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MICRODYNAMICS: CONTINUUM MODELLING THE SIMPLE COMPOSITE MATERIALS

CZESŁAW WOŹNIAK

Center of Mechanics

Institute of Fundamental Technological Research, Warsaw

In this paper there is analyzed a certain general modelling procedure for the investigation of non-stationary processes in periodic composites. The resulting macro-models take into account the effect of the microstructure length dimension on the dynamic behaviour of the body and are simple enough to be applied to the analysis of engineering problems. The modelling procedure under consideration was recently applied in a series of papers to some selected composite materials and structures. The aim of this study is to generalize the obtained partial results and to formulate the above macro-models within a framework of the nonlinear continuum mechanics for an arbitrary simple material.

1. Introduction

The formulation of different modelling approaches in composite mechanics is motivated by the well known fact that the exact analysis of periodic heterogeneous materials within the framework of solid mechanics (viz. the micromechanics of periodic composites) can be carried out only for a few special problems. In general, the equations of micromechanics, due to the noncontinuous and highly oscillating form of functions describing material properties of a composite body, cannot be taken as the basis for obtaining useful information on most of the problems met in engineering practice. That is why in composite mechanics we deal with a variety of macro-modelling methods leading to different approximate mathematical models of periodic heterogeneous materials and structures. Mathematical models of this kind are often called *the macro-models* and describe the effects of constituents only as averaged apparent properties of a body within a framework of the macromechanics of composite materials. Nevertheless, many macro-modelling procedures make

it possible to detect also the micro-mechanical behaviour of a composite on the basis of solutions to problems of macromechanics. The list of contemporary and updated macro-modelling approaches is very extensive; in order to interrelate the macro-modelling procedure developed in this contribution with the existing methods we outline below some trends in the formulation of approximate theories for periodic composite materials.

Generally speaking, the known methods of macro-modelling can be separated into two main groups. To the first belong the general procedures, in which there are no restrictions imposed on distribution of constituents within the periodicity cell. The second group consists of methods developed independently for special types of composite materials, namely for laminated composites, fibrous composites and for solids with inclusions and cavities of various shapes (i.e., for particulate composites). Thus, we can deal with *general* and *special macro-modelling procedures*. Obviously, the general macro-models have the practical meaning if they can be applied to the analysis of special types of composites. In this contribution the main attention is concentrated on the macro-modelling of nonstationary processes in periodic composites, where the effect of the length dimensions of the periodicity cell on the dynamic behaviour of a body plays an important role. That is why we discern below *the length-scale macro-models* and *the local macro-models*, in which this effect is neglected.

The main efforts in constructing the macro-models in dynamics of composites were posed on the special macro-modelling procedures. The list of references to this subject is very extensive. We can mention here the effective stiffness theories for periodically laminated elastic composites, cf Sun, Achenbach and Hermann (1968), Achenbach and Hermann (1968), Grot and Achenbach (1970), the investigations into dynamics of fibre-reinforced composites, cf Aboudi (1981), Tolf (1983), and those related to media with voids, cf Nunziato and Covin (1979). In the framework of this paper we shall restrict our considerations to the general models; for the discussion of special models in the framework of elastodynamics, cf the forthcoming review paper by Baczyński (1995).

Among the general macro-modelling procedures we can mention those based on the asymptotic homogenization approach, cf Bensoussan, Lions and Papanicolaou (1987), Sanchez-Palencia and Zaoui (1985), and the extensive list of references therein. The resulting macro-models are described by equations involving constant coefficients (called the effective modulae) and time dependent functions (for nonstationary processes). These mathematical objects have to be determined independently for every periodic structure by obtaining solutions to certain variational problems posed on the periodicity

cell as well as certain initial value problems for materials with a memory (e.g. for visco-elastic materials). Hence, the formulation of macro-models by the asymptotic homogenization methods for any specific composite materials requires rather lengthy numerical computations. For this reason the asymptotic approach, as a rule, is restricted only to the first approximation. Within this approximation we deal with the local macro-models, in which the effect of the size of the periodicity cell on the behaviour of the body is neglected. To describe length-scale phenomena (i.e. to formulate the length-scale macro-models) by the asymptotic homogenization approach, the higher steps in the formal asymptotic procedure have to be considered. This line of procedure is not accepted by most of researchers interested mainly in engineering applications of the resulting theories, due to considerable difficulties on the stage of formulation of governing equations of macromechanics for a selected composite body. Free from this drawback are general macro-modelling methods, based on theories of material continua with microstructure suggested by Mindlin (1964), Eringen and Suhubi (1964) and others; for the thermomechanics of microstructural materials cf also Woźniak (1967), (1968). Models of this kind are called the microstructural models and belong to the length-scale macro-models. They can be formulated without any reference to the boundary-value problems on the periodicity cell. The pertinent modelling procedures are specified by certain *a priori* assumptions related to the expected class of micro-deformations and a certain smoothing operations. For the elastic materials they lead to systems of the second-order partial differential equations for three fields representing macro-kinematics and for extra unknown fields describing the micro-kinematic behaviour of a composite. The existence of many unknown independent kinematic variables results in serious difficulties related to a complicated form of boundary-value problems. Moreover, we can also deal with essential discrepancies between the number of boundary conditions required by the mathematical structure of the theory and the number of these conditions describing the boundary interactions for composite materials from the viewpoint of engineering applications of the theory. For example, on the boundary interfaces of laminated materials only three displacement conditions have the physical sense. The microstructural models were successfully applied to the investigations of the wave propagation in unbounded media. A certain alternative of microstructural models constitute macro-models based on the mixture and interacting continuum theories, developed by Green and Naghdi (1965), (1966) and (1967), Green and Steel (1966), Steel (1967) and (1968), Bedford and Stern (1971) and (1972), Hegemier (1972), Tiersten and Jahanmir (1977) and others. We deal here with the length-scale macro-models which are often oriented towards the investigations of selected dynamic problems.

In investigations of nonstationary processes for composites, we are often interested only in certain aspects of the micro-dynamic behaviour of periodic solids. In these cases we deal with certain special space distributions of expected micro-deformations caused by the heterogeneity of the medium. We are also interested in the length-scale macro-models which are physically reasonable and simple enough to be applied to an analysis of engineering problems. The aim of this study is a general description of the macro-modelling procedure satisfying these requirements. The macro-models of this kind were recently applied to selected dynamic problems in a series of papers by Woźniak (1993) and (1994), Woźniak et al.(1993) and (1994), Wierzbicki (1994), Mazur-Śniady (1993), Baron and Woźniak (1994), Michalak and Woźniak (1994), Wągrowska and Woźniak (1994), Konieczny and Woźniak (1995), Jędrysiak and Woźniak (1995), Cielecka (1995), Matysiak and Nagórko (1995). In the above references mainly linear elastic and linear viscoelastic composite materials and structures were analyzed. In this paper the pertinent modelling procedure will be formulated within a framework of the nonlinear continuum mechanics and for an arbitrary simple material. Moreover, the analysis will be carried out in the general abstract form without any exact physical specification of the fields and processes under consideration.

The results of the macro-modelling procedure previously applied in the aforementioned papers, were referred to as *the refined macrodynamics* of periodic materials and structures. The term *refined* is related to the fact that the obtained equations describe in the explicit form the effect of the microstructure length parameter (i.e. the maximum characteristic length dimension of the periodicity cell) on the dynamic behaviour of the body. In order to evaluate this effect there were introduced also local models, obtained from the refined macrodynamics by scaling the microstructure down. These local models coincide with the macro-models analyzed within the framework of theories with microlocal parameters, developed by Woźniak (1987), Matysiak and Woźniak (1987), Jakubowska and Matysiak (1987), Wągrowska (1987), Naniewicz (1987), Lewiński (1987), Kaczyński and Matysiak (1988), Matysiak and Nagórko (1989), Woźniak (1991), Matysiak (1992), Kaczyński (1993) and others.

The refined macromechanics has some advantages compared to the general macro-models outlined above. First, it is able to describe nonlinear problems related to the finite deformations of an arbitrary simple material. Secondly, the formulation of governing equations of the pertinent refined macro-models does not require any solutions to the boundary-value problem on the basic cell as well as solutions to the initial-value problems (if we deal with viscoelastic materials). Thirdly, the extra unknown fields in the refined macromechanics

(i.e., the fields which do not have their counterparts in solid mechanics of heterogeneous media) are governed by relations involving exclusively time derivatives and/or time dependent functionals. It follows that the extra unknowns do not enter the boundary conditions and play a role of certain dynamical internal variables. This fact is very essential in the applications of the theory, since the boundary-value problems of the refined macrodynamics are similar to those met in solid mechanics. At last, using the refined models we are able to evaluate *a posteriori* the relation between the size of the periodicity cell and the accuracy of the obtained solution. The refined macro-models can be formulated on the various levels of accuracy and hence are useful for modelling procedures of the adaptive type. The main drawback of the refined macro-models is the restriction of microdynamics to the postulated *a priori* class of micro-motions.

Denotations. Index α runs over $1,2,3$ and is related to the materials coordinates. We shall also introduce a non-tensorial superscript A which runs over $1,...,N$. Summation convention holds for both kinds of the aforementioned indices.

2. Preliminaries

Let Ω_R be a regular region in the physical 3-space occupied by the solid body (or its part) in the reference position. The points of Ω_R will be referred to as the material points and denoted by $X = (X^1, X^2, X^3)$ where X^α, $\alpha = 1,2,3$ are the material coordinates. By $\rho_R(X)$ we denote the mass density at X, related to the reference position. The body is subjected to a certain process understood as the sufficiently regular mapping $\Omega_R \times R \ni (X,t) \to q(X,t) \in R^m$, where every $q(\cdot,t)$ is said to be the configuration of the body at the time t (we often restrict the domain of t to a certain time interval (t_0, t_f)). Setting $m = 4$ we can interpret $q^i(X,t)$, $i = 1,2,3$, as the displacement components of the material point at time t from the reference configuration, and $q^4(X,t)$ as the values of the temperature field; in this case $q(\cdot,t)$ is the deformation-temperature configuration of the body. In an arbitrary process the body is interacting with external fields; the density of these fields (per unit mass of the body) at time t and for $X \in \Omega_R$ will be denoted by $e(X,t)$. It is assumed that $e(X,t) \in R^m$ and $e(\cdot,t)$ is referred to as the supply field (at time t). For $m = 4$ the values $e_i(X,t)$, $i = 1,2,3$, can be interpreted as the body forces and $e_4(X,t)$ stands for a density of the internal heat sources. The interactions inside the body at every

t are assumed to be uniquely described by the flux field $\boldsymbol{s}_R^\alpha(\cdot,t)$; the intensity of these interactions is given by $\boldsymbol{s}_R \equiv \boldsymbol{s}_R^\alpha(X,t)n_{R\alpha}(X)$, where $n_{R\alpha}(X)$ is a unit normal determining the oriented area element at the material point X, $X \in \Omega_R$. In the case $m = 4$ we can interpret $s_R^{i\alpha}(X,t)$, $i = 1,2,3$, as the components of the first Piola-Kirchhoff stress tensor and $s_R^{4\alpha}(X,t)$ as the components of the heat flux vector (related to the reference position of the body). The boundary interactions are given by $t_R(X,t) = \boldsymbol{s}_R^\alpha(X,t)n_{R\alpha}(X)$, $X \in \partial\Omega_R$, where $n_{R\alpha}(X)$ is the unit outward normal to $\partial\Omega_R$ at X.

Let Δ_R be an arbitrary regular subregion of Ω_R. Define by $da_R = da_R(X)$ the element of $\partial\Delta_R$ at $X \in \partial\Delta_R$ and let $dv_R(X) \equiv dX^1dX^2dX^3$. For the time being let us assume that the components of the fields $\boldsymbol{q}(\cdot,t)$ are independent for the class of processes under consideration. In this case the flux and supply fields, for every instant t, are interrelated by the general balance equations

$$\frac{1}{2}\int_{\Delta_R} \boldsymbol{k}_R\, dv_R = \oint_{\partial\Delta_R} \boldsymbol{s}_R^\alpha n_{R\alpha}\, da_R + \int_{\Delta_R} \boldsymbol{e}\rho_R\, dv_R \tag{2.1}$$

which hold for every Δ_R and where the values of the balanced quantity $\boldsymbol{k}_R$, $\boldsymbol{k}_R(X,t) \in R^m$, are interrelated with the processes by the relation

$$\boldsymbol{k}_R(X,t) = \hat{\boldsymbol{k}}_R\Big(X, \dot{\boldsymbol{q}}(X,t), \boldsymbol{q}(X,t), \nabla\boldsymbol{q}(X,t)\Big) \tag{2.2}$$

In Eq (2.2) $\hat{\boldsymbol{k}}_R(\cdot)$ is the known function defining the balanced quantity. For $m = 4$ we can assume that $k_R^i(X,t) = \rho_R(X)\dot{q}^i(X,t)$, $\dot{q}^i(X,t)$ being the velocity of the material point X, $i = 1,2,3$, and $k_R^4(X,t) = 0.5\rho_R(X)\dot{q}^i(X,t)\dot{q}^i(X,t) + \rho_R(X)\varepsilon(X,\nabla q^i,q^4)$, where $\varepsilon(X,\cdot)$ is, for every $X \in \Omega_R$, the strain energy function. The material properties of the body are assumed to be determined by the constitutive relation for a simple material

$$\boldsymbol{s}_R^\alpha(X,t) = \hat{\boldsymbol{s}}_R^\alpha\Big(\boldsymbol{X}, \nabla\boldsymbol{q}(X,t-\tau), \boldsymbol{q}(X,t-\tau)\Big) \qquad \tau \geq 0 \tag{2.3}$$

where $\hat{\boldsymbol{s}}_R^\alpha(X,\cdot)$ is the response functional. For elastic materials, under the aforementioned interpretation of $\boldsymbol{q}(X,t)$, we have $s_R^{i\alpha}(X,t) = \hat{s}_R^{i\alpha}\Big(X, \nabla q^i(X,t), q^4(X,t)\Big)$, where $\hat{s}_R^{i\alpha}(\cdot)$ are the known functions.

From the condition (2.1), which is assumed to hold for every Δ_R, we obtain the local form of the balance equation

$$\frac{d}{dt}\boldsymbol{k}_R(X,t) = \boldsymbol{s}_{R,\alpha}^\alpha(X,t) + \boldsymbol{e}(X,t)\rho_R(X) \tag{2.4}$$

which holds for almost every $\boldsymbol{X} \in \Omega_R$. Let us multiply the above equation by an arbitrary regular vector function $\delta\boldsymbol{q}(\cdot)$ defined on Ω_R, such that $\delta\boldsymbol{q}(\boldsymbol{X}) \in \boldsymbol{R}^m$; the scalar product in $\boldsymbol{R}^m$ will be denoted by a dot. Integrating the obtained result over Ω_R, introducing the boundary interactions $\boldsymbol{t}_R(\boldsymbol{X}, t)$, $\boldsymbol{X} \in \partial\Omega_R$ and bearing in mind Eq (2.2), we arrive at the weak form of the general balance equation, represented by the condition

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_R} \hat{\boldsymbol{k}}_R(\dot{\boldsymbol{q}}, \boldsymbol{q}, \nabla\boldsymbol{q}) \cdot \delta\boldsymbol{q}\, dv_R = \oint_{\partial\Omega_R} \boldsymbol{t}_R \cdot \delta\boldsymbol{q}\, da_R + \\ &- \int_{\Omega_R} \boldsymbol{s}_R^{\alpha} \cdot \delta\boldsymbol{q}_{,\alpha}\, dv_R + \int_{\Omega_R} \boldsymbol{e} \cdot \delta\boldsymbol{q}\rho_R\, dv_R \end{aligned} \tag{2.5}$$

On the assumption that the components of fields $\boldsymbol{q}(\cdot, t)$ are independent, Eq (2.4) is assumed to hold for an arbitrary regular trial vector function $\delta\boldsymbol{q}(\cdot)$. However, if components of fields $\boldsymbol{q}(\cdot, t)$ are constrained by certain given *a priori* conditions, then instead of the general balance equation in the form (2.4) (or in the equivalent form of condition (2.1) which holds for every Δ_R) we have to postulate the balance equation in the form (2.5), which is satisfied for trial functions $\delta\boldsymbol{q}(\cdot)$ restricted by conditions similar to those constraining the components of the fields $\boldsymbol{q}(\cdot, t)$.

Let $V_R \equiv (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ be the region in the physical 3-space. Setting $l \equiv \sqrt{(l_1)^2 + (l_2)^2 + (l_3)^2}$ and denoting by L the smallest characteristic length dimension of Ω_R, we shall assume that the ratio $\lambda \equiv l/L$, from a numerical viewpoint, can be treated as negligible small compared to 1. In the sequel we restrict considerations to heterogeneous solids for which there are known: the reference position (and hence the region Ω_R) and the volume element V_R, such that $\rho_R(\cdot)$, $\hat{\boldsymbol{k}}_R(\cdot, \dot{\boldsymbol{q}}, \boldsymbol{q}, \nabla\boldsymbol{q})$ and $\hat{\boldsymbol{s}}_R^{\alpha}(\cdot, \nabla\boldsymbol{q}, \boldsymbol{q})$ are V_R-periodic functions. Bearing in mind that $\lambda \ll 1$, we shall refer these solids to as the micro-periodic composites and l as the microstructure length parameter. In this case Eqs (2.1)÷(2.5) describe the micromechanics of micro-periodic composites made of simple materials. The right-hand sides of Eqs (2.1), (2.2) as well as the mass density $\rho_R(\cdot)$, met in engineering problems, are highly-oscillating piecewise constant (constant for every material constituent) V_R-periodic functions of material coordinates $\boldsymbol{X}^{\alpha}$. Hence, the micromechanics does not constitute the basis for the computational analysis of special problems, and has to be replaced by a certain approximate model of the micro-periodic composite. Various examples of these models were discussed in Introduction.

3. Modelling concepts

The macro-modelling procedure, leading to the general form of the refined models of composites, will be based on certain modelling assumptions. In order to formulate these assumptions we shall introduce in this Section two modelling concepts. From now on we shall tacitly assume that Ω_R and V_R (and hence also l) are known.

Let $h^A(\cdot)$, $A = 1,...,N$, $N \geq 1$, be a system of continuous linear independent V_R-periodic functions (and hence depending on the microstructure length parameter l) having piecewise continuous first order derivatives and satisfying conditions

(i) $<h^A>= 0$

(ii) $h^A(Z) \in \mathcal{O}(l)$

(iii) $h^A{}_{,\alpha}(Z) \in \mathcal{O}(1)$

for $A = 1,...,N$ and for almost every Z; by $\mathcal{O}(1)$ we have denoted a set of functions (depending on the parameter l) the maximum value of which remains constant with $l \to 0$. Every N-tuple $(h^1(\cdot),...,h^N(\cdot))$ will be called *the micro-shape function system.* As the example of this system we can take N-tuple of real or imaginary parts of functions $h^A(X) = l\exp[ig_\alpha(A)X^\alpha]$, $A = 1,...,N$, where $g_\alpha(A) = 2\pi n_\alpha(A)/(\mu_\alpha l)$ (no summation over α); μ_α are arbitrary constants from the interval $(0,1)$ and $n_\alpha(A)$, $\alpha = 1,2,3$, is the triple of positive integers assigned to the index A. It has to be remembered that the microstructure length parameter l (maximum characteristic length dimension of V_R) is known for every periodic composite material; hence, setting $l \to 0$ we deal with a certain class of these materials in which the microstructure is scaled down.

Let $F(\cdot)$ be a real valued function defined on Ω_R; in the sequel we tacitly assume that $F(\cdot)$ can also depend on time coordinate t. Let us introduce the positive real ε_F as computation accuracy of the values $F(X)$ at every $X \in \Omega_R$. $F(\cdot)$ will be called *the macro-function* (related to the computation accuracy ε_F and to the microstructure length parameter l) if for every $X, Y \in \Omega_R$, the condition $\|X - Y\| < l$ implies $|F(X) - F(Y)| < \varepsilon_F$. If $F(\cdot)$ is sufficiently regular and conditions of the above form hold also for all derivatives of $F(\cdot)$ (as well as for all time derivatives) with the pertinent accuracy parameters $\varepsilon_{\nabla F}, \varepsilon_{\dot{F}}, ...$, then $F(\cdot)$ will be called *the regular macro-function.* The choice of parameters $\varepsilon_F, \varepsilon_{\nabla F}, \varepsilon_{\dot{F}}$ depends on the accuracy of calculations. Roughly speaking, oscillations of regular macro-functions and all

their derivatives within an arbitrary but fixed cell $V_R(\boldsymbol{X})$. $V_R(\boldsymbol{X}) \equiv \boldsymbol{X} + V_R$, $V_R(\boldsymbol{X}) \subset \Omega_R$, from the computational viewpoint can be neglected. Every class of macro-functions under consideration has to include functions which are constant on Ω_R.

Let $f(\cdot, \alpha)$ for every $\alpha \in \boldsymbol{R}$ be an integrable V_R-periodic function of $\boldsymbol{Z} = (Z^1, Z^2, Z^3)$ and $f(Z, \cdot)$ for every Z be a continuous function. Moreover, let $F(\cdot)$ stands for a continuous macro-function defined on Ω_R. Let us approximate the region Ω_R by a sum $\bigcup V_R(\boldsymbol{X})$, $\boldsymbol{X} \in \Lambda$, of mutually disjointed cells $V_R(\boldsymbol{X})$, where Λ is a lattice of points on Ω_R such that $V_R(\boldsymbol{X}) \subset \Omega_R$ for every $\boldsymbol{X} \in \Lambda$. On these assumptions we obtain the formulae

$$\begin{aligned}
&\int_{\Omega_R} f\Big(\boldsymbol{X}, F(\boldsymbol{X})\Big)\, dv_R(\boldsymbol{X}) = \sum_{X \in \Lambda} \int_{V_R(X)} f\Big(\boldsymbol{Z}, F(Z)\Big)\, dv_R(Z) + \mathcal{O}(\lambda) \\
&\sum_{X \in \Lambda} \int_{V_R(X)} f\Big(Z, F(Z)\Big)\, dv_R(Z) = \int_{\Omega_R} <f>(\boldsymbol{X})\, dv_R(\boldsymbol{X}) + \mathcal{O}(\lambda) + \mathcal{O}(\varepsilon_F) \\
&\int_{V_R(X)} f\Big(Z, F(Z)\Big)\, dv_R(Z) = \int_{V_R(X)} f\Big(Z, F(\boldsymbol{X})\Big)\, dv_R(Z) + \mathcal{O}(\varepsilon_F)
\end{aligned} \tag{3.1}$$

where we have denoted

$$<f>(\boldsymbol{X}) = \frac{1}{l_1 l_2 l_3} \int_{V_R(X)} f\Big(Z, F(\boldsymbol{X})\Big)\, dv_R(Z) \tag{3.2}$$

It has to be remembered that all manipulations carried out inside the brackets $<\cdot>$ (such as the differentiation $<f_{,\alpha}>(\boldsymbol{X})$) hold for constant $\boldsymbol{X} \in \Omega_R$. Moreover

$$<f>(\boldsymbol{X}) = <f>(\boldsymbol{Z}) + \mathcal{O}(\varepsilon_F) \qquad \forall\, Z \in V_R(\boldsymbol{X}) \tag{3.3}$$

If f is independent of the second argument then

$$<f> = \frac{1}{l_1 l_2 l_3} \int_{V_R} f(Z)\, dv_R(Z)$$

is the averaged (constant) value of a V_R-periodic function $f(\cdot)$. The formulae of the forms (3.1)÷(3.3) also hold if $f = f\Big(\boldsymbol{X}, \boldsymbol{F}(\boldsymbol{X})\Big)$, where $\boldsymbol{F}(\boldsymbol{X}) = \Big(F^1(\boldsymbol{X}), ..., F^n(\boldsymbol{X})\Big)$ and $F^1(\cdot), ..., F^n(\cdot)$ are continuous macro-functions defined on Ω_R (related to accuracy parameters $\varepsilon_{F^1}, ..., \varepsilon_{F^n}$, respectively); in this case $\mathcal{O}(\varepsilon_F) = \mathcal{O}(\varepsilon_{F^1}) + ... + \mathcal{O}(\varepsilon_{F^n})$. Moreover, if $F(\cdot)$ and

$\nabla F(\cdot)$ are macro-functions and $h(\cdot)$ is an arbitrary micro-shape function, then

$$\nabla\Big[h(X)F(X)\Big] = F(X)\nabla h(X) + \mathcal{O}(\varepsilon_F) \tag{3.4}$$

since $h(X)\nabla F(X) \in \mathcal{O}(\varepsilon_F)$. In the sequel reals $\lambda \equiv l/L, \varepsilon_F, \varepsilon_{\nabla F}, ...,$ where $F(\cdot)$ is an arbitrary regular macro-function, will be treated as certain small parameters.

4. Modelling hypotheses

Using the concept of regular macro-function and that of the micro-shape function system, we shall introduce three macro-modelling hypotheses. These hypotheses are formulated on the extra assumption that to the set of all admissible configurations $\boldsymbol{q}(\cdot,t)$ (at a certain time t) belong the constant mappings: $\boldsymbol{q}(X,t) = \text{const}$ for every $X \in \Omega_R$.

- *Macro Localization Hypothesis* (MLH). *The configurations* $\boldsymbol{q}(\cdot,t)$ *in every class of processes under consideration can be restricted by the condition*

$$\boldsymbol{q}(X,t) = \boldsymbol{Q}(X,t) + h^A(X)\boldsymbol{W}^A(X,t) \qquad X \in \Omega_R \tag{4.1}$$

where components of $\boldsymbol{Q}(\cdot,t)$, $\boldsymbol{W}^A(\cdot,t)$ *are arbitrary independent regular macro-functions and* $h^A(\cdot)$, $A = 1,...,N$, *is the postulated a priori micro-shape function system.*

The term *macro-localization* in the above hypothesis has to be understood as the representation of an arbitrary configuration $\boldsymbol{q}(\cdot,t)$ of the body by a system of $m(N+1)$ independent macro-functions (components of $\boldsymbol{Q}(\cdot,t)$ and $\boldsymbol{W}^A(\cdot,t)$). The choice of micro-shape functions $h^A(\cdot)$ from the qualitative point of view determines the space distribution of micro-disturbances $h^A\boldsymbol{W}^A$, which can be expected during every process under consideration. Eq (4.1) describes the superimposition of these disturbances on macro-configurations $\boldsymbol{Q}(\cdot,t)$ of the body. They can be caused either by a periodic heterogeneous material structure of the composite or by a certain initial configuration, which can be postulated in the form (4.1) for $t = t_0$. Hence, the fields $\boldsymbol{W}^A(\cdot,t)$ describe, from the quantitative point of view, distribution of disturbances in Ω_R. The fields $\boldsymbol{Q}(\cdot,t)$ and $\boldsymbol{W}^A(\cdot,t)$ represent the new basic unknown fields which will be referred to as the macro-configurations and macro-variables,

respectively. Let us also observe that by increasing the number N of micro-shape functions (and hence, also the number of macro-variables) we can obtain more detailed descriptions of the investigated processes. It will be shown that specifications of micro-shape functions in Eq (4.1) lead to certain macro-models of the periodic composite under consideration.

Under the MLH configurations $\boldsymbol{q}(\cdot, t)$ of the body are not independent; hence, in the forthcoming analysis we have to use the balance equation in the weak form (2.5) (cf the remarks at the end of Section 2). This fact will be noted by the second modelling hypothesis.

- *Macro-Balance Assumption* (MBA). *The balance equation (2.5) is assumed to hold for*

$$\delta \boldsymbol{q}(X) = \delta \boldsymbol{Q}(X) + h^A(X)\delta \boldsymbol{W}^A(X) \qquad X \in \Omega_R \qquad (4.2)$$

where components of $\delta \boldsymbol{Q}(\cdot)$ and $\delta \boldsymbol{W}^A(\cdot)$ are arbitrary linear independent regular macro-functions defined on Ω_R.

The form of MBA is implied by that of MLH. Components of $\delta \boldsymbol{Q}(\cdot)$, $\delta \boldsymbol{W}^A(\cdot)$ are related (as macro-functions) to accuracy parameters ε_Q, ε_W of components of $\boldsymbol{Q}(\cdot)$, $\boldsymbol{W}^A(\cdot)$, respectively.

After substituting the right-hand sides of Eqs (4.2) into the formula (2.5) we obtain the condition in which components of $\delta \boldsymbol{Q}(\cdot)$, $\delta \boldsymbol{W}^A(\cdot)$ are linear independent trial macro-functions.

The modelling hypothesis that follows is strictly related to the concept of a macro-function and to the fact that the length dimensions of the periodicity cell V_R are negligible small compared to the minimum characteristic length dimension of the region Ω_R. For the sake of simplicity we assume, that the components of supply fields $\boldsymbol{e}(\cdot, t)$ are macro-functions. In the sequel we use the denotation $\boldsymbol{W} \equiv (\boldsymbol{W}^1, ..., \boldsymbol{W}^N)$. Moreover, $\varepsilon_Q, \varepsilon_W, \varepsilon_{\nabla Q}, ...$, stand for calculational accuracy parameters related to the components of $\boldsymbol{Q}(\cdot, t), \boldsymbol{W}(\cdot, t), \nabla \boldsymbol{Q}(\cdot, t), ...$, respectively.

- *Macro-Modelling Approximation* (MMA). *In the course of the macro-modelling procedure there will be neglected:*

  (i) Terms $\mathcal{O}(\lambda)$, $\mathcal{O}(\varepsilon_F)$ in formulae of the form (3.1), where F runs over $\boldsymbol{Q}$, $\nabla \boldsymbol{Q}$, $\dot{\boldsymbol{Q}}$, $\boldsymbol{W}$, $\nabla \boldsymbol{W}$, $\dot{\boldsymbol{W}}$

(ii) *Terms* $\mathcal{O}(\varepsilon_F)$ *in formulae of the form (3.3), where* F *runs over* $\boldsymbol{Q}, \nabla\boldsymbol{Q}, \dot{\boldsymbol{Q}}, \boldsymbol{W}, \nabla\boldsymbol{W}, \dot{\boldsymbol{W}}$

(iii) *Terms* $\mathcal{O}(\varepsilon_F)$ *in formulae of the form (3.4), where* F *runs over* $\boldsymbol{W}, \nabla\boldsymbol{W}$.

If $\varepsilon_F \to 0$ then the macro-function $F(\cdot)$ tends to a constant mapping defined on Ω_R.

5. Modelling procedure

The modelling procedure, leading from Eqs (2.2) $\div$ (2.4) of micromechanics to the refined macro-model of a composite body, will be based on Eqs (2.2), (2.3) and (2.5) combined with the modelling hypotheses MLII, MBA, MMA. Substituting the right-hand sides of Eqs (4.1) into Eqs (2.2), (2.3) and denoting

$$\tilde{\boldsymbol{k}}_R(Z,\dot{\boldsymbol{Q}},\dot{\boldsymbol{W}},\boldsymbol{Q},\boldsymbol{W}) \equiv \hat{\boldsymbol{k}}_R\Big(Z,\dot{\boldsymbol{Q}} + h^A(Z)\dot{\boldsymbol{W}}^A,\boldsymbol{Q} + h^A(Z)\boldsymbol{W}^A\Big)$$

$$\tilde{\boldsymbol{s}}_R^\alpha(Z,\nabla\boldsymbol{Q},\boldsymbol{Q},\boldsymbol{W}) \equiv \hat{\boldsymbol{s}}_R^\alpha\Big(Z,\nabla\boldsymbol{Q} + \nabla h^A(Z)\boldsymbol{W}^A,\boldsymbol{Q} + h^A(Z)\boldsymbol{W}^A\Big)$$

we obtain for $\boldsymbol{Z} = X$

$$\begin{aligned}
\boldsymbol{k}_R(X,t) &= \tilde{\boldsymbol{k}}_R\Big(Z,\dot{\boldsymbol{Q}}(X,t),\boldsymbol{Q}(X,t),\dot{\boldsymbol{W}}(X,t),\boldsymbol{W}(X,t)\Big) \\
\boldsymbol{s}_R^\alpha(X,t) &= \tilde{\boldsymbol{s}}_R^\alpha\Big(Z,\nabla\boldsymbol{Q}(X,t-\tau),\boldsymbol{Q}(X,t-\tau),\boldsymbol{W}(X,t-\tau)\Big) + \qquad (5.1) \\
&+ \mathcal{O}(\varepsilon_W) + \mathcal{O}(\varepsilon_{\nabla W}) \qquad\qquad \tau \geq 0
\end{aligned}$$

In Eq $(5.1)_2$ we have taken into account the formula (3.4), setting $(h^A\boldsymbol{W}^A)_{,\alpha} = h^A{}_{,\alpha}\boldsymbol{W}^A + \mathcal{O}(\varepsilon_W) + l\mathcal{O}(\varepsilon_{\nabla W})$. On the right-hand side of Eqs (5.1) both $\tilde{\boldsymbol{k}}_R$ and $\tilde{\boldsymbol{s}}_R^\alpha$ are V_R-periodic functions of the first argument; components of all remaining arguments are macro-functions related to the computation accuracy parameters ε_Q, $\varepsilon_{\dot{Q}}$, ε_W, $\varepsilon_{\nabla Q}$. Bearing this in mind and using Eqs (3.1), (3.4), (4.1), (4.2), we arrive at the following formulae for the integrals over Ω_R in the balance equation (2.5)

$$\int_{\Omega_R} \boldsymbol{k}_R \cdot \delta \boldsymbol{q}\, dv_R = \int_{\Omega_R} \left(<\boldsymbol{k}_R> \cdot \delta \boldsymbol{Q} + <\tilde{\boldsymbol{k}}_R h^A> \cdot \delta \boldsymbol{W}^A \right) dv_R +$$
$$+ \mathcal{O}(\lambda) + \mathcal{O}(\varepsilon_Q) + \mathcal{O}(\varepsilon_W) + \mathcal{O}(\varepsilon_{\dot{Q}}) + \mathcal{O}(\varepsilon_{\dot{W}})$$
$$\int_{\Omega_R} \boldsymbol{s}_R^\alpha \cdot \delta \boldsymbol{q}_{,\alpha}\, dv_R = \int_{\Omega_R} \left(<\tilde{\boldsymbol{s}}_R^\alpha> \cdot \delta \boldsymbol{Q}_{,\alpha} + <\tilde{\boldsymbol{s}}_R^\alpha h^A{}_{,\alpha}> \cdot \delta \boldsymbol{W}^A \right) dv_R + \tag{5.2}$$
$$+ \mathcal{O}(\lambda) + \mathcal{O}(\varepsilon_{\nabla Q}) + \mathcal{O}(\varepsilon_Q) + \mathcal{O}(\varepsilon_{\nabla W}) + \mathcal{O}(\varepsilon_W)$$
$$\int_{\Omega_R} \boldsymbol{e} \cdot \delta \boldsymbol{q} \rho_R\, dv_R = \int_{\Omega_R} \left(<\rho_R> \boldsymbol{e} \cdot \delta \boldsymbol{Q} + <\rho_R h^A> \boldsymbol{e} \cdot \delta \boldsymbol{W}^A \right) dv_R +$$
$$+ \mathcal{O}(\lambda) + \mathcal{O}(\varepsilon_Q) + \mathcal{O}(\varepsilon_W)$$

where the averages $< \cdot >$ are taken over the argument $\boldsymbol{Z}$ of V_R-periodic functions on the right-hand sides of Eqs (5.1) as well as functions $h^A(\boldsymbol{Z})$, $\rho_R(\boldsymbol{Z})$. Let us introduce new fields $\boldsymbol{S}_R^\alpha(\cdot, t)$, $\boldsymbol{H}_R^A(\cdot, t)$, defined on Ω_R for every t by means of the equations

$$\boldsymbol{S}_R^\alpha(\boldsymbol{X}, t) = <\tilde{\boldsymbol{s}}_R^\alpha> \left(\nabla \boldsymbol{Q}(\boldsymbol{X}, t-\tau), \boldsymbol{Q}(\boldsymbol{X}, t-\tau), \boldsymbol{W}(\boldsymbol{X}, t-\tau) \right)$$
$$\boldsymbol{H}_R^A(\boldsymbol{X}, t) = <\tilde{\boldsymbol{s}}_R^\alpha h^A{}_{,\alpha}> \left(\boldsymbol{W}(\boldsymbol{X}, t-\tau), \nabla \boldsymbol{Q}(\boldsymbol{X}, t-\tau), \boldsymbol{Q}(\boldsymbol{X}, t-\tau) \right) \tag{5.3}$$

Let us observe that the fields $\boldsymbol{S}_R^\alpha(\cdot, t)$, $\boldsymbol{H}_R^A(\cdot, t)$ satisfy condition of the form (3.3) (provided that the response functionals $\tilde{\boldsymbol{s}}_R^\alpha(\boldsymbol{X}, \cdot)$ in Eq (3.3) are sufficiently regular) i.e., they are not oscillating within cells $V_R(\boldsymbol{X}) \cap \Omega_R$, $\boldsymbol{X} \in \Omega_R$. Taking into account Eqs (5.2), (5.3) and applying MMA, from Eq (2.5) we obtain the following weak form of the general balance equation

$$\frac{d}{dt} \int_{\Omega_R} \left(<\tilde{\boldsymbol{k}}_R> \cdot \delta \boldsymbol{Q} + <\tilde{\boldsymbol{k}}_R h^A> \cdot \delta \boldsymbol{W}^A \right) dv_R = \oint_{\partial \Omega_R} \boldsymbol{t}_R \cdot \delta \boldsymbol{q}\, da_R +$$
$$- \int_{\Omega_R} \left(\boldsymbol{S}_R^\alpha \cdot \delta \boldsymbol{Q}_{,\alpha} + \boldsymbol{H}_R^A \cdot \delta \boldsymbol{W}^A \right) dv_R + \tag{5.4}$$
$$+ \int_{\Omega_R} \left(<\rho_R> \boldsymbol{e} \cdot \delta \boldsymbol{Q} + <\rho_R h^A> \boldsymbol{e} \cdot \delta \boldsymbol{W}^A \right) dv_R$$

which has to hold for fields $\delta \boldsymbol{Q}$, $\delta \boldsymbol{W}^A$ defined on Ω_R, components of which are arbitrary independent regular macro-functions. From the above condition

we derive the following local form of the balance equation (restriction of $\delta \boldsymbol{Q}$, $\delta \boldsymbol{W}^A$ to m-tuples of macro-functions is irrelevant here)

$$\begin{aligned} &\frac{d}{dt} <\tilde{\boldsymbol{k}}_R> (\dot{\boldsymbol{Q}}, \dot{\boldsymbol{W}}, \boldsymbol{Q}, \boldsymbol{W}) - \boldsymbol{S}^{\alpha}_{R,\alpha} = <\rho_R> \boldsymbol{e} \\ &\frac{d}{dt} <\tilde{\boldsymbol{k}}_R h^A> (\dot{\boldsymbol{Q}}, \dot{\boldsymbol{W}}, \boldsymbol{Q}, \boldsymbol{W}) + \boldsymbol{H}^A_R = <\rho_R h^A> \boldsymbol{e} \end{aligned} \tag{5.5}$$

which holds at every material point and every instant in the time interval we investigate the process under consideration. This process is now described by the mappings

$$\begin{aligned} &\Omega_R \times \boldsymbol{R} \ni (\boldsymbol{X}, t) \to \boldsymbol{Q}(\boldsymbol{X}, t) \in \boldsymbol{R}^m \\ &\Omega_R \times \boldsymbol{R} \ni (\boldsymbol{X}, t) \to \boldsymbol{W}^A(\boldsymbol{X}, t) \in \boldsymbol{R}^m \qquad A = 1, ..., N \end{aligned} \tag{5.6}$$

where the fields $\boldsymbol{Q}(\cdot, t)$ represent the macro-configurations of the body and the internal variables $\boldsymbol{W}^A(\cdot, t)$ describe the spatial distributions of disturbances in these configurations, related to the expected form of processes postulated by the MLH.

Eqs (5.3) and (5.5) constitute the final result of the modelling procedure and represent a certain macro-model of the periodic composites under consideration. This statement is implied by the fact that all functions in the aforementioned equations are either macro-functions, like the components of $\boldsymbol{Q}(\cdot, t)$, $\boldsymbol{W}^A(\cdot, t)$, $\dot{\boldsymbol{Q}}(\cdot, t)$, $\dot{\boldsymbol{W}}^A(\cdot, t)$, or are not oscillating within an arbitrary cell $V_R(\boldsymbol{X})$ in Ω_R, like the components of $\boldsymbol{S}^{\alpha}_R(\cdot, t)$, $\boldsymbol{H}^A_R(\cdot, t)$ and $\boldsymbol{e}(\cdot, t)$. That is why Eqs (5.5) are called the macro-balance equations and Eqs (5.3) are referred to as the macro-constitutive equations. It has to be emphasized, that Eqs (5.3) and (5.5) describe the whole class of macro-models. Every special model belonging to this class is uniquely specified by the choice of micro-shape functions. The discussion of the above obtained macro-models will be carried out in Section 7; in the subsequent section we shall pass to a certain approximation of these models.

6. Passage to the local models

Eqs (5.3) and (5.5) represent a certain length-scale macro-model of composites, since terms $< \tilde{\boldsymbol{k}}_R >$, $< \tilde{\boldsymbol{k}}_R h^A >$ and $< \rho_R h^A >$ depend on the

microstructure length parameter l, $h^A(X) \in \mathcal{O}(l)$. Considering the class of composites depending on l and scaling the microstructure down, $l \to 0$, we obtain $<\tilde{\boldsymbol{k}}_R h^A> = 0$, $<\rho_R h^A> = 0$ and the macro-balance equations (5.5) reduce to the form

$$\frac{d}{dt}<\tilde{\boldsymbol{k}}_R>(\dot{Q},\dot{\boldsymbol{W}},Q,\boldsymbol{W}) - \boldsymbol{S}^{\alpha}_{R,\alpha} = <\rho_R>\boldsymbol{e} \qquad \boldsymbol{H}^A_R = 0 \tag{6.1}$$

Under denotation

$$\bar{\boldsymbol{s}}^{\alpha}_R(X,\nabla Q,Q,W) \equiv \tilde{\boldsymbol{s}}^{\alpha}_R\Big(X,\nabla Q + \nabla h^A(X)\boldsymbol{W}^A,Q\Big)$$

the macro-constitutive equations read

$$\begin{aligned} \boldsymbol{S}^{\alpha}_R(X,t) &= <\bar{\boldsymbol{s}}^{\alpha}_R>\Big(\nabla Q(X,t-\tau),Q(X,t-\tau),\boldsymbol{W}(X,t-\tau)\Big) \\ \boldsymbol{H}^A_R(X,t) &= <\bar{\boldsymbol{s}}^{\alpha}_R h^A{}_{,\alpha}>\Big(\nabla Q(X,t-\tau),Q(X,t-\tau),\boldsymbol{W}(X,t-\tau)\Big) \end{aligned} \tag{6.2}$$

The limit passage $l \to 0$ has to be interpreted as the extra assumption (introduced instead of the MMA) which states that the length-scale effect on the behaviour of the composite can be neglected. Hence, Eqs (6.1) and (6.2) represent the local macro-model of the composites under consideration. Comparing Eqs $(6.1)_2$ and $(5.5)_2$ it is easy to conclude that the fields $\boldsymbol{H}^A_R$ are due to the effect of the size of periodicity cell on the behaviour of the body. Hence, these fields can be referred to as the generalized length-scale forces.

7. Conclusions and final remarks

Macro-models of periodic composites, described by Eqs (5.3) and (5.5) were specified and discussed for elastic, thermoelastic and viscoelastic materials in the series of papers [1] ÷ [21]. Models of this kind were referred to as the refined models being a certain generalization of the previously introduced local models, governed by Eqs (6.1) and (6.2), cf [33] ÷ [38], [40], [48], [52]. The comparison of solutions to special problems, obtained within the framework of both refined and local models, made it possible to evaluate the effect of the size of microstructure on the behaviour of the body. Carrying out this analysis we have to specify the refined and the local models, using the same micro-shape function system. Moreover, introducing different systems of micro-shape function, we can analyze various aspects of the micro-dynamic

behaviour of the composite, investigated on different levels of accuracy. Substituting the right-hand sides of Eqs (5.3) into Eqs (5.5), and assuming that the supply fields $\boldsymbol{e}(\cdot, t)$ are known, we obtain the system of equations for the mapping (5.6), which describe the macro-configurations $\boldsymbol{Q}(\cdot, t)$ and macro-variables $\boldsymbol{W}^A(\cdot, t)$ for every time t in the time interval under consideration. The main characteristic feature of the resulting equations is that the equations for the macro-variables $\boldsymbol{W}^A(\cdot, t)$ (obtained by combining Eqs $(5.5)_2$ and $(5.3)_2$) do not involve material derivatives of these fields, but exclusively time derivatives and functionals in $\boldsymbol{W}^A(\boldsymbol{X}, \cdot)$. It means that the macro-variables $\boldsymbol{W}^A(\cdot, t)$ are independent of the boundary conditions and hence can be interpreted as certain internal variables. This fact is essential for the applications of the theory, since for the boundary-value problems formulated within a framework of the refined models, we deal with boundary conditions imposed only on macro-configurations $\boldsymbol{Q}(\cdot, t)$. The number and physical sense of these conditions are similar to those met in pertinent problems of solid mechanics. It has to be emphasized that the solutions to problems formulated for refined and local models have the physical sense only if configurations and internal macro-variables are represented by regular macro-functions. This requirement imposes certain restrictions on the class of problems described by the macro-models under consideration.

Summarizing the obtained results, we shall outline the general line of modelling procedure in the analysis of special problems. This procedure, applied to different formulations of refined and local models, has to be carried out in the following steps:

(i) Formulation of the weak form of the balance equation (2.5) for the class of investigated problems

(ii) Specification of the constitutive relations (2.3) for the material constituents of the body

(iii) Formulation of the MLII by the specification of functions $h^A(\cdot)$, $A = 1, ..., N$

(iv) Calculations of the averages $< \tilde{\boldsymbol{s}}_R^\alpha >$, $< \tilde{\boldsymbol{s}}_R^\alpha h^A{}_{,\alpha} >$, $< \tilde{\boldsymbol{k}}_R >$, $< \tilde{\boldsymbol{k}}_R h^A >$, $< \rho_R >$ and $< \rho_R h^A >$, determining the refined model of the composite by means of Eqs (5.3) and (5.5); functions $\tilde{\boldsymbol{k}}_R$ and functionals $\tilde{\boldsymbol{s}}_R^\alpha$ are obtained from Eqs (5.1)

(v) Passage to the local model, governed by Eqs (6.1) and (6.2), by neglecting terms $\mathcal{O}(l)$ in Eqs (5.3) and (5.5).

At the end of this paper we shall formulate some final remarks.

Remark 1. The formula (4.1) and hence also Eq (4.2), have the physical meaning only in the macro-interior of the periodic body under consideration; this macro-interior is defined as the set of points $X \in \Omega_R$ such that $V_R(X) \subset \Omega_R$. In the macro-boundary layer, i.e., for every $X \in \Omega_R$ such that $V_R(X) \cap \partial\Omega_R \neq \emptyset$, the formula (4.1) as well as the formula (4.2) have only formal meaning and for $X \in \Omega_R$ they cannot be accepted. That is why the trial functions in the surface integral of Eq (5.4) were not specified by Eq (4.2). Without this specification we are not able to obtain the natural macro-boundary conditions for Eqs (5.5).

Remark 2. Introducing into Eqs (5.4) the boundary conditions of micromechanics $\boldsymbol{t}_R = \boldsymbol{s}_R^\alpha n_{R\alpha}$ and using the MBA and MMA, we shall arrive at the formulae $S_{R,\alpha}^\alpha = <\tilde{\boldsymbol{s}}_{R,\alpha}^\alpha>$, $H_R^A = <\tilde{\boldsymbol{s}}_{R,\alpha}^\alpha h^A>$ the right-hand sides of which are functionals in $\tau \in [0,\infty)$ (the second formula is implied by Eq $(5.3)_2$ and $<(\tilde{\boldsymbol{s}}_R^\alpha h^A)_{,\alpha}> = 0$). It follows that Eqs (5.5) can be interpreted as the averaged (over $V_R(X)$, $X \in \Omega_R$) form of Eqs (2.4) combined with Eqs (2.2) and (2.3).

Remark 3. It can be shown that rejecting in the MMA condition (iii) and using the procedure similar to that established above, we obtain the balance equations

$$\frac{d}{dt} <\tilde{\boldsymbol{k}}_R> - S_{R,\alpha}^\alpha = <\rho_R> \boldsymbol{e}$$

$$\frac{d}{dt} <\tilde{\boldsymbol{k}}_R h^A> - G_R^{A\alpha}{}_{,\alpha} + H_R^A = <\rho_R h^A> \boldsymbol{e}$$

where

$$S_R^\alpha = <\hat{\boldsymbol{s}}_R^\alpha> \qquad G_R^{A\alpha} = <\hat{\boldsymbol{s}}_R^\alpha h^A>$$

$$H_R^A = <\hat{\boldsymbol{s}}_R^\alpha h^A{}_{,\alpha}>$$

are functionals in $\nabla \boldsymbol{Q}(X, t-\tau)$, $\boldsymbol{Q}(X, t-\tau)$, $\nabla \boldsymbol{W}^A(X, t-\tau)$, $\boldsymbol{W}^A(X, t-\tau)$, $\tau \geq 0$. The above equations represent a certain microstructural model of the periodic composite. We have stated in Introduction that the models of this kind in most cases are too complicated to be successfully applied to the analysis of special problems. Moreover, they can lead to certain physical ambiguities related to the formulation of boundary conditions.

Remark 4. Setting $\delta \boldsymbol{q} = \delta \boldsymbol{Q}$ in the surface integral of Eqs (5.4), we obtain

$$\boldsymbol{S}_R^\alpha(X, t) n_{R\alpha}(X) = \boldsymbol{t}_R(X, t) \qquad X \in \partial\Omega_R$$

This formal procedure leads to certain restrictions imposed on the boundary interactions t_R, due to the fact that the left-hand side of the above equation represents the boundary values of fields which are not oscillating within an arbitrary but fixed periodicity cell $V_R(X)$ in Ω_R.

Remark 5. Notice, that from the formal point of view the averaging procedure over $V_R(X)$ due to the V_R-periodicity of functions depending on Z, $Z \in R^3$, can be carried out for every $X \in \Omega_R$ since the averages over V_R are well defined. However, for points $X \in \Omega_R$ which belong to the macro-boundary layer (i.e., satisfy the condition $V_R(X) \cap \partial\Omega_R \neq \emptyset$), this formal averaging operation has not any physical interpretation. It follows that Eqs (5.5), from a formal viewpoint being defined for every $X \in \Omega_R$, can not be properly interpreted for $X \in \Omega_R$ belonging to the macro-boundary layer. However, by means of the condition $\lambda \ll 1$, this layer occupies a negligible small part of Ω_R.

Appendix: macro- and micro-coordinates

The governing equations (5.3) and (5.5) of the refined micro-models can be also obtained by the formal procedure in which the periodic composite is treated as a body with the internal structure, cf Woźniak (1969). To this end let us introduce two kinds of arguments: $Z \in R^3$ and $X \in \Omega_R$. Let us notice, that in Eqs (2.2) $\div$ (2.4) as well as in Eqs (4.1) and (4.2), we deal with functions which are V_R-periodic and hence they can be treated as functions of an argument $Z \in R^3$. At the same time we shall introduce functions depending on the material coordinates $X = (X^1, X^2, X^3) \in \Omega_R$, which are not V_R-periodic. The coordinates X^α, $\alpha = 1,2,3$, will be treated now as the macro-coordinates, i.e., they specify a certain cell $V_R(X)$ in Ω_R (or a part of this cell if $V_R(X) \cap \partial\Omega_R \neq \emptyset$). If $Z \in V_R$ then Z^α, $\alpha = 1,2,3$, will be interpreted as certain micro-coordinates specifying points in the cell $V_R(X)$. In the subsequent formal procedure we introduce fields depending both on $X \in \Omega_R$ and $Z \in R^3$ and we shall treat X and Z as independent variables. The fields depending both on X and Z, for $X = Z$ are assumed to reduce to those introduced in Sections $2 \div 5$. To this end we shall define ${}^*f(Z,X) = f(X)$ for $X = Z$; hence

$$f_{,\alpha} = \left(\frac{\partial}{\partial X^\alpha} + \frac{\partial}{\partial Z^\alpha}\right){}^*f$$

Under these notations, the local balance equation (2.4) reads

$$\frac{d}{dt}{}^{*}\boldsymbol{k}_R(Z,X,t) = \Big(\frac{\partial}{\partial X^{\alpha}} + \frac{\partial}{\partial Z^{\alpha}}\Big){}^{*}\boldsymbol{s}_R^{\alpha}(Z,X,t) + \boldsymbol{e}(X,t)\rho_R(Z) \tag{A.1}$$

For the fixed $X \in \Omega_R$ and $Z \in V_R(X)$ the above equation can be treated as related to the cell $V_R(X) \cap \Omega_R$ of the body. Using the MHL, Eqs (4.1) can be written down in the form

$${}^{*}\boldsymbol{q}(Z,X,t) = Q(X,t) + h^A(Z)W^A(X,t)$$

and the trial functions are

$${}^{*}\delta\boldsymbol{q}(Z,X) = \delta Q(X) + h^A(Z)\delta W^A(X)$$

Let us multiply Eq (A.1) by ${}^{*}\delta\boldsymbol{q}(Z,X)$ and integrate the resulting scalar product over $V_R(X) \times \Omega_R$. Since functions of Z are V_R-periodic and $\delta Q(\cdot)$, $\delta W^A(\cdot)$ are arbitrary independent macro-functions, then using the MBA and MMA, after a series of manipulations and bearing in mind that

$$\oint_{\partial V_R(X)} {}^{*}\widehat{\boldsymbol{s}}_R^{\alpha} \cdot {}^{*}\delta\boldsymbol{q}\, dv_R(Z) = 0$$

we obtain

$$\begin{aligned} &\frac{d}{dt} <{}^{*}\boldsymbol{k}_R>(X,t) - <{}^{*}\boldsymbol{s}_R^{\alpha}>_{,\alpha}(X,t) - <\rho_R>\boldsymbol{e}(X,t) = 0 \\ &\frac{d}{dt} <{}^{*}\boldsymbol{k}_R h^A>(X,t) - <{}^{*}\boldsymbol{s}_R^{\alpha} h^A{}_{,\alpha}>(X,t) - <\rho_R h^A>\boldsymbol{e}(X,t) = 0 \end{aligned} \tag{A.2}$$

It can be seen that Eqs (A2) combined with Eqs (5.1) and (5.3), coincide with the macro-balance equation (5.5).

The procedure outlined in this Appendix emphasizes the role of macro- and micro-coordinates in the description of periodic material structures. Let us observe that in the non-asymptotic modelling of periodic composites proposed in this contribution, where the microstructure length parameter is the known structural constant l, the concept of macro-coordinates is closely related to that of the macro-function: arguments X^{α} of an arbitrary macro-function can be referred to as the macro-coordinates. At the same time it has been tacitly assumed, that for any fixed triple $X = (X^1, X^2, X^3) \in \Omega_R$ of macro-coordinates and for an arbitrary instant t, the functions under consideration are V_R-periodic in coordinates $Z = (Z^1, Z^2, Z^3) \in R^3$. In this case we

can restrict the domain of Z to the cell V and refer Z^1, Z^2, Z^3 to as the micro-coordinates, related to a cell $V_R(X)$, where $V_R(X) \subset \Omega_R$.

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Mikrodynamika: ciągłe modele kompozytów z materiałów porowatych

Streszczenie

W pracy podano ogólny sposób modelowania niestacjonarnych procesów w kompozytach periodycznych. Otrzymane makro-modele uwzględniają wpływ wymiarów mikrostruktury na dynamikę ciała i są na tyle dogodne by mogły znaleźć zastosowanie do analizy zagadnień inżynierskich. Rozpatrywana metoda modelowania była już ostatnio stosowana do badania pewnych szczególnych materiałów i konstrukcji kompozytowych. Celem tego opracowania jest uogólnienie otrzymanych poprzednio wyników na dowolny materiał prosty w ramach nieliniowej mechaniki kontinuum.

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