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CRITICAL EULER LOAD FOR A CANTILEVER TAPERED BEAM

Jerzy Jaroszewicz

Mechanical Faculty Technical University of Bialystok
e-mail: mass@cksr.ac.bialystok.pl

Longin Zoryj

Mechanical Engineering Faculty Technical University of Lvov, Ukraine

In this paper the method of influence function is applied to solving the problem of critical Euler load of a cantilever tapered beam. In such a case the critical load results form a dead load of a beam. A general form of a characteristic equation is obtained by means of the Cauchy function in a power series form which enables evaluation of critical Euler loads for sharp cone and frustum of cone cantilever beams. The accuracy of calculations of a critical Euler load of cantilever beams subjected to compressive loading is tested by comparison with the well-known theoretical solutions. It is shown that a very good approximation of the exact solution can be obtained using only two first terms of a characteristic series.

Key words: critical Euler load, tapered beam, dead load

1. Introduction

Many structures such as chimneys, towers, headframes, masts and domes can be modelled by means of tapered cantilever beams with a variable cross-section subjected to dead load. In such a case the dead load is of axial direction and is non-uniformly distributed along the length of beam. Determination of the critical load due to dead loading for a beam with a variable cross-section leads to a boundary value problem defined by differential equations with variable parameters. The exact solution of a such a problem in a closed form is possible only in a few cases when a variable cross-section of a beam is taken into consideration. Many examples of solutions of problem of stability in

Euler's sense for beams with binomial distribution of flexural rigidity can be found in the literature (cf Dynnyk (1952)). The books by Timoshenko (1971) and by Rzhanicyn (1955) contain solutions for sharp tapered beams obtained by means of the Bessel function. Jaroszewicz and Zoryj (1983) discussed the influence of variable dead load on the natural frequency of transverse vibrations of a bar with a constant cross-section.

However the literature does not provide any analytical solutions even for such a simple case as a frustum of cone beam. Of course this and similar problems can be solved by means of numerical methods, e.g. the finite element method, using the commercial programs; like, NISA, NASTRAN, ABACUS, etc. However lack of the exact solution does not allow us to estimate the accuracy of approximate analytical and numerical solutions.

In this paper the Cauchy influence function being a solution of the general problem of stability has been proposed in a form of power series. Basic advantage of this method is a general form of the characteristic equation. The method has been effectively applied to investigation of transverse vibrations and deflection of a beam with a variable cross-section (cf Jaroszewicz and Zoryj (1994a,b)) as well as stability of a cantilever beam (cf Jaroszewicz and Zoryj (1994a)).

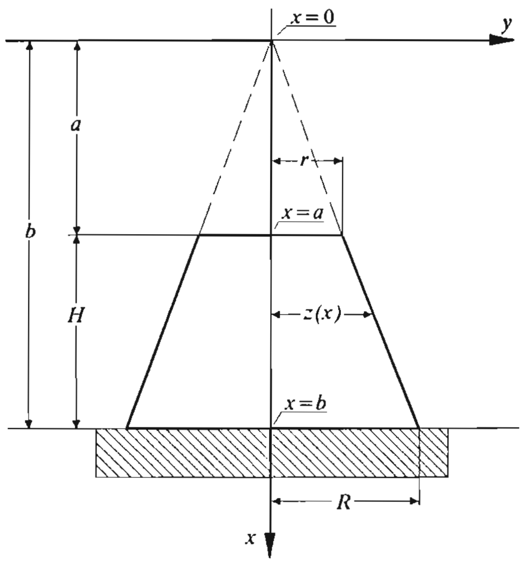

Fig. 1.

Universal model of tapered cantilever beam in a frustum cone form is shown in Fig.1, where the following notation is used:

$z(x)$ – variable radius

r, R – radii of the upper and lower bases of a cone, respectively

H – altitude of a cone

k – beam taper parameter, $k = (R - r)/H$.

Radius, volume and plane moment of inertia, respectively, can be defined by the following formulas

$$z(x) = k(x - a) + r \tag{1.1}$$

$$V(x) = \frac{\pi}{3}(x - a)[k^2(x - a)^2 + 3kr(x - a) + 3r^2] \tag{1.2}$$

$$J(x) = \frac{1}{4}\pi[k(x - a) + r]^4 \tag{1.3}$$

Taking into consideration Eqs (1.2) and (1.3) one can find the dead load for of a tapered beam

$$G = \rho V(x)\Big|_{x=b} \tag{1.4}$$

and continuous loads caused by the dead weight of the beam

$$N(x) = p_0 V(x) \tag{1.5}$$

where

$$p_0 = \frac{GH^2}{EJ_0} \qquad J_0 = J(x)\Big|_{x=b} \qquad r > ka$$

and ρ denotes density. The flexural rigidity is given by

$$f(x) = EJ(x) \tag{1.6}$$

where E – Young modulus of elasticity. The function $f^{-1}(x)$ should be continuous, positive definite and should have a finite value and integral $[a, b]$.

The absolute value of Euler critical load can be expressed as follows

$$G_{cr} = p\frac{EJ_0}{H^2} \tag{1.7}$$

where p denotes the critical load parameter.

2. Formulation of the problem

The problem of finding the Euler critical load for a model shown in Fig.1 leads to determination the first mode of the following boundary value problem

$$[f(x)\varphi']' + N(x)\varphi = 0 \tag{2.1}$$

$$f(a)\varphi'(a) = 0 \qquad\qquad \varphi(b) = 0 \tag{2.2}$$

where $\varphi = y'(x)$ and the prime denotes derivative with respect to x.

3. Characteristic equation

The fundamental solution of Eq (2.1) is proposed in following form (cf Jaroszewicz and Zoryj (1994b))

$$\varphi(x,\alpha) = c_0K(x,\alpha) + c_1\widetilde{K}(x,\alpha) \tag{3.1}$$

where $K(x,\alpha)$, $\widetilde{K}(x,\alpha)$ are the Cauchy function of Eq (2.1) and its derivative with respect to parameter α, respectively, c_0 and c_1 are arbitrary constants.

Taking into consideration

$$\alpha = a \qquad\qquad K'(a,a) = \frac{1}{a^4} \qquad\qquad \widetilde{K}'(a,a) = 0 \tag{3.2}$$

and substituting Eq (3.1) into boundary conditions (2.2) yields

$$c_0 = 0 \qquad \text{and} \qquad c_1 \neq 0 \tag{3.3}$$

The find form of characteristic equation is given by

$$\widetilde{K}(b,a) = 0 \tag{3.4}$$

4. Terms of the characteristic equation

Equation (3.5) can be written in a power series form in terms of load parameters p

$$\widetilde{K}(x,\alpha) = \sum_{i=0}^{\infty}(-1)^i\widetilde{K}_i(x,\alpha)p^i \tag{4.1}$$

Taking into consideration the formulas obtained by Jaroszewicz and Zoryj (1994b)

$$K_0(x,\alpha) = \int_{\alpha}^{x} \frac{ds}{f(s)} \tag{4.2}$$

$$\widetilde{K}_0(x,\alpha) = -\frac{1}{f(\alpha)} \tag{4.3}$$

$$\widetilde{K}_i(x,\alpha) = \int_{\alpha}^{x} K_0(x,s)V(s)\widetilde{K}_{i-1}(s,\alpha)\, ds \tag{4.4}$$

after suitable transformations the characteristic equation takes the following form

$$-\frac{1}{f(a)}\Big[1 - pF(b,a) + p^2 S(b,a) - p^3 W(b,a) + ...\Big] = 0 \tag{4.5}$$

where

$$F(x,\alpha) = \int_{\alpha}^{x} K_0(x,s)V(s)\, ds \tag{4.6}$$

$$S(x,\alpha) = \int_{\alpha}^{x} K_0(x,\alpha)V(s)F(s,\alpha)\, d\alpha \tag{4.7}$$

$$W(x,\alpha) = \int_{\alpha}^{x} K_0(x,\alpha)V(s)S(s,\alpha)\, d\alpha \tag{4.8}$$

5. Selected particular cases

Resolving integrals in Eqs (4.6), (4.7) for homogenous frustum of cone cantilever beam yields

$$\begin{aligned} F(b,a) &= \frac{1+\chi}{4(1+\chi+\chi^2)} \\ S(b,a) &= \frac{\frac{1}{5}(1+9\chi^2+9\chi^5+\chi^7) - \chi(1+\chi^2+\chi^3+\chi^5)}{8(1+\chi+\chi^2)^2(1-\chi)^6} \end{aligned} \tag{5.1}$$

where $\chi = r/R$ – beam taper parameter.

The analytical formula for $W(x,\alpha)$ and successive terms of the characteristic series are very complicated but can be evaluated by means of numerical integration.

For a sharp tapered cantilever beam $(\chi \equiv 0)$, characteristic equation (4.5) reduces to the following form

$$1 + \sum_{n=1}^{\infty} (-1)^n a_n p^n = 0 \tag{5.2}$$

where

$$a_0 = 1 \qquad\qquad a_n = a_{n-1}\frac{1}{n(n+3)} \qquad\qquad n = 1,2,3... \tag{5.3}$$

In this case the values of coefficients of the first two terms agree with those obtained from Eqs (5.1)

$$F(b,a) = a_1 = \frac{1}{4} \qquad\qquad S(b,a) = a_2 = \frac{1}{40}$$

In the case of cylindrical beam $(\chi \to 1)$ the coefficients of series (5.2) take the form

$$a_0 = 1 \qquad\qquad a_n = a_{n-1}\frac{1}{3n(3n-1)} \qquad\qquad n = 1,2,3... \tag{5.4}$$

In this case

$$F(b,a) = \frac{1}{6}$$

and after resolving the indeterminancy of 0/0 type

$$S(b,a) = \frac{1}{180}$$

The values of $S(b,a)$ and $F(b,a)$ for a cylinder obtained using Eq (5.4) are equal to those given by Jaroszewicz and Zoryj (1983).

6. Results of calculations

The exact values of critical load parameter for a sharp tapered beam $(\chi \equiv 0)$, and a cylinder $(\chi \to 0)$ are $p = 10.18$ and $p = 7.84$, respectively. The above values were obtained by means of the characteristic equation in a power series form, Eq (5.2). Diagrams shown in Fig.2 and Fig.3 illustrate the

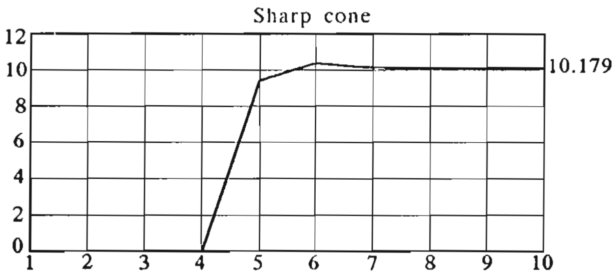

Fig. 2.

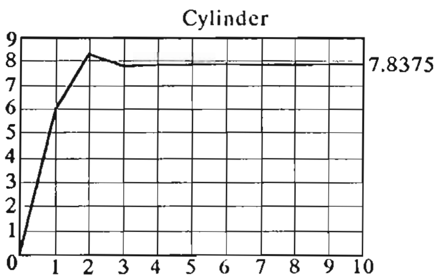

Fig. 3.

dependence of calculations accuracy of the critical load on a number of terms of the characteristic series (Eq (5.2)) taken into consideration.

In the case of a cylinder (Fig.3) it is possible to obtain dual estimators of the critical load with ssatisfactory accuracy using only the first two terms of a series by means of Bernstein estimators (cf Bernstein and Keropian (1960))

$$\frac{1}{\sqrt{F^2 - 2S}} < p < \frac{2}{F + \sqrt{F^2 - 4S}} \tag{6.1}$$

where: $F = F(x, \alpha)$, $S = S(x, \alpha)$ can be calculated from Eqs (4.6) and (4.7) or (5.1).

After substitution one obtains

$$p_+ = 8.94 \qquad\qquad p_- = 7.75$$

In the case of a sharp cone beam $(\chi \equiv 0)$ and frustum of cone cantilever beam $(0 < \chi < 1)$ only the upper estimator can be found. In order to obtain the other it is necessary to evaluate four terms of a series.

For a beam in a form of a frustum cone $(0 < \chi < 1)$ only lower estimator can be found from Eq (6.1), because $F^2 - 4S < 0$. The values of lower estimator for selected values of χ are given in Table 1. These values were calculated using Eqs (5.1).

Table 1

χ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
p_-	8.944	7.729	7.159	6.896	6.808	6.831	6.930	7.082	7.275	7.498	7.720

7. Conclusions

1. The exact values of critical load for a cantilever beam in a form of sharp cone ($p = 10.179$) and cylinder ($p = 7.839$) were obtained. These values are in agreement with the well-known solutions (cf Rzhanicyn (1955)).

2. The general characteristic equation which allows one to take into consideration arbitrary geometry of a cantilever (4.5) was obtained. The case of a cantilever in a form of a frustum cone was considered in detail (Table 1) and the first two terms of characteristic series were obtained in a closed form, Eqs (5.1).

3. The number of terms of the characteristic series, Eq (5.2), necessary to obtain an exact solution for a cantilever in a form of a sharp cone and a cylinder was evaluated (Fig.2 and Fig.3).

4. In the case of a beam of variable cross section the proposed formulas, Eqs (4.1) $\div$ (4.4), can be directly used for numerical calculations. A simple computer program has been produced which allows one to calculate the critical load for a cantilever with complex geometry under dead load.

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Obciążenie krytyczne Eulera w przypadku zbieżnej belki wspornikowej

Streszczenie

W pracy zastosowano metodę funkcji wpływu do rozwiązania zagadnienia obciążenia krytycznego Eulera w przypadku zbieżnej belki wspornikowej. Rozpatrzono obciążenie pochodzące od ciężaru własnego belki. Otrzymano ogólną postać równania charakterystycznego przy pomocy funkcji Cauchy zapisanej w postaci szeregu potęgowego, która uwzględnia zmienną siłę krytyczną Eulera w przypadku wspornika w kształcie stożka ostrego i ściętego. Pokazano dużą dokładność obliczeń siły krytycznej Eulera w przypadku ściskanych wsporników w porównaniu ze znanymi wynikami rozwiązań teoretycznych. Pokazano, że możliwe jest otrzymanie dobrego przybliżenia do rozwiązania ścisłego, wykorzystując jedynie dwa pierwsze człony szeregu charakterystycznego.

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