MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
1, 33, 1995

APPLICATION OF TWO-DIMENSIONAL APPROACH IN MECHANICAL SYSTEMS SUBJECT TO LONGITUDINAL LOADINGS

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The survey of literature carried out from the point of view of the application possibilities of two-dimensional approach in mechanical systems subject to longitudinal loadings is in the paper. The exact theory as well as the approximate ones, and problems formulated for semi-infinite and finite circular cylinders are considered. These problems reduce to analysis of two equations of motion in two spatial variables according to the exact theory, and to the discussion of equations in a simplified form according to suitable approximate theories. The two-dimensional approach reflects better the physical phenomena in a cylinder than the one-dimensional approach, however appropriate methods for its application in discrete-continuous mechanical systems consisting of several rigid bodies and elastic cylindrical elements have not been elaborated yet. Many mathematical difficulties are overcome in the one-dimensional wave approach. So, in the final part of the paper an effective method based on the utilization of the one-dimensional longitudinal waves is described.

1. Introduction

Mechanical systems represent various mechanisms and machines. They usually consist of many elements of diverse shapes, dimensions and mechanical properties. Mechanical systems are subject to different loadings, periodic as well as nonperiodic. They may be longitudinally, torsionally or transversely deformed. The deformations can be elastic or inelastic, small or large; however large and inelastic deformations are neglected in the present paper. In the technical literature, the discussion of mechanical systems is conducted using discrete models, models with continuously distributed parameters and discrete-continuous models. The discrete models are most popular.

In the paper we confine ourselves to the discussion of the application possibilities of elastic waves to dynamic investigations of mechanical systems subject to longitudinal loadings. It should be pointed out that the application of elastic waves in systems torsionally deformed is studied by Pielorz (1992).

Elements subject to longitudinal loadings appear in many machines and mechanisms exploited in various branches of the industry, e.g. in engineering industry, metallurgical industry, mining industry and in the transport. Investigations into such elements reduce to solving one-, two- or three-dimensional dynamic problems. The way of the application of one-dimensional longitudinal waves is worked out and employed, e.g. by Pielorz (1980) and (1986), Mioduchowski et al. (1983), Pielorz and Nadolski (1989), Nadolski and Pielorz (1990). However, the problem of the use of two-dimensional and three-dimensional descriptions of mechanical systems subject to longitudinal loadings remains still not overcome. Elastic elements in mechanical systems have often the shape of a cylinder, for this reason the survey of hitherto results concerning two-dimensional problems for cylindrical elements subject to longitudinal loadings can appear to be useful.

The review contains the exact theory (cf Love (1944), Kolsky (1963), Graff (1975)) as well as the approximate theories (cf Mindlin and Herrman (1950), Mindlin and McNiven (1960)) and problems for a semi-infinite and a finite circular cylinders studied by Davies (1948), Miklowitz (1957), Miklowitz and Nisewanger (1957), Skalak (1957), Bertholf (1967), Saito and Chonan (1976), Saito and Wada (1977a,b). Below, the results given in these papers are briefly presented from the point of view of the possibilities of their application to mechanical systems consisting of cylindrical elements. In the final part of the paper a one-dimensional wave approach is presented.

2. General equations of motion

Structural elements being subject to various loadings can be considered as one-dimensional, two-dimensional and three-dimensional elements. From the literature it follows that two-dimensional and three-dimensional dynamic problems for structural elements are rather feebly investigated. In the case of an elastic isotropic homogeneous medium with small displacements and small deformations, i.e. for the strain tensor e_{ij} and the stress tensor σ_{ij} in the rectangular coordinates x_1, x_2, x_3 expressed by (cf Love (1944), Kolsky

(1963), Graff (1975))

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{2.1}$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij} \tag{2.2}$$

equations of motion take the form

$$\sigma_{ij,j} + X_i = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{2.3}$$

where

u_i	–	displacement components
t	–	time
λ, G	–	material constants
ρ	–	density
X_i	–	mass forces
δ_{ij}	–	Kronecker delta

and the comma denotes partial differentiation.

When solving specific problems, one has to add to the above equations appropriate initial and boundary conditions.

Depending on the geometry of considered body it may be convenient to employ Eqs (2.1) ÷ (2.3) in cylindrical coordinates. Namely, in the case of circular cylinder the equations of motion in the coordinate system r, θ, z neglecting mass forces take the form (cf Love (1944))

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\\
\rho\frac{\partial^2 u_\theta}{\partial t^2} &= \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r}\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}
\end{aligned} \tag{2.4}$$

where u_r, u_θ, u_z are the displacement components, and the components of the stress tensor are

$$\begin{aligned}
\sigma_{rr} &= \lambda\Delta + 2G\frac{\partial u_r}{\partial r}\\
\sigma_{\theta\theta} &= \lambda\Delta + 2G\Big(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\Big)\\
\sigma_{zz} &= \lambda\Delta + 2G\frac{\partial u_z}{\partial z}
\end{aligned} \tag{2.5}$$

$$\sigma_{r\theta} = G\Big(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big)$$
$$\sigma_{rz} = G\Big(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\Big)$$
$$\sigma_{\theta z} = G\Big(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\Big)$$

where Δ is the dilatation which in the cylindrical coordinates is expressed by

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{2.6}$$

The equations of motion (2.4) can be written using the dilatation Δ and rotation components $\omega_r, \omega_\theta, \omega_z$ or in terms of potentials. In the first case Eqs (2.4) take the form (cf Love (1944), Kolsky (1963))

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= (\lambda+2G)\frac{\partial \Delta}{\partial r} - \frac{2G}{r}\frac{\partial \omega_z}{\partial \theta} + 2G\frac{\partial \omega_\theta}{\partial z}\\
\rho\frac{\partial^2 u_\theta}{\partial t^2} &= (\lambda+2G)\frac{1}{r}\frac{\partial \Delta}{\partial \theta} - 2G\frac{\partial \omega_r}{\partial z} + 2G\frac{\partial \omega_z}{\partial r}\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= (\lambda+2G)\frac{\partial \Delta}{\partial z} - \frac{2G}{r}\frac{\partial (r\omega_\theta)}{\partial r} + \frac{2G}{r}\frac{\partial \omega_r}{\partial \theta}
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
2\omega_r &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \qquad\qquad 2\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\\
2\omega_z &= \frac{1}{r}\Big[\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\Big]
\end{aligned} \tag{2.8}$$

and $\omega_r, \omega_\theta, \omega_z$ satisfy the identity

$$\frac{1}{r}\frac{\partial (r\omega_r)}{\partial r} + \frac{1}{r}\frac{\partial \omega_\theta}{\partial \theta} + \frac{\partial \omega_z}{\partial z} = 0 \tag{2.9}$$

On the other hand, employing the scalar potential ϕ and the vector potential $\boldsymbol{H}$, with displacement components given by

$$\begin{aligned}
u_r &= \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z}\\
u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\\
u_z &= \frac{\partial \phi}{\partial z} + \frac{1}{r}\Big[\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\Big]
\end{aligned} \tag{2.10}$$

Eqs (2.4) are satisfied if the functions ϕ and $\boldsymbol{H}$ satisfy the wave equations

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \qquad \nabla^2\boldsymbol{H} = \frac{1}{c_2^2}\frac{\partial^2\boldsymbol{H}}{\partial t^2} \tag{2.11}$$

where

$$\begin{aligned}
\nabla^2\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}\\
\nabla^2\boldsymbol{H} &= \Big(\nabla^2 H_r - \frac{1}{r^2}H_r - \frac{2}{r^2}\frac{\partial H_\theta}{\partial\theta}\Big)\boldsymbol{e}_r + \\
&+ \Big(\nabla^2 H_\theta - \frac{1}{r^2}H_\theta + \frac{2}{r^2}\frac{\partial H_r}{\partial\theta}\Big)\boldsymbol{e}_\theta + \nabla^2 H_z\boldsymbol{e}_z
\end{aligned} \tag{2.12}$$

and

$$c_1^2 = \frac{\lambda + 2G}{\rho} \qquad c_2^2 = \frac{G}{\rho} \tag{2.13}$$

represent the velocities of a dilatation wave and of a distorsion wave, respectively, while $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z$ are versors in the cylindrical coordinates system.

The equations of motion (2.3), (2.4), (2.7) or (2.11) with appropriate initial and boundary conditions are the base for the determination of displacement components in structural elements, in rectangular or cylindrical coordinates, respectively. However, the use of these equations in practical problems usually involves serious difficulties of the mathematical nature and it may not be useful because many problems can be reduced to solving specific cases of these equations.

For example, for a cylindrical element with a circular cross-section undergoing only torsional deformations $(u_r = u_z = 0,\ u_\theta = u_\theta(r,z,t))$ Eqs (2.4) reduce to the following single equation of motion

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{1}{c_2^2}\frac{\partial^2 u_\theta}{\partial t^2} \tag{2.14}$$

with two nonzero components of the stress tensor

$$\sigma_{r\theta} = G\Big(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big) \qquad \sigma_{z\theta} = G\frac{\partial u_\theta}{\partial z} \tag{2.15}$$

On the other hand, for the cylindrical element subject to longitudinal loadings $(u_r = u_r(r,z,t),\ u_\theta = 0,\ u_z = u_z(r,z,t))$ Eqs (2.7) take the form

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= (\lambda + 2G)\frac{\partial\Delta}{\partial r} + 2G\frac{\partial\omega_\theta}{\partial z}\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= (\lambda + 2G)\frac{\partial\Delta}{\partial z} - \frac{2G}{r}\frac{\partial(r\omega_\theta)}{\partial r}
\end{aligned} \tag{2.16}$$

Utilizing the potentials ϕ and $\psi = H_\theta$, for which

$$u_r = \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z} \qquad\qquad u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r} \tag{2.17}$$

equations of motion (2.11) reduce to the following two equations

$$\begin{aligned} &\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \\ &\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \end{aligned} \tag{2.18}$$

The case of a circular cylinder upon a bending loading is more complicated than problems for torsional and longitudinal waves in the cylindrical element. In the latter cases the motion is symmetrical about the axis of the cylinder and nonzero displacements are independent of θ. For flexural waves, however, all three displacement components must be considered, and all three involve θ.

All three cases of waves in the cylindrical element are important in the dynamic analysis of mechanical systems because elastic elements in these systems very often have the shape of a circular cylinder. Moreover, one should realize that in mechanical systems, apart from elastic cylindrical elements, also rigid bodies may occur.

Depending on geometrical dimensions and loadings, the determination of displacements, strains and velocities in elastic elements of mechanical systems is reduced to solving one-dimensional, two-dimensional or three-dimensional problems, respectively.

Below, the two-dimensional approach in dynamic investigations of cylinders subject to longitudinal loadings is discussed. A one-dimensional approach utilizing longitudinal waves will be presented in the final part of the paper.

The dynamic analysis of the circular cylinder subject to a longitudinal loading is performed in the literature using Eqs (2.16) or (2.18) resulting from the exact theory of elasticity or using approximate theories.

In the present paper, the frequency equation for a circular cylinder will be discussed first, next the approximate theories most often applied to the circular cylinder subject to longitudinal loadings will be presented. Studying selected examples we shall get to know methods for investigations of cylinders using elastic waves. The examples concern an impact or a step-function type of loadings for semi-infinite cylinders (cf Davies (1948), Skalak (1957)), and the analysis of a finite rod with a rigid body situated on one end, connected with an elastic half-space (cf Saito and Chonan (1976), Saito and Wada (1977a,b)).

3. The Pochhammer equation of frequency

Consider an infinite elastic circular cylinder with the radius of cross-section a, the surface of which is free from stresses. The analysis is carried out in cylindrical coordinates (r, θ, z) with the z-axis overlapping the axis of the cylinder. If the corresponding displacement components are denoted by u_r, u_θ, u_z, then in the case of longitudinal waves u_θ vanish everywhere, u_r, u_z are independent of θ, and the equations of motion for an elastic medium (2.7) reduce to two Eqs (2.16) with boundary conditions

$$\sigma_{rr} = \sigma_{rz} = 0 \qquad \text{for} \qquad r = a \tag{3.1}$$

In order to derive the equation of frequency for the considered cylinder the displacements u_r and u_z are sought in the form

$$\begin{aligned} u_r(r,z,t) &= U(r)\exp[\mathrm{i}(\gamma z - \omega t)] \\ u_z(r,z,t) &= W(r)\exp[\mathrm{i}(\gamma z - \omega t)] \end{aligned} \tag{3.2}$$

where

γ – wave number
ω – phase frequency
U, W – unknown functions of the coordinate r.

Upon substituting Eqs (3.2) into (2.16) we obtain appropriate Bessel equations for Δ and ω_θ. From these equations it follows that Δ is proportional to the Bessel function $J_0(h'r)$, and ω_θ is proportional to the Bessel function $J_1(k'r)$ where

$$h'^2 = \frac{\omega^2}{c_1^2} - \gamma^2 \qquad\qquad k'^2 = \frac{\omega^2}{c_2^2} - \gamma^2 \tag{3.3}$$

Substituting next Eqs (3.2) into (2.6) and $(2.8)_2$ we get the following formulae for the functions $U(r)$ and $W(r)$

$$\begin{aligned} U(r) &= A\frac{\partial}{\partial r}J_0(h'r) + C\gamma J_1(k'r) \\ W(r) &= A\mathrm{i}\gamma J_0(h'r) + C\mathrm{i}\frac{1}{r}\frac{\partial}{\partial r}[rJ_1(k'r)] \end{aligned} \tag{3.4}$$

where A and C are constants. Upon substituting formulas (3.4), using Eqs (2.5) and (3.2), into boundary conditions (3.1) we obtain the equation of frequency in the form (cf Love (1944), Kolsky (1963))

$$\Big[2G\frac{\partial^2 J_0(h'a)}{\partial a^2}-\frac{\lambda\omega^2}{c_1^2}J_0(h'a)\Big]\Big(2\gamma^2-\frac{\omega^2}{c_2^2}\Big)J_1(k'a)-4\gamma^2G\frac{\partial J_0(h'a)}{\partial a}\frac{\partial J_1(k'a)}{\partial a}=0 \tag{3.5}$$

where $\partial/\partial a$ denotes $[\partial/\partial r]_{r=a}$.

From Eq (3.5) the phase velocity $c=\omega/\gamma$ for sinusoidal waves propagating along an infinitely long cylinder for any wave length $\Lambda = 2\pi/\gamma$ may be determined. The solutions sought in the form of Eqs (3.2) are not exact for a cylinder of finite length with ends free from stresses. However, when the length of cylinder is great compared with the radius a, the appropriate stresses on the cylinder ends become very small (cf Love (1944)).

Expanding the Bessel functions into power series, one can find from Eq (3.5) that for small a (i.e. for $J_0(h'a)\cong 1$, $\partial J_0(h'a)/\partial a \cong h'^2a/2$, $J_1(k'a)\cong k'a/2$ etc.) the phase speed $c=\omega/\gamma$ approaches the velocity of longitudinal wave in a rod $c_0^2=E/\rho$. Taking into account the terms with a^2 in the power series representing the Bessel functions a better approximation for the first dispersion curve may be obtained

$$\frac{c}{c_0}=1-\frac{\nu^2\pi^2a^2}{\Lambda^2} \tag{3.6}$$

where ν is the Poisson ratio. The corrective curve of Pochhammer, described by Eq (3.6), is marked by a broken line $1A$ in Fig.1.

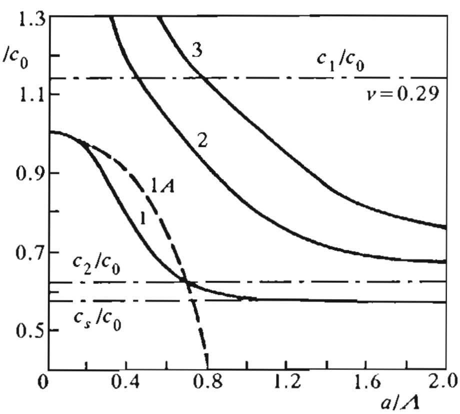

Fig. 1. The first three dispersion curves for longitudinal waves in an elastic cylinder

The Pochhammer equation was published in 1876, however numerical results for dispersion curves according to Eq (3.5) have been known since 1941.

Namely, Bancroft (1941) determined the first dispersion curve for longitudinal waves in the cylinder for several values of the Poisson ratio ν, and Davies (1948) found the first three dispersion curves for $\nu = 0.29$. These curves are shown in Fig.1. In this figure are also marked the values of c_1/c_0, c_2/c_0 and c_s/c_0, respectively, where c_s is the velocity of Rayleigh surface wave which for $\nu = 0.29$ is $c_s = 0.5764c_0$.

From Fig.1 it follows that for long wavelengths, $a/\Lambda < 0.1$, the phase velocity of longitudinal waves differs only slightly from the velocity of a longitudinal wave in a rod c_0 and from the velocity represented by the Rayleigh curve $1A$ (Eq (3.6)). However, the difference between the velocities c and c_0 increases with the growth in a/Λ. It means that accuracy of the classical wave theory for the rod decays with the decrease in the wavelength. For great values of a/Λ the phase velocity c approaches asymptotically the velocity of Rayleigh wave c_s. From Fig.1 it also follows that the Rayleigh corrective curve $1A$ describes quite adequately the first dispersion curve for $a/\Lambda < 0.7$.

In the above considerations, Eq (2.16) was employed in order to derive the equation of frequency. In a slightly different way the equation of frequency for longitudinal waves in a cylinder was obtained by Onoe et al. (1962) using Eqs (2.18) and assuming potentials ϕ and ψ in the form

$$
\begin{aligned}
\phi &= AJ_0(h'r)\exp[\mathrm{i}(\gamma z - \omega t)] \\
\psi &= -BJ_1(k'r)\exp[\mathrm{i}(\gamma z - \omega t)]
\end{aligned}
\tag{3.7}
$$

Detailed analysis of the frequency equation derived substituting Eqs (3.7) into the boundary conditions (3.1) is performed by Onoe et al. (1962) taking into account real, imaginary and complex wave numbers, respectively.

4. Approximate equations of motion

In the literature, apart from Eqs (2.16) and (2.18) obtained from the theory of elasticity, i.e. the exact theory, one can also find approximate equations describing the motion of a cylinder subject to longitudinal loadings. Such equations are developed using streght-of-material relations (cf Love (1944)), or introducing some simplifications into the equations of motion (2.16) (cf Mindlin and Herrmann (1950), Mindlin and McNiven (1960)).

The examples of equations derived from the strength-of-material relations are the classical wave equation

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.1}$$

where u is a longitudinal displacement and the Love equation presented below which completes the equation (4.1) by a term representing the effect of lateral inertia. In the both cases it is assumed that the x-axis coincides with the axis of the rod.

As it is shown in Section 3, Rayleigh assessed the effect of lateral inertia on longitudinal vibrations for a cylinder employing the exact theory of elasticity. This effect was also assessed by Love using, however, the Hamilton's principle. Namely, as a result of the longitudinal displacements u and the Poisson effect, lateral displacements v and w will occur at a certain point of the cross-section having coordinates y and z. The lateral strains existing in the cylinder are determined from the Hooke's law as

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \tag{4.2}$$

In the case of the uniaxial stress so that $\sigma_y = \sigma_z = 0$, from Eqs (4.2) we have

$$\varepsilon_y = \varepsilon_z = -\frac{\nu}{E}\sigma_x = -\nu\frac{\partial u}{\partial x} \tag{4.3}$$

and the lateral displacements v and w are given by

$$v = y\varepsilon_y = -\nu y\frac{\partial u}{\partial x} \qquad\qquad w = -\nu z\frac{\partial u}{\partial x} \tag{4.4}$$

Applying the Hamilton's principle, upon taking into account Eqs (4.3) and (4.4) in expressions for the kinematic and potential energy, and after applying numerous mathematical transformations the Love equation of motion can be derived in the form (cf Love (1944), Graff (1975))

$$\frac{\partial^2 u}{\partial x^2} + \frac{\nu^2 k^2}{c_0^2}\frac{\partial^4 u}{\partial x^2 \partial t^2} = \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} \tag{4.5}$$

where k^2 is the polar radius of gyration of the cross-section.

In order to constitute the influence of the lateral inertia on the dispersion curves of the rod, the solution to the Love equation (4.5) is sought in the form

$$u = A\exp[\mathrm{i}\gamma(x - ct)] \tag{4.6}$$

where c is the phase velocity. Substituting Eq (4.6) into (4.5) gives

$$-\gamma^2 + \frac{\nu^2 k^2}{c_0^2}\gamma^4 c^2 + \frac{\gamma^2 c^2}{c_0^2} = 0 \tag{4.7}$$

Upon introduction of the nondimensional quantities

$$\bar{c} = \frac{c}{c_0} \qquad\qquad \bar{\gamma} = k\nu\gamma \tag{4.8}$$

the relation (4.7) becomes

$$\bar{c} = \frac{1}{\sqrt{1 + \bar{\gamma}^2}} \tag{4.9}$$

If $\bar{\gamma} = 0$, from Eq (4.9) we obtain the inertialess result for the classical wave equation (4.1). The dispersion curve corresponding to Eq (4.9) is shown in Fig.2 together with the dispersion curves for the classical wave equation and for the exact theory, i.e. for the frequency equation (3.5) for longitudinal vibrations of the cylinder having a circular cross-section.

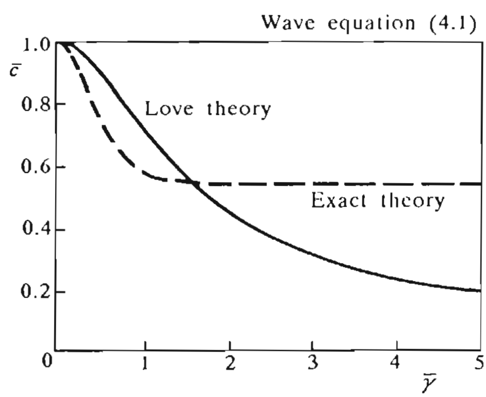

Fig. 2. Dispersion curve for the Love equation for a rod

From relation (4.9) it follows that for low frequencies and long wavelengths the wave in the rod described by the Love equation (4.5) propagates at the speed being close to c_0. With the increase in the wavenumber the dispersion curves shown in Fig.2 diverge rapidly. It concerns especially the classical wave theory with $\bar{\gamma} > 0.3$, while the Love theory quite well approximates the exact theory to the vicinity of $\bar{\gamma} = 2$.

On the basis of dispersion curves shown in Fig.2 one can assume $\bar{\gamma} < 0.3$ as the wavenumber limit for the classical wave theory. For $\bar{\gamma} = 0.3$ we have $\bar{c} \cong 1$, and the frequency $\bar{\omega} = \omega\nu k/c_0 \cong 0.3$. As an example, for the steel rod with a circular cross-section of the diameter $d = 0.0254$ m and for $c_0 = 5080$ m/s, $\nu = 0.29$, $k = d/\sqrt{2^3}$ we obtain: the radial frequency $\omega = \bar{\omega}c_0/(\nu k) = 5.85 \cdot 10^5$ rad/s, the cyclic frequency $f = \omega/(2\pi) = 93186.0$ Hz, the wave number $\gamma = \bar{\gamma}/(\nu k) = 115.19\ \mathrm{m}^{-1}$, the wave length $\Lambda = 2\pi/\gamma = 0.0545$ m. From the above it follows that the classical wave theory can be employed in the discussion of dynamic problems for a circular rod if the cyclic frequency is lower than 90000 Hz and the wavelength Λ is greater than 0.05 m.

The classical wave theory can be also applied to the study of mechanical systems consisting of rigid bodies and rods undergoing longitudinal deformations. As an example, in the case of a rod having the finite length $l = 0.254$ m, one end of which is free and the other is rigidly fixed, the first radial frequency and the wavelength are (cf Kaliski et al. (1966)), $\omega_1 = \pi c_0/(2l) = 31917$ rad/s, $f_1 = \omega_1/2\pi = 5.1 \cdot 10^3$ Hz, $\gamma_1 = \omega_1/c_0 = \pi/(2l) = 6.28\ \mathrm{m}^{-1}$, $\Lambda_1 = 2\pi/\gamma_1 = 1.0$ m. On the other hand, for the rod having one end free and the other end connected to a rigid body with the mass M from the equation of frequency $\omega \tan(\omega l/c_0) - K_0 c_0/l = 0$, where $K_0 = A\rho l/M$, we have (cf Kaliski et al. (1966))

— for $K_0 = 0.25$

$$\omega_1 = 0.48\frac{c_0}{l} = 9.75 \cdot 10^3\mathrm{rad/s} \qquad f_1 = 1553.1\mathrm{Hz} \qquad \Lambda_1 = \frac{c_0}{f_1} = 3.2\mathrm{m}$$

— for $K_0 = 1$

$$\omega_1 = 0.86\frac{c_0}{l} = 1.74 \cdot 10^4\mathrm{rad/s} \qquad f_1 = 2782\mathrm{Hz} \qquad \Lambda_1 = \frac{c_0}{f_1} = 1.8\mathrm{m}$$

In an analogous way one can determine cyclic frequencies and wavelengths for systems consisting of several rods and several rigid bodies.

Other approximate theories for longitudinal waves in rods are given by Mindlin and Herrmann (1950), Mindlin and McNiven (1960) (cf Graff (1975)).

Mindlin and Herrmann (1950) assumed that the components of cylinder displacements have the form

$$u_r = \frac{r}{a}u(z,t) \qquad u_\theta = 0 \qquad u_z = w(z,t) \tag{4.10}$$

Then, after using the relation between energy and work and employing the equations of motion for the theory of elasticity (2.16) taking into account (4.10) we obtain the following two equations

$$
\begin{aligned}
&a^2\kappa^2 G\frac{\partial^2 u}{\partial z^2} - 8\kappa_1^2(\lambda + G)u - 4a\kappa_1^2\lambda\frac{\partial w}{\partial z} = \rho a^2\frac{\partial^2 u}{\partial t^2} \\
&2a\lambda\frac{\partial u}{\partial z} + a^2(\lambda + 2G)\frac{\partial^2 w}{\partial z^2} = \rho a^2\frac{\partial^2 w}{\partial t^2}
\end{aligned}
\tag{4.11}
$$

where κ, κ_1 are the constant correction coefficients. The first equation is mainly associated with the radial shear and inertia effects. The second equation, after omitting the first term, is similar to the classical wave equation with the speed of dilatational wave instead of the longitudinal wave speed in a rod. However, for long wavelengths the phase speed determined from the dispersion relation for Eqs (4.11) approaches the longitudinal wave speed in a rod: $c^2 \rightarrow E/\rho$. Correction coefficients in Eqs (4.11) are selected in such a way that the dispersion curve for Eqs (4.11) is a good approximation of the first dispersion curve obtained from the exact theory, Fig.2.

Mindlin and McNiven (1960) assumed the components of cylinder displacement in the form of the following series

$$
\begin{aligned}
&u_r(r,z,t) = \sum_{n=0}^{\infty} U_n\left(\frac{r}{a}\right)u_n(z,t) \qquad\qquad u_\theta = 0 \\
&u_z(r,z,t) = \sum_{n=0}^{\infty} W_n\left(\frac{r}{a}\right)w_n(z,t)
\end{aligned}
\tag{4.12}
$$

where U_n and W_n are the Jacobi polynomials. Confining ourselves to three displacement functions u_0, w_0, w_1 (Mindlin and Herrmann (1950) took only two displacement functions into account) and performing appropriate transformations we obtain the following three equations of motion

$$
\begin{aligned}
&(u_0'' - 4w_1') - 8(\lambda + G)\kappa_1^2 u_0 - 4\lambda\kappa_1 w_0' = \rho a^2\kappa_3^2 u_0 \\
&(\lambda + 2G)w_0'' + 2\lambda\kappa_1 u_0' = \rho a^2 w_0 \\
&(\lambda + 2G)w_1'' + 6G\kappa_2^2(u_0' - 4w_1) = \rho a^2\kappa_4^2 w_1
\end{aligned}
\tag{4.13}
$$

where κ_i are the correction coefficients being so selected that suitable dispersion curves differ insignificantly from the first three dispersion curves for the exact theory of elasticity.

From the above considerations it follows that in approximate theories for the circular elastic cylinder one, two or three displacement functions are used.

The dispersion curves obtained for the simplified equations of motion are close to the first or to the first three dispersion curves for the exact theory within a limited range of variability of the wavenumber.

Various approaches concerning the description of composite materials have been developed recently (cf Christensen (1979), Aboudi (1986) and (1987)). For example Aboudi (1986) and (1987) propose a high-order continuum theory for the analysis of harmonic and transient waves in three-dimensional media taking into account anisotropy. This theory covers longitudinal waves as a particular case. Aboudi (1986) and (1987) study a model consisting of cells having the form of a rectangular parallelepiped. Cells are divided into subcells. In the analysis the high-order continuum theory is employed which is based on the expansion of the displacement vector at a point in a subcell of the representative cell in terms of the coordinates of that point with respect to the local system. This expansion is expressed in terms of the Legendre polynomials permitting the modelling of increasing complex deformation patterns within the subcell. Such an approach enables one to determine assumed functions for displacements in the range of the orthogonality of the Legendre polynomials and to derive the appropriate dispersion relations. It should be pointed out that mathematical formulas obtained in the case of dilatational waves are very complicated.

The classical wave equation (4.1) also belongs to approximate equations of motion for a circular cylinder. The simple form of this equation in the comparison with Eqs (4.5), (4.11) and (4.13) enables one to use it in the dynamic analysis of mechanical systems consisting of several rods and rigid bodies subject to longitudinal deformations (cf Pielorz (1980) and (1986), Mioduchowski et al. (1983), Pielorz and Nadolski (1989), Nadolski and Pielorz (1990)). The application of equation (4.1) will be shown in the final part of the present paper. The classical wave theory for rods is, moreover, a useful and efficient tool for determination of the mechanical properties of materials (cf Lundberg and Blanc (1988), Lundberg et al. (1990)).

5. Solutions to selected problems

In the foregoing considerations the equations of motion according to the theory of elasticity as well as to appropriate approximate theories for the circular cylinder subject to longitudinal loadings are given. The equations of the theory of elasticity are employed, e.g. by Skalak (1957), Bertholf (1967), Saito and Chonan (1976), Saito and Wada (1977a,b), Svärdh (1984), Downey and Bogy (1987), the Love equation by Davies (1948), Conway and Jakubowski (1969), and the approximate theory of Mindlin-Herrmann by Mindlin and Herrmann (1950), Miklowitz (1957), Miklowitz and Nisewanger (1957), Mindlin and McNiven (1960). Below, the results of some of these papers are briefly reviewed focusing attention mainly on the methods for solution of appropriate equations.

The paper by Skalak (1957) concerns the impact of two semi-infinite elastic circular cylinders with the radius a and having lateral surfaces free of traction. After impact at instance $t = 0$ the cylinders are assumed to behave as a single, solid, infinite cylinder. The problem of strain determination reduces to the solution to two equations of motion (2.16) with the following boundary conditions

$$\begin{aligned} &\sigma_{rr}(a,z,t) = \begin{cases} \lambda v_0/c_1 & \text{for} \quad -c_1 t < z < c_1 t \\ 0 & \text{for} \quad |z| > c_1 t \end{cases} \\ &\sigma_{rz}(a,z,t) = 0 \end{aligned} \tag{5.1}$$

where v_0 is the impact speed.

The solution to Eqs (2.16) with (4.14) is obtained by means of the method of double-integral transforms. Evaluation of the solution is given using approximations which are valid for significant values of the time after the impact. Namely, after applying numerous transformations connected with the Fourier-Laplace transforms, the following formula for the strain $\varepsilon_z = \partial u_z/\partial z$ is derived for large values of time

$$\varepsilon_z = \frac{\partial u_z}{\partial z} = \frac{v_0}{c_0}\Big[\frac{1}{6} + \int\limits_0^{\alpha} \mathrm{Ai}(\alpha)\, d\alpha + \frac{1}{6} + \int\limits_0^{\alpha'} \mathrm{Ai}(\alpha)\, d\alpha\Big] \tag{5.2}$$

where $\mathrm{Ai}(\alpha)$ is the Airy integral and

$$\begin{aligned} &\alpha' = \frac{z'}{\sqrt[3]{3et}} \qquad \alpha'' = \frac{z''}{\sqrt[3]{3et}} \qquad e = \frac{\nu^2 a^2 c_0}{4} \\ &z' = z - c_0 t \qquad z'' = -z - c_0 t \end{aligned} \tag{5.3}$$

The diagram of the function $-\varepsilon_z c_0/v_0$ versus $z'/\sqrt[3]{et}$ is presented in Fig.3. It can be seen that $z' = 0$ corresponds to the occurence time of the disturbance caused by an external loading according to the classical wave theory. The analysis carried out by Skalak (1957) shows that the strain function begins to rise earlier, attaining the maximum amplitude later than it is predicted

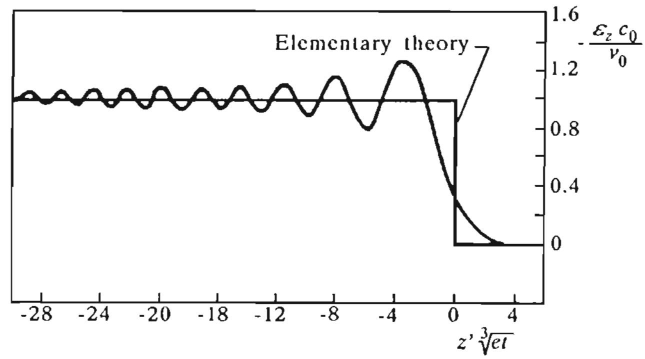

Fig. 3. Strain $-\varepsilon_z c_0/v_0$ versus $z'(et)^{-1/3}$

by the classical theory. Next, the strain function oscillates with a decreasing amplitude about the solution for the classical wave theory.

Analogous results are obtained by Davies (1948) employing the Love equation (4.5) in the case when the end of the rod is subject to a step pulse. The solution is derived by means of the Laplace transforms. From the both papers i.e. Davies (1948) and Skalak (1957) it follows that the lateral inertia effects are the main reason for the oscillatory behaviour of the solution according to the exact elasticity theory about the solution predicted by the classical wave theory. It appears that they are the only effects taken into account in the Love theory for the rod by the incorporation them into the classical wave equation (4.1). These conclusions concern significant values of the time. The oscillatory character of the solution about the solution for the classical theory is also obtained by Conway and Jakubowski (1969) in the case of the axial impact of circular cylinders with finite lengths analyzed using the Love equation.

A semi-infinite cylinder described by Eqs (4.11), i.e. by the equations derived from the approximate theory of Mindlin-Herrmann, the end of which is subject to a step loading is considered by Miklowitz (1957) using the Laplace transforms. Theoretical results are compared with experimental results presented by Miklowitz and Nisewanger (1957). The behaviour of the formulae for strains is similar to that obtained by Davies (1948) and Skalak (1957).

The case of a semi-infinite and a finite elastic cylinders subject to a step-function loading is also discussed by Bertholf (1967). In his paper equations of the theory of elasticity (2.16) are used and they are solved by means of the method of finite differences. A direct numerical integration of the equations of motion for a cylinder enables one to obtain information about displacements and strains within the cylinder as well as on its surface, at points close to as well as for far from the loaded end of the cylinder. It is shown that theoretical results remain in an agreement with experimental results presented by Miklowitz and Nisewanger (1957).

The problem of propagation of longitudinal elastic waves in a semi-infinite and a finite hollow elastic cylinders with radia a and b is investigated e.g. by Fitch (1963), Heimann and Kolsky (1966), Nigul (1967), Chong et al. (1971), Svärdh (1984). As an example, Svärdh (1984) using a double integral transform technique obtained the asymptotic solutions of equations of motion (2.16) for a semi-infinite hollow cylinder in two cases of boundary conditions, i.e., when axial pressure is applied to a radially clamped end and when a prescribed axial velocity is applied to an end being free from shear stress. From the discussion presented by Svärdh (1984) it follows that the influence of the radia ratio a/b is not great and dispersion curves depend on the distance from the end of the cylinder.

Davies (1948), Miklowitz (1957), Miklowitz and Nisewanger (1957), Skalak (1957), Bertholf (1967) consider semi-infinite and finite circular cylinders. Elastic cylindrical elements having a finite length can be parts of mechanical systems, however mechanical systems usually consist of more than a single element. Systems which can be included to mechanical systems are discussed by Saito and Chonan (1976), Saito and Wada (1977a,b), Downey and Bogy (1987). Investigations carried out by Saito and Chonan (1976), Saito and Wada (1977a) concern a finite rod connected to an elastic half-space with the other end being free or being attached to a rigid body loaded by a variable external loading. Saito and Wada (1977b) consider a rigid body connected to an elastic half-space by a spring. Saito and Chonan (1976), Saito and Wada (1977a,b) assume that the half-space can represent the plane surface of an elastic element of sufficient width and thickness. During vibrations the elastic waves are reflected and refracted at the interface between a half-space and a rod or a spring, and dissipation of energy by the waves radiating to infinity produces a damping of the motion of the system. At the interface between the rod and the half-space two kinds of specific boundary conditions are assumed, i.e. uniform normal stress distribution over the interface and uniform normal displacement distribution over the interface. Below, as an example, the main results of Saito and Wada (1977a) are presented.

Consider an elastic circular rod of length l and radius a, one end of which is attached to a rigid body with the mass M and the other end to an elastic half-space, as it is shown in Fig.4. The rigid body is subject to a harmonic longitudinal external loading $P\exp(\omega t)$ acting in the direction coinciding with the rod axis direction, where ω is the loading frequency. The rod axis is taken as the z-axis of cylindrical coordinates (r, θ, z), and the

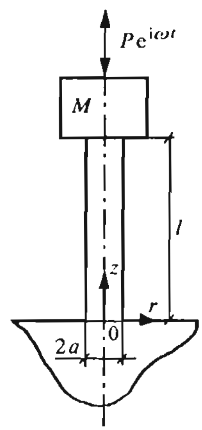

Fig. 4. Rod with a rigid body connected to an elastic half-space

plane $z = 0$ is the boundary of the half-space. It is assumed that the rod undergoes only longitudinal deformations, each plane cross-section of the rod remains plane during the motion and the stress distribution over it is uniform. In that case the equation of longitudinal vibrations of the rod is the classical wave equation

$$\frac{\partial^2 u_R}{\partial t^2} = c_R^2 \frac{\partial^2 u_R}{\partial z^2} \tag{5.4}$$

where

u_R – rod displacement in the z direction

E_R, ρ_R – elastic modulus and the density of the rod, respectively, and $c_R^2 = E_R/\rho_R$.

The solution to Eq (5.4) is sought in the form

$$u_R(z,t) = U_R(z)\exp(i\omega t) \tag{5.5}$$

Upon substituting Eq (5.5) into (5.4) we have

$$U_R(z) = D_1 \cos\Big[\frac{\omega}{c_R}(l-z)\Big] + D_2 \sin\Big[\frac{\omega}{c_R}(l-z)\Big] \tag{5.6}$$

where D_1 and D_2 are unknown functions of ω.

The deformation of the half-space is symmetrical with respect to the z-axis, so any variables are independent of θ. If u_r and u_z are the displacement components of the half-space in the r and z directions, respectively, then equations of motion for the half-space take the form (2.18) in the potentials ϕ

and ψ, where all constants E, G, ρ, c_1, c_2 concern the considered half-space. Solutions to equations (2.18) are sought in the form

$$\begin{aligned}\phi &= \exp(\mathrm{i}\omega t)\int_0^\infty A(\xi)\exp(\alpha z)J_0(\xi r)\xi\,d\xi\\ \psi &= \exp(\mathrm{i}\omega t)\int_0^\infty B(\xi)\exp(\beta z)J_1(\xi r)\xi\,d\xi\end{aligned} \tag{5.7}$$

where

$$\alpha^2 = \xi^2 - \frac{\omega^2}{c_1^2} \qquad\qquad \beta^2 = \xi^2 - \frac{\omega^2}{c_2^2}$$

$J_n(\xi r)$ is the first kind and nth order Bessel function, and $A(\xi)$, $B(\xi)$ are unknown functions of ξ.

To Eqs (5.4) and (2.18) boundary conditions have to be added:

— for $z = l$

$$M\frac{\partial^2 u_R(l,t)}{\partial t^2} - P\exp(\mathrm{i}\omega t) + \pi a^2 E_R\frac{\partial u_R(l,t)}{\partial z} = 0 \tag{5.8}$$

— for $z = 0$

$$\sigma_{zr}(r,0,t) = 0 \qquad \text{for} \quad 0 \le r < \infty \tag{5.9}$$

$$\sigma_{zz}(r,0,t) = \begin{cases} E_R\dfrac{\partial u_R(0,t)}{\partial z} & \text{for} \quad 0 \le r \le a\\ 0 & \text{for} \quad a < r\end{cases} \tag{5.10}$$

$$u_z(r,0,t) = u_R(0,t) \qquad \text{for} \quad 0 \le r \le a \tag{5.11}$$

It is, however, impossible to satisfy the boundary conditions (5.10) and (5.11) simultaneously when the stress and the displacement in the cross-section of the rod are assumed to be uniform. For this reason Saito and Wada (1977a) consider two kinds of extreme boundary i.e. uniform normal stress distribution over the interface and uniform normal displacement in the interface. In the last case the stress at the interface is maximum at $r = a$ and minimum at $r = 0$.

In the first case, i.e. for the uniform normal stress distribution over the interface, boundary conditions (5.10) are applicable, however instead of (5.11) we write

$$u_1(t) = u_R(0,t) \qquad \text{for} \quad 0 \le r \le a \tag{5.12}$$

where u_1 is the approximate mean displacement

$$u_1(t) = \frac{1}{\pi a^2} \int_0^a 2\pi r u_z(r,0,t)\, dr \tag{5.13}$$

In the case of the uniform normal displacement over the interface analysis presented by Saito and Wada (1977a) is confined to the static problem ($\omega \to 0$).

Substituting the formulas (5.6) and (5.7) into the appropriate boundary conditions, the unknown quantities D_1, D_2, $A(\xi)$, $B(\xi)$ can be determined. They are expressed by very complicated formulas (cf Saito and Wada (1977a)). Numerical results given by Saito and Wada (1977a) are presented for three kinds of rod and half-space material combinations: (a) the rod is plastic and the half-space is aluminium, (b) the rod and the half-space are of the same material, (c) the rod is aluminium and the half-space is plastic. These results are obtained for the both cases of boundary conditions, i.e. for the uniform normal stress distribution as well as uniform normal displacement in the interface between the rod and the half-space.

Exemplary amplitude-frequency curves for the displacements of the rigid body $|u_R(l,t)k_R/P|$, where $k_R = \pi a^2 E_R/l$, are plotted in Fig.5 for $l = 20a$, $K = \rho_R \pi a^2 l/M = 0.1, 1.0, 10, \infty$, for the rod made of plastic and for the half-space made of aluminium. They concern the case of the uniform stress in the interface. Other diagrams concerning both cases of the boundary conditions are given by Saito and Wada (1977a). From these diagrams it follows that the effect of boundary conditions for $z = 0$ is greater when the rod is made of aluminium and the half-space of plastic.

Downey and Bogy (1987) consider the problem of the normal impact of an elastic rod on an elastic half-space. It is assumed that a circular rod has the length much greater than its diameter, and the upper end of the rod is connected with a rigid body. The motion of the rod is described by the classical wave equation and for the elastic half-space Eqs (2.16) are used. The displacement of the rod in the contact region is assumed to be the mean displacement of the half-space analogously as in the paper by Saito and Wada (1977a). In analytical considerations the Laplace and the Hankel transforms are employed. Numerical results are presented by Downey and Bogy (1987) for various half-space materials, rod lengths and masses of the rigid body, respectively. It is found that in the absence of the rigid body the maximum contact stress depends entirely on the rod material, but with the rigid body added the contact stress can become much greater and depends on the rod material, the half-space material and on the mass of the rigid body. The contact time for rods without a rigid body is dependent mainly on the length of

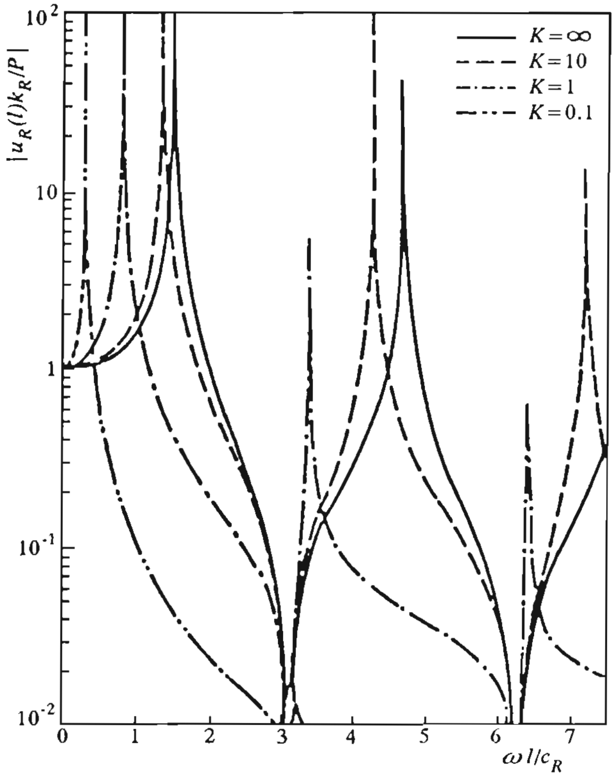

Fig. 5. Amplitude-frequency curves for $l = 20a$

the rod, and in all considered cases the contact of the both bodies is terminated before a second reflection from the free end occurs.

6. Two-dimensional and one-dimensional descriptions in mechanical systems

The approach employed by Davies (1948), Mindlin and Herrmann (1950), Miklowitz (1957), Miklowitz and Nisewanger (1957), Skalak (1957), Mindlin and McNiven (1960), Fitch (1963), Heimann and Kolsky (1966), Bertholf (1967), Nigul (1967), Conway and Jakubowski (1969), Chong et al. (1971), Saito and Chonan (1976), Saito and Wada (1977a,b), Svärdh (1984), Aboudi (1986) and (1987), Downey and Bogy (1987) for the investigation of two-dimensional problems for the semi-infinite and finite cylinders has rather limited practical application, for instance in mechanical systems subject to longitudinal loadings. These systems usually consist of several cylindrical elements of finite lengths having various mechanical properties and variable cross-sections which may be connected by means of elements with high stiffness treated as rigid bodies. Thus, the limitations of the two-dimensional approach in mechanical systems are connected not only with the solutions having the form of complicated integral expressions but also with the possibilities of taking into account several elastic elements and rigid bodies.

Difficulties of such a type do not occur when the elementary wave theory can be applied. From the papers by Davies (1948), Miklowitz (1957), Miklowitz and Nisewanger (1957), Skalak (1957), Bertholf (1967), Saito and Chonan (1976), Saito and Wada (1977a,b) it follows that the solution for this theory has a fundamental meaning because the solutions for various two-dimensional problems oscillate about the solution predicted by the classical wave theory. Moreover, in order to avoid mathematical difficulties, Saito and Wada (1977a), Downey and Bogy (1987) discussing the rod with the rigid body confine their considerations to the case when the motion of the rod is described by the classical wave equation. It is supposed that two-dimensional descriptions better than one-dimensional ones describe physical phenomena in the circular cylinder subject to longitudinal loadings; however, the appropriate methods for their applications to dynamic investigations of discrete-continuous mechanical systems have not been elaborated yet.

The method using one-dimensional longitudinal elastic waves is presented by Pielorz (1980), (1986) and (1989), Mioduchowski et al. (1983), Pielorz and Nadolski (1989), Nadolski and Pielorz (1990). It is based on the solution of the d'Alembert type for the equations of motion and allows one to determine displacements, strains and velocities in transient as well as in steady states in any cross-section of mechanical systems consisting of an arbitrary number of elastic elements connected with a suitable number of rigid bodies. Davies (1948), Mindlin and Herrmann (1950), Miklowitz (1957), Miklowitz and Nisewanger (1957), Skalak (1957), Mindlin and McNiven (1960), Bertholf (1967), Conway and Jakubowski (1969), Chong et al. (1971), Saito and Chonan (1976), Saito and Wada (1977a,b), Svärdh (1984), Downey and Bogy (1987) dealing with two-dimensional problems neglect damping, while using the one-dimensional approach damping can be easily taken into account by means of the equivalent damping (cf Pielorz (1989), Pielorz and Nadolski (1989), Nadolski and Pielorz (1990)).

As an example, consider the longitudinal impact at the instant $t = 0$ of two systems consisting of s_1 and s_2 elastic elements having variable cross-sections A_i connected suitably with $s_1/2$ and $s_2/2$ rigid bodies (s_1, s_2 are even numbers). It is assumed that the x-axis coincides with the axis of elastic elements. The determination of displacements $u_i(x,t)$ of the ith elastic element is reduced to solving $N = s_1 + s_2$ equations of motion

$$u_{i,tt} - c_0^2\Big(u_{i,xx} + \frac{A_i'}{A_i}u_{i,x}\Big) = 0 \qquad \text{for} \quad x = 1,2,...,N \tag{6.1}$$

with the following boundary conditions

$$\begin{array}{lll} u_{1,x} = 0 & \text{for} \quad x = 0 & \\ u_{N,x} = 0 & \text{for} \quad x = L_N & \\ M_i u_{i,tt} + EA_i u_{i,x} + & & \\ -EA_{i+1}u_{i+1,x} + D_i(u_{i,t} - v_j) = 0 & \text{for} \quad x = L_i & i = 1,3,...,N-1 \\ A_i u_{i,x} = A_{i+1}u_{i+1,x} & \text{for} \quad x = L_i & i = 2,4,...,N-2 \\ u_i = u_{i+1} & \text{for} \quad x = L_i & i = 1,2,...,N-1 \end{array} \tag{6.2}$$

and initial conditions

$$u_i(x,0) = 0 \qquad\qquad u_{i,t}(x,0) = v_j \qquad\qquad i = 1,2,...,N \tag{6.3}$$

where

M_i	–	mass of the ith rigid body
l_i	–	length of the ith elastic element
v_j	–	appropriate velocities of elements before the impact
D_i	–	coefficients of the equivalent damping

and comma denotes partial differentiation, $L_i = l_1 + l_2 + ... + l_i$.

The solutions to Eqs (6.1) in the case when A_i = const are sought in the form

$$u_i(x,t) = f_i[c_0(t-t_{0i})-x+x_{0i}]+g_i[c_0(t-t_{0i})+x-x_{0i}] \qquad i = 1,2,...,N \tag{6.4}$$

where the functions f_i and g_i represent waves caused by the collision propagating in the ith element of the considered system in the direction of x-axis and the opposite one, respectively. The constants t_{0i} and x_{0i} in the arguments of the functions f_i and g_i denote the instant and the location of one of the ends of the ith element at which the first disturbance reaches this element. Moreover, the functions f_i and g_i are equal to zero for negative arguments, i.e. before the occurence of the first disturbance. Another form for the solutions to equations (6.1) is given by Pielorz (1986) and (1989).

The functions f_i and g_i are the functions of a single argument. Their forms are determined by the boundary conditions (6.2). Upon substituting (6.4) into

(6.2) we obtain for the functions f_i and g_i a system of ordinary differential equations with a retarded argument of the neutral type. This system can be solved analytically or numerically by means of the method of finite differences. Solutions for specific examples of colliding systems are presented by Pielorz (1980) and (1986), Mioduchowski et al. (1983), both taking into account and neglecting local deformations.

The wave method using a solution of the d'Alembert type can be also applied to dynamic investigations of systems subject to longitudinal loadings having different mechanical properties (cf Pielorz and Nadolski (1989), Nadolski and Pielorz (1990)). Moreover, it is employed by Pielorz (1980), (1986) and (1989), Mioduchowski et al. (1983), Pielorz and Nadolski (1989), Nadolski and Pielorz (1990) for the discussion of systems much more complicated than the systems analyzed by Vardy and Alsarraj (1989) using the method of characteristics.

7. Final remarks

From papers reviewed in the present paper it follows that dynamic investigations of circular cylinders subject to longitudinal loadings or longitudinal collisions very often require the use of the two-dimensional approach. Only when the radius of the cylinder is much smaller than the cylinder length, the one-dimensional approach is permissible. Theoretical results are confirmed by experiments (cf Davies (1948), Miklowitz and Nisewanger (1957), Kolsky (1963), Lundberg and Blanc (1988)).

The above conclusion concerns the system consisting of a single or two cylinders. However, the problem of the application of the two-dimensional description to mechanical systems subject to longitudinal loadings is rather a difficult question. From the reviewed literature it follows that this problem is still not overcome. Davies (1948), Miklowitz (1957), Miklowitz and Nisewanger (1957), Skalak (1957), Fitch (1963), Heimann and Kolsky (1966), Bertholf (1967), Nigul (1967), Conway and Jakubowski (1969), Chong et al. (1971), Saito and Chonan (1976), Saito and Wada (1977a,b), Svärdh (1984), Aboudi (1986) and (1987), Downey and Bogy (1987) discuss two-dimensional problems for a single or two elastic cylinders not showing, however, the way for the use of the two-dimensional approach in the dynamic analysis of systems consisting, for example, of several cylindrical elements connected with rigid bodies. Solutions having the form of complicated analytical expressions seem to be the main reason for that. Moreover, in the case of a rod connected to a rigid body, probably to avoid mathematical difficulties, Saito and Wada (1977a), Downey and Bogy (1987) describe the motion of the rod by the classical wave equation.

The one-dimensional wave approach appeared to be a very useful and efficient tool in the investigations of various mechanical systems longitudinally deformed. In this approach the solution of the d'Alembert type is used and it can be employed for the discussion of various elements of machines and mechanisms subject to longitudinal loadings and collisions (cf Pielorz (1980), (1986) and (1989), Mioduchowski et al. (1983), Pielorz and Nadolski (1989)). The elementary wave theory can be also efficient for the determination of mechanical properties of materials (cf Lundberg and Blanc (1988), Lundberg et al. (1990)).

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Wykorzystanie podejścia dwuwymiarowego w układach mechanicznych poddanych obciążeniom wzdłużnym

Streszczenie

W pracy dokonano przeglądu literatury z punktu widzenia możliwości zastosowania podejścia dwuwymiarowego w układach mechanicznych poddanych obciążeniom wzdłużnym. Przegląd obejmuje ścisłą teorię sprężystości i teorie przybliżone oraz problemy dla walców kołowych o długości półnieskończonej i skończonej. Problemy te sprowadzają się do rozwiązywania dwóch równań ruchu względem dwóch zmiennych przestrzennych według teorii ścisłej oraz do równań w postaci uproszczonej według odpowiednich teorii przybliżonych. Podejście dwuwymiarowe raczej lepiej odzwierciedla zjawiska fizyczne występujące w walcu aniżeli opis jednowymiarowy, jednakże dotychczas nie opracowano jeszcze efektywnych metod wykorzystania podejścia dwuwymiarowego w badaniach dynamicznych dyskretno-ciągłych układów mechanicznych złożonych z wielu brył sztywnych i sprężystych elementów walcowych. Wiele trudności typu matematycznego zostało opanowanych w przypadku stosowania podejścia jednowymiarowego. W związku z tym, w zakończeniu przedstawiono efektywną metodę opartą na wykorzystaniu jednowymiarowych fal podłużnych.

Manuscript received March 7, 1994; accepted for print April 14, 1994