$\widehat{\alpha}$ – angle representing position of the vector $\vec{r}$ in the $0xy$ system (system at rest)

$\Delta\alpha$ – increment of angle α

φ – angle of rotation of the driven shaft

h – distance between the drive shaft axis and that of the driven shaft

l – length of the chain segment between the points of contact with the driving element and driven wheel

l_0 – initial length of the chain.

The solution is sought as a linear approximation of the solution for the transmission with a circular driving element. The angle of rotation of the drive shaft (in the $0x_1y_1$ system) denoted as φ, and following substitutions were made

$$\widehat{\eta} = \eta_0 + \varepsilon\eta \qquad \widehat{\psi} = \psi_0 + \psi + \widehat{\eta} \qquad \widehat{\alpha} = \alpha_0 + \varepsilon\alpha$$

$$\varphi = \varphi_0 + \varepsilon\varphi \qquad \rho = R + e(\eta) \qquad l = l_0 + \varepsilon l \tag{4.5}$$

$$\Delta\alpha = \widehat{\alpha} - \alpha_0$$

where

$\psi_0, \eta_0, \alpha_0, \varphi_0, l_0$ – values assumed by the coordinates $\widehat{\psi}$, $\widehat{\eta}$, $\widehat{\alpha}$, $\widehat{\varphi}$, $\widehat{l}$, respectively, for the transmission with a circular driving element

$\varepsilon\eta, \varepsilon\alpha, \varepsilon\varphi, \varepsilon l$ – their minor deviations of respective coordinates

R – radius of a hypothetical circular element on the drive shaft

$e(\eta)$ – shape deviation of the circular element (in further formulas, for the sake of simplicity, the argument η was omitted).

For the transmission with a circular driving element

$$\eta_0 = -\psi \tag{4.6}$$

Substituting Eqs (3.2) ÷ (3.5) and (4.1) ÷ (4.3) into Eqs (2.15) ÷ (2.17) and (3.1), we have

$$(R+e)\cos(\psi_0+\varepsilon\eta) + (l_0+\varepsilon l)\sin(\alpha_0+\varepsilon\alpha) - r\cos(\alpha_0+\varepsilon\alpha) - h = 0$$

$$(R+e)\sin(\psi_0+\varepsilon\eta) - (l_0+\varepsilon l)\cos(\alpha_0+\varepsilon\alpha) - r\sin(\alpha_0+\varepsilon\alpha) = 0$$

$$\Big[\frac{de}{d\eta}\sin(\psi_0+\varepsilon\eta) + (R+e)\cos(\psi_0+\varepsilon\eta)\Big]\sin(\alpha_0+\varepsilon\alpha) + \tag{4.7}$$

$$+\Big[\frac{de}{d\eta}\cos(\psi_0+\varepsilon\eta) - (R+e)\sin(\psi_0+\varepsilon\eta)\Big]\cos(\alpha_0+\varepsilon\alpha) = 0$$

$$-\int\limits_0^{-\psi+\varepsilon\eta}\sqrt{\Big(\frac{de}{d\eta}\Big)^2 + (R+e)^2}\, d\eta + l_0 + \varepsilon l + r\varepsilon\alpha - r\varphi_0 - r\varepsilon\varphi = l_0$$

The values of $\varepsilon\eta$, $\varepsilon\alpha$, $\varepsilon\varphi$, εl and e are considered as small ones (of the order of ε). The functions in Eqs (4.7) were expanded into power series around zero values of $\varepsilon\eta$, $\varepsilon\alpha$, $\varepsilon\varphi$, εl, e. After neglecting small components, two sets of equations of the order greater than zero have been obtained. One of then contains only large components (without small ones, of the order of ε) and describes the kinematics of transmission with a circular driving element

$$\begin{aligned}
&R\cos\psi_0 + l_0\sin\alpha_0 - r\cos\alpha_0 - h = 0\\
&R\sin\psi_0 - l_0\cos\alpha_0 - r\sin\alpha_0 = 0\\
&R\sin(\psi_0-\alpha_0) = 0\\
&R\psi - r\varphi_0 = 0
\end{aligned} \tag{4.8}$$

The second one consists of equations of the order of ε. It describes the influence of a non-circular driving element on the transmission kinematics

$$e\cos\psi_0 - \varepsilon\eta R\sin\psi_0 + \varepsilon l\sin\alpha_0 + \varepsilon\alpha l_0\cos\alpha_0 - \varepsilon\alpha r\sin\alpha_0 = 0$$

$$e\sin\psi_0 + \varepsilon\eta R\cos\psi_0 - \varepsilon l\cos\alpha_0 + \varepsilon\alpha l_0\sin\alpha_0 - \varepsilon\alpha r\cos\alpha_0 = 0 \tag{4.9}$$

$$\frac{de}{d\eta}\cos(\psi_0-\alpha_0) - e\sin(\psi_0-\alpha_0) + \varepsilon\alpha R\cos(\psi_0-\alpha_0) - \varepsilon\eta R\cos(\psi_0-\alpha_0) = 0$$

$$-R\varepsilon\eta\int\limits_0^{-\psi} e(\eta)\, d\eta + \varepsilon l + r\varepsilon\alpha - r\varepsilon\varphi = 0$$

The solution to Eqs (4.8) is straighforward since for a regular transmission

$$\begin{aligned} \psi_0 &= \alpha_0 = \arccos\frac{R-r}{h} \\ l_0 &= \sqrt{h^2-(R-r)^2} \\ \varphi_0 &= \frac{R}{r}\psi \end{aligned} \tag{4.10}$$

while the solutions to Eqs (4.9) assume the following forms

$$\begin{aligned} \varepsilon\eta &= -\frac{1}{R}\frac{de(-\psi)}{d\psi} - \frac{e(-\psi)}{l_0} \\ \varepsilon\alpha &= -\frac{e(-\psi)}{l_0} \\ \varepsilon l &= -\frac{de(-\psi)}{d\psi} - (R-r)\frac{e(-\psi)}{l_0} \\ \varepsilon\varphi &= -\frac{1}{R}\int\limits_0^{-\psi} e(\eta)\,d\eta \end{aligned} \tag{4.11}$$

Thus, the relation between the drive shaft and driven shaft rotation angles, respectively, being sought after is

$$\varphi = \frac{R}{r}\Big[\psi - \frac{1}{R}\int\limits_0^{-\psi} e(\eta)\,d\eta\Big] \tag{4.12}$$

from where the transmission ratio, bu virtue of Eq (2.14), is

$$i(\psi) = \frac{R}{r}\Big[1 + \frac{1}{R}e(-\psi)\Big] \tag{4.13}$$

The transmission ratio was obtained as a sum of two components. The first one constant and depends on the value of a hypothetical radius R; the second one depends on a shape of the driving element, i.e., varies with the function $e(\eta)$.

5. Sample mechanical system

A mechanical system with an elliptical driving element (Fig.3) will be used as an example. In the general, the equation of an ellipse, in the $0x_1y_1$ system is

$$\frac{x_1^2}{A^2} + \frac{y_1^2}{B^2} = 1 \tag{5.1}$$

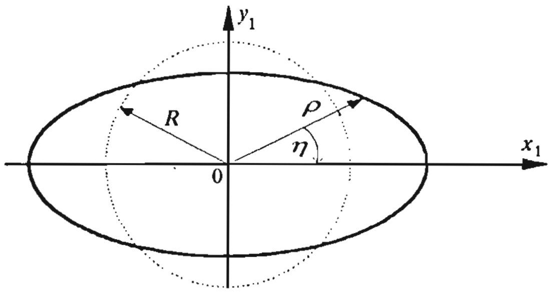


Fig. 3.

The values $A = R + a$ and $B = R - a$ are assumed for the semimajor and semiminor axes, respectively, where a is the parameter defining the shape (oblateness) of the ellipse.

The polar coordinates $x_1 = \rho\cos\eta$, $y_1 = \rho\sin\eta$ were introduced. After taking into account Eq (4.5), on the assumption that the parameter a is small (of the order of ε), the equation of ellipse (with components less than ε neglected) has been obtained in the form

$$\rho = R + a\cos 2\eta \tag{5.2}$$

Eq (4.12) can be therefore rewritten as follows

$$\varphi = \frac{R}{r}\Big(\psi + \frac{a}{R}\sin 2\psi\Big) \tag{5.3}$$

while the transmission ratio Eq (4.13) is

$$i(\psi) = \frac{R}{r}\Big(\psi + \frac{a}{R}\cos 2\psi\Big) \tag{5.4}$$

and the average transmission ratio Eq (2.14) is the same as that for the transmission with a circular driving element with the radius R

$$i = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{R}{r}\Big(\psi + \frac{a}{R}\cos 2\psi\Big)\,d\psi = \frac{R}{r} \tag{5.5}$$

It was assumed that the driving moment is a periodic function with the harmonic component

$$M(\psi) = M_0\Big[1 - \frac{M_1}{M_0}\cos(\psi - \gamma)\Big] \tag{5.6}$$

where M_0, M_1 are coefficients defining constant and variable parts of the moment, respectively. The parameter γ represents the angle of phase shift of the driving moment in relation to the angle of rotation of the driving shaft (coordinate ψ). This parameter depends on a position of the driving motor shaft (the source of moment) relative to the driving element; consequently, this parameter can be considered as a parameter of the transmission λ_ν. By substituting Eqs (5.4) and (5.6) into Eq (3.5), an equation for motion was obtained in the form

$$\begin{aligned} &J\ddot{\psi} + \Big(2J\frac{a}{R}\frac{1}{1+\frac{a}{R}\cos 2\psi}\sin 2\psi\Big)\dot{\psi}^2 = \\ &= \Big(\frac{r}{R}\Big)^2 \frac{M_0}{\Big(1+\frac{a}{R}\cos 2\psi\Big)^2}\Big[1 - \frac{M_1}{M_0}\cos 2(\psi-\gamma)\Big] - \frac{r}{R}\frac{M_c}{1+\frac{a}{R}\cos 2\psi} - c\dot{\psi} \end{aligned} \tag{5.7}$$

6. Analytical solution

It is impossible to arrive at an analytical solution to Eq (5.7). A numerical solution, however, does not give general information about the characteristics of the system. For this reason, an approximate analytical solution to the simplified equation has been used. This solution, together with the numerical solution, allows one to recognize real properties of the system. Simplified solutions have been obtained for two extreme values of the moment of inertia (i.e. at zero and tending to infinity). The remaining solutions, for finite values of the moment of inertia, would be located between the two extremes.

The first case represents the system where the moment of inertia was omitted $J = 0$. Thus, it is easy to identify the drive shaft velocity as a function of position (angle ψ)

$$\begin{aligned} \omega &= \dot{\psi} = \frac{1}{c}\Big(\frac{r}{R}\Big)^2 \frac{M_0}{\Big(1+\frac{a}{R}\cos 2\psi\Big)^2}\Big[1 - \frac{M_1}{M_0}\cos 2(\psi-\gamma)\Big] + \\ &- \frac{1}{c}\frac{r}{R}\frac{M_c}{1+\frac{a}{R}\cos 2\psi} \end{aligned} \tag{6.1}$$

However, the solution is sought in the form of a function of time, for only such a solution allows one to find the average velocity, see Eq (2.14). The solution assumes the following forms (cf Gierulski and Wójcik (1995a,b))

$$\begin{aligned} \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 \\ \psi &= \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 \end{aligned} \tag{6.2}$$

where, as before, the symbol ε denotes the order of magnitude of successive elements.

Additionally, such proportions were established between the components of the driving moment that M_1/M_0 is small, of the order of ε. After using expansion into power series and neglecting minor components of an order greater than ε^2, Eq (6.1) can be rewritten as

$$\begin{aligned} \omega = \frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big\{1 - 2\frac{a}{R}\cos 2\psi + 2\frac{a}{R}\frac{M_1}{M_0}\cos 2(\psi-\gamma)\cos 2\psi + \\ +3\Big(\frac{a}{R}\Big)^2\cos^2 2\psi - \frac{M_1}{M_0}\cos 2(\psi-\gamma) - \frac{R}{r}\frac{M_c}{M_0}\Big[1 - \frac{a}{R}\cos 2\psi + \Big(\frac{a}{R}\Big)^2\cos^2 2\psi\Big]\Big\} \end{aligned} \tag{6.3}$$

The first approximation step is

$$\begin{aligned} \omega_0 &= \frac{1}{c}\Big(\frac{r}{R}\Big)^2 M_0\Big(1 - \frac{R}{r}\frac{M_c}{M_0}\Big) \\ \psi_0 &= \omega_0 t \end{aligned} \tag{6.4}$$

the velocity is constant, therefore it is at the same time the average velocity. The next approximation step (of the order of ε) yields the following solution

$$\begin{aligned} \varepsilon\omega_1 &= -\frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big[2\frac{a}{R}\cos 2\omega_0 t + \frac{M_1}{M_0}\cos 2(\omega_0 t-\gamma) - \frac{R}{r}\frac{M_c}{M_0}\frac{a}{R}\cos 2\omega_0 t\Big] \\ \varepsilon\psi_1 &= -\frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big[2\frac{a}{R}\sin 2\omega_0 t + \frac{M_1}{M_0}\sin 2(\omega_0 t-\gamma) - \frac{R}{r}\frac{M_c}{M_0}\frac{a}{R}\sin 2\omega_0 t\Big]\frac{1}{2\omega_0} \end{aligned} \tag{6.5}$$

which allows one to find the next component of series $(6.2)_2$ being of the order of ε^2

$$\begin{aligned} \varepsilon^2\omega_2 = \frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big[3\Big(\frac{a}{R}\Big)^2\cos^2 2\omega_0 t + \\ +2\frac{a}{R}\frac{M_1}{M_0}\cos 2(\omega_0 t-\gamma)\cos 2\omega_0 t - \frac{R}{r}\frac{M_c}{M_0}\Big(\frac{a}{R}\Big)^2\cos^2 2\omega_0 t + \\ -\frac{1}{1-\frac{R}{r}\frac{M_c}{M_0}}\Big(\frac{M_1}{M_0}\sin 2(\omega_0 t-\gamma) + 2\frac{a}{R}\sin 2\omega_0 t - \frac{R}{r}\frac{M_c}{M_0}\frac{a}{R}\sin 2\omega_0 t\Big)^2\Big] \end{aligned} \tag{6.6}$$

Substituting Eqs (6.4) and (6.5) into Eq $(6.2)_1$, the average velocity, Eq (2.15), of the drive shaft is calculated

$$\begin{aligned}\tilde{\omega}(\lambda_\nu) = \frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big\{1 - \frac{R}{r}\frac{M_c}{M_0} + \\ -\frac{1}{1-\frac{R}{r}\frac{M_c}{M_0}}\Big[\frac{1}{2}\Big(\frac{M_1}{M_0}\Big)^2 + \frac{1}{2}\Big(\frac{a}{R}\Big)^2 + \frac{M_1}{M_0}\frac{a}{R}\cos 2\gamma\Big]\Big\}\end{aligned} \tag{6.7}$$

where the transmission parameters λ_1, λ_2 are a/R and γ. The average velocity attains maximum for those values for parameters λ_1, λ_2 which meet conditions (2.16), that is when

$$\gamma = \frac{\pi}{2} \qquad\qquad \frac{a}{R} = \frac{M_1}{M_0} \tag{6.8}$$

From the above, the maximum average velocity is

$$\tilde{\omega}_{max} = \frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big(1 - \frac{R}{r}\frac{M_c}{M_0}\Big) \tag{6.9}$$

For comparison, in the case of transmission with a circular driving element with a transmission ratio equal to the average one Eq (2.15), the average velocity is

$$\breve{\omega} = \frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big[1 - \frac{R}{r}\frac{M_c}{M_0} - \frac{1}{2}\frac{1}{1-\frac{R}{r}\frac{M_c}{M_0}}\Big(\frac{M_1}{M_0}\Big)^2\Big] \tag{6.10}$$

Fig.4 and Fig.5 show the average velocity versus the transmission parameters courses for different values of M_c/M_0, i.e. the figures show varying contribution of the constant component of transmission load. It is possible to attain a higher average velocity, than that given by Eq (6.10), when the angle of phase shift γ is greater than $\pi/4$. The second case refers to the system in which the moment of inertia tends to infinity $J \to \infty$. Then, the velocity of the driven shaft is constant, $\dot{\varphi} =$ const. There is no need to integrate Eq (5.7) in order to calculate the average velocity.

In accordance with Eq (2.1), work of the driving moment (5.6) in one revolution of the drive shaft is

$$L_{(2\pi)} = \int\limits_0^{2\pi} M_0\Big[1 - \frac{M_1}{M_0}\cos 2(\psi - \gamma)\Big]\,d\psi = 2\pi M_0 \tag{6.11}$$

Work of the resisting moment (3.4), including both the constant component and that dependent on the velocity for one revolution of the drive shaft, and

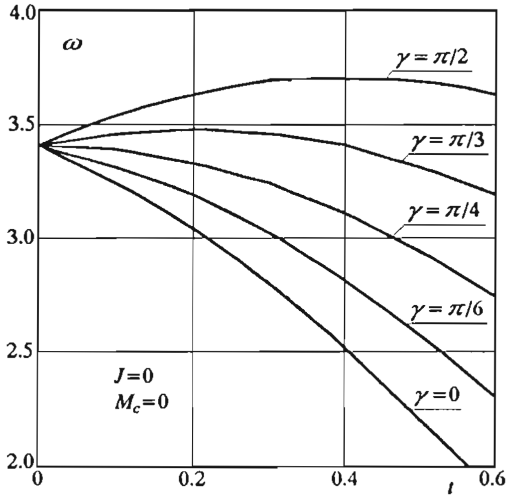

4.0
3.5
3.0
2.5
2.0
ω
γ = π/2
γ = π/3
γ = π/4
γ = π/6
γ = 0
J=0
M_c=0
0
0.2
0.4
t
0.6

Fig. 4.

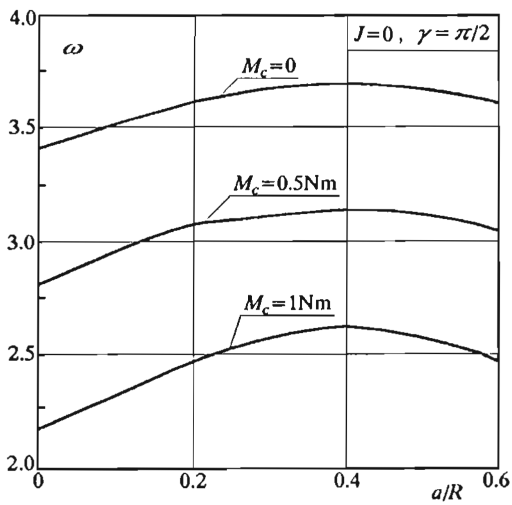

4.0
3.5
3.0
2.5
2.0
ω
J=0, γ = π/2
M_c=0
M_c=0.5Nm
M_c=1Nm
0
0.2
0.4
a/R
0.6

Fig. 5.

taking into account the value for average transmission ratio (5.5), is as follows

$$L_{M_c} = \int\limits_0^{2\pi\frac{R}{r}} (M_c + c\dot{\varphi})\, d\varphi = 2\pi\frac{R}{r}(M_c + c\dot{\varphi}) \tag{6.12}$$

On can see that $L_{(2\pi)} = L_{M_c}$, thus

$$M_0 = \frac{R}{r}(M_c + c\dot{\varphi}) \tag{6.13}$$

Taking into account the following relationship between the average velocity of the drive shaft and that of the driven shaft

$$\dot{\varphi} = \frac{R}{r}\widetilde{\omega} \tag{6.14}$$

the average velocity of the drive shaft is as follows

$$\widetilde{\omega} = \frac{1}{c}M_0\Big(\frac{r}{R}\Big)^2\Big(1 - \frac{R}{r}\frac{M_c}{M_0}\Big) \tag{6.15}$$

This value of velocity is depends neither on the shape of the driving element, nor on the harmonic component of the driving moment. The value of average velocity of the drive shaft is equal to the maximum velocity for a system where the moment of inertia is equal to zero.

7. Computer simulation of the transmission work

As a part of a computer simulation, Eq (5.7) was solved for a variety of values assumed for the transmission parameters and taking the moment of inertia into account (cf Gierulski and Wójcik (1994), (1995a)). Only the states of stabilized motion were analysed. Fig.6 ÷ Fig.12 illustrate the results of simulation, with the moment of inertia neglected, for different values assumed for the parameter a/R, at different values for the angle γ. Fig.13 ÷ Fig.18 show the results of simulation with the moment of inertia included. Results were obtained for the following values of the parameters: $M_0 = 10\,\mathrm{Nm}$, $M_1/M_0 = 0.4$, $r/R = 1/3$, $c = 0.3\,\mathrm{Nms}$, $M_c = 0$. The time-dependent courses are shown in figures. Table 1 shows the set of results obtained.

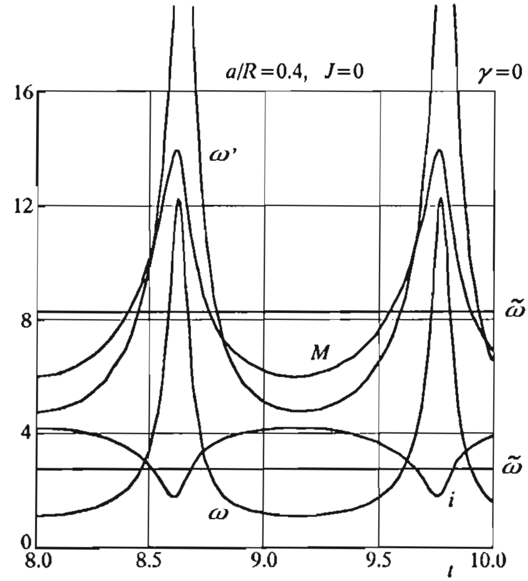

a/R = 0.4, J = 0
γ = 0
16
12
8
4
0
ω'
M
ω̃
ω̃
i
ω
8.0
8.5
9.0
9.5
10.0
t

Fig. 6.

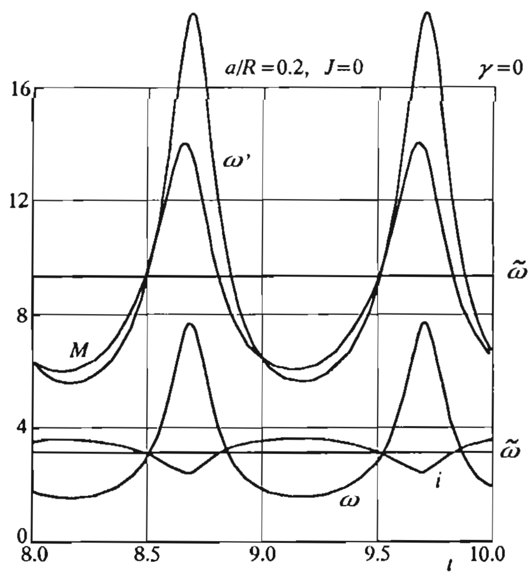

a/R = 0.2, J = 0
γ = 0
16
12
8
4
0
ω'
ω̃'
M
ω̃
i
ω
8.0
8.5
9.0
9.5
10.0
t

Fig. 7.

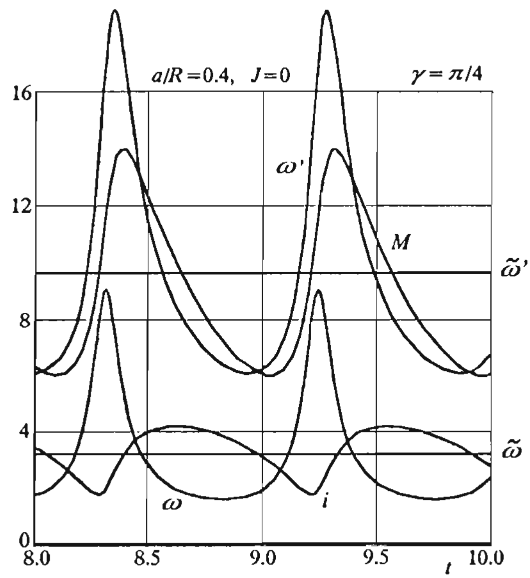

a/R=0.4, J=0
γ=π/4
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
ω'
M
ω̃'
ω̃
ω
i

Fig. 8.

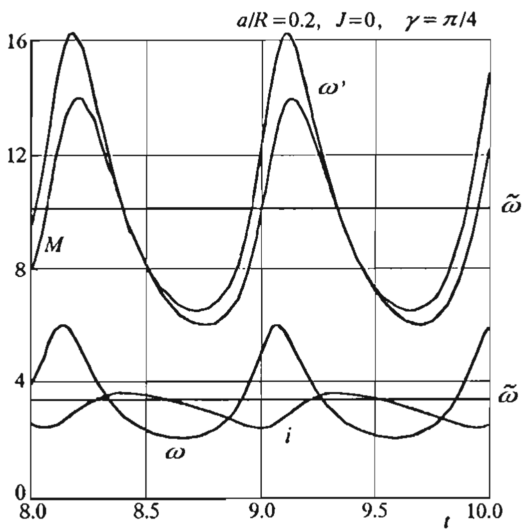

a/R=0.2, J=0, γ=π/4
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
ω'
M
ω̃'
ω̃
ω
i

Fig. 9.

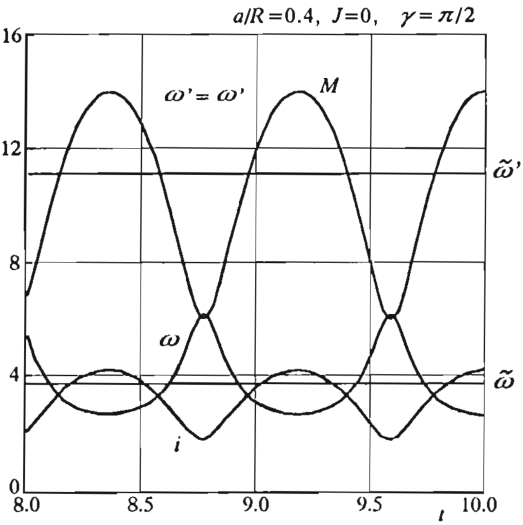

a/R=0.4, J=0, γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
t
10.0
ω'= ω'
M
ω̃'
ω
ω̃
i

Fig. 10.

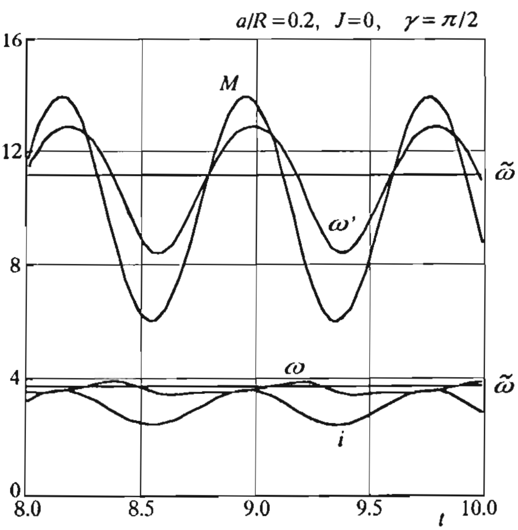

a/R=0.2, J=0, γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
t
10.0
M
ω̃'
ω'
ω
ω̃
i

Fig. 11.

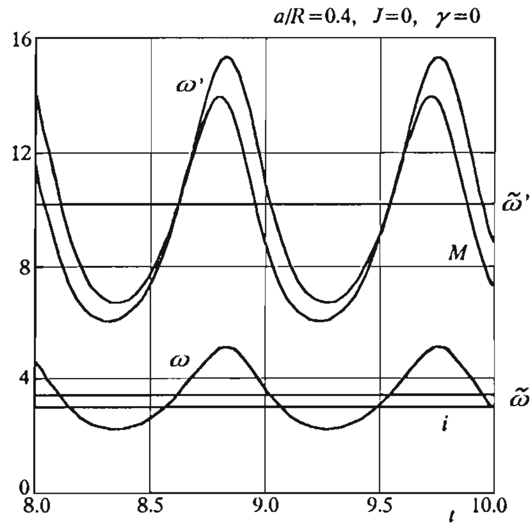

a/R=0.4, J=0, γ=0
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
ω'
ω̃'
M
ω
ω̃
i

Fig. 12.

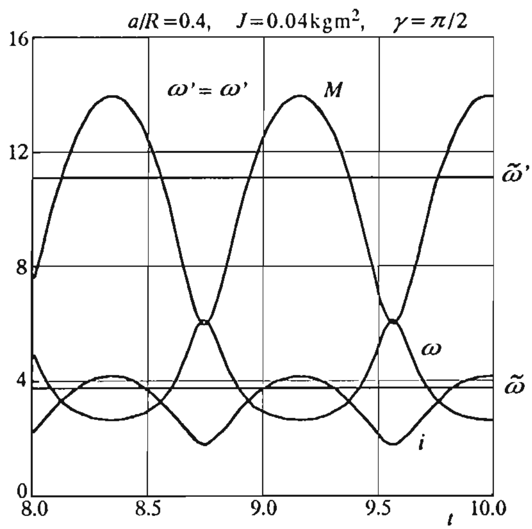

a/R=0.4, J=0.04kgm², γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
ω'= ω'
M
ω̃'
ω
ω̃
i

Fig. 13.

Table 1

	$\widetilde{\omega}$ – average velocity			
J	γ	$\frac{a}{R}=0.4$	$\frac{a}{R}=0.2$	$\frac{a}{R}=0$
0	0	2.57	3.04	3.41
0	$\pi/4$	3.11	3.33	3.41
0	$\pi/2$	3.72	3.63	3.41
0.04	$\pi/2$	3.79	3.67	3.55
0.1	$\pi/2$	3.81	3.69	3.66

Fig.19 shows the dependence of the average velocity on the value of parameter a/R, for angle $\gamma = \pi/2$, and various values of the inertia moment.

Fig.19 presents the results of numerical solutions. Comparing them with those obtain by means of analytical methods a good agreement can be found. Consequently, it is possible to make generalized conclusions regarding the properties of the transmission.

8. Practical applications

The proposed transmission offers adventages which can be evaluated by comparison with a transmission containing a circular driving element. A transmission with a non-circular driving element enables one to attain a higher driven shaft velocity at the same velocity of the drive shaft. This can be brought about by proper selection of the transmission parameters. For example, for a transmission with optimal parameters, Eqs (6.8), the drive shafts of transmissions being compared, will rotate at the same average velocity when a transmission with a circular driving element has a driven gear with a greater radius. On the assumption that the moment of resistance does not include the constant component $M_c = 0$, the value of this radius $\widehat{r}$ is

$$\widehat{r} = r\frac{1}{\sqrt{1-\frac{1}{2}\Big(\frac{M_1}{M_0}\Big)^2}} \tag{8.1}$$

from where the average velocity of the driven gear $\widehat{\omega}'$ is

$$\widehat{\omega}' = \widetilde{\omega}'\sqrt{1-\frac{1}{2}\Big(\frac{M_1}{M_0}\Big)^2} \tag{8.2}$$

The radius of driven gear and the average velocity of driven shaft are expressed as functions of the same quantities as for the transmission with a non-circular

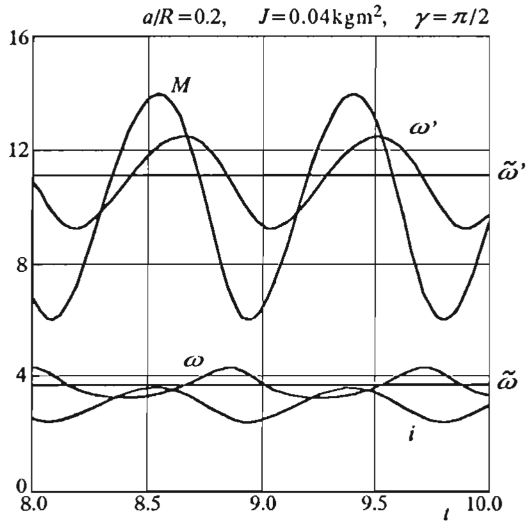

a/R=0.2, J=0.04kgm², γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
M
ω'
ω̃'
ω
ω̃
i

Fig. 14.

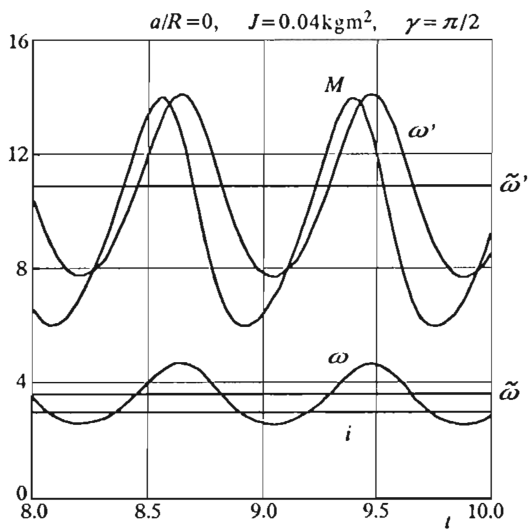
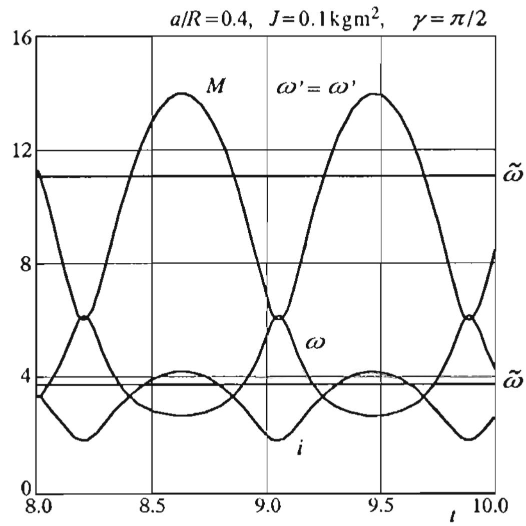
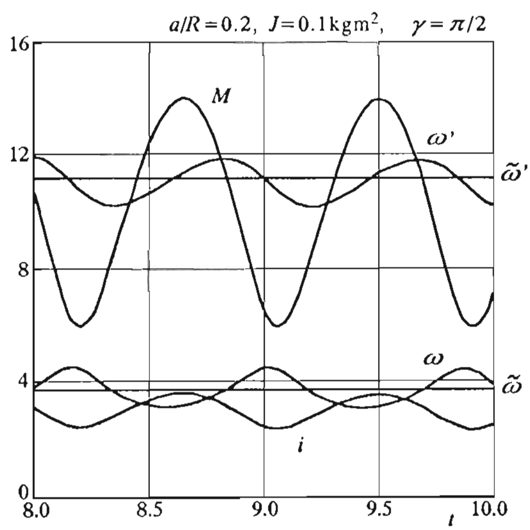

a/R=0, J=0.04kgm², γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
M
ω'
ω̃'
ω
ω̃
i

Fig. 15.



a/R=0.4, J=0.1kgm², γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
M
ω'=ω'
ω̃'
ω
ω̃
i

Fig. 16.



a/R=0.2, J=0.1kgm², γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
M
ω'
ω̃'
ω
ω̃
i

Fig. 17.

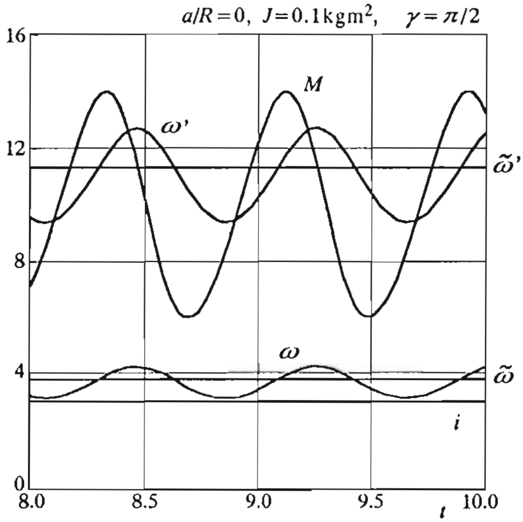

a/R=0, J=0.1kgm², γ=π/2
16
12
8
4
0
8.0
8.5
9.0
9.5
10.0
t
M
ω'
ω̃'
ω
ω̃
i

Fig. 18.

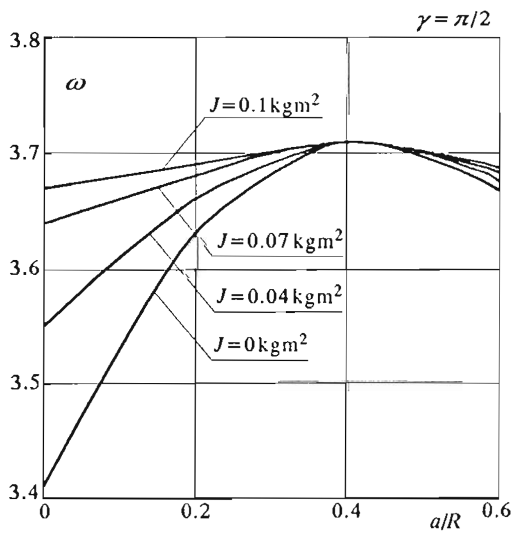

γ=π/2
3.8
3.7
3.6
3.5
3.4
ω
J=0.1kgm²
J=0.07kgm²
J=0.04kgm²
J=0kgm²
0
0.2
0.4
0.6
a/R

Fig. 19.

driving element $(r,\tilde{\omega}')$. The reduction in the driven shaft velocity depends on the harmonic of the driving moment. The above statements apply to average values of the velocity. In Eqs (8.1) and (8.2) the quantities $(\hat{r},\hat{\omega}')$ apply to the transmission with a circular driving element.

9. Final notes

The transmission with a periodical variable transmission ratio contributes to the optimization of energy reception in systems with a periodical variable driving moment. As a result, the average velocity can increase due to creation of the environment conducive to an increase in power received from the motor. The smaller the reduced moment of inertia in the system in motion, the greater the effect of using such a transmission. One possible application of this kind of transmission is a biomechanical power transmission system. Transmissions of this kind can be found in bicycles; however, there is a lack of literature describing the theoretical analysis and conditions of optimization.

It might be particularly advantageous to use this type of transmission in man-powered aircrafts, considering the small values of the moment of inertia found in such systems.

Application of the transmission with a periodically variable transmission ratio in power transmission systems is subject to many limitations. Such a transmission may cause an unbalance in the motion of the machine or one of its elements; it may also cause additional unwanted vibrations in the system. Finally, in real systems, it is not always possible to implement the transmission ratio which ensures optimal operation of the system as determined by the theoretical analysis.

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Optymalizacja odbioru energii w układach napędzanych okresowym momentem

Streszczenie

W mechanicznych układach przenoszenia energii ważną rolę może spełniać przekładnia. W pracy przedstawiono analizę teoretyczną przekładni specjalnego typu. Analizę poparto wynikami sumulacji komputerowej pracy układu zawierającego tego typu przekładnię.

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