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THERMOELASTICITY AND HOMOGENIZATION

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The problem of homogenization of a nonlinear thermoelastic periodic composite is treated by the method of the two-scale asymptotic expansions. Effective material coefficients are given in the general form, different from that obtained in the standard linearized theory. The local problems are however the same as in the linearized theory. The Francfort's remark on a shift of the initial conditions remains to be valid in a modified form.

1. Introduction

So far the most satisfying discussion of a linear thermoelastic composite has been done by Francfort (1982) (cf also Francfort (1983) and Brahim-Otsmane et al. (1988)) Francfort obtained the homogenized material coefficients of such a body and indicated the necessity for modification of the initial temperature condition for this body. By contrast to an earlier trial by Ene (1983) where a temperature – displacement picture was used, an essential role in Francfort's analysis was played by an entropy – displacement approach to the thermoelastic homogenization.

As in the linear thermoelasticity the entropy s is a linear function of temperature T (cf Nowacki (1966)), the homogenization procedures by Ene and Francfort are essentially the same (apart from the effect on shift of the initial temperature conditions). It appears, however, that the linear thermoelasticity in which the entropy – temperature relation is linear, and the nonlinear term $T\dot{s}$ in the energy equation is replaced by the linear term $T_0\dot{s}$ (T_0 – being the reference temperature), is overlinearized as far as the a homogenization procedure is concerned. In the present paper we are to outline the homogenization procedure for a periodic thermoelastic composite based on the quasi-linear thermoelasticity, in which the Duhamel-Neumann relations and the Fourier law for heat conduction are linear, but the entropy is a nonlinear function of temperature. Hence there is no need to linearize term $T\dot{s}$ in the energy equation. Such an approach to other problems of thermomechanics has been previously proposed by Ignaczak (1990) (cf also Landau and Lifshitz (1958), Stecki (1971)). We are to show that the above quasi-linear thermoelastic homogenization leads to some results obtained by Francfort (1982) and (1983), as well as to certain new results; e.g. two thermoelastic (stress-temperature) coefficients $\underline{\gamma}^H$ and $\underline{\gamma}^{HL}$ are obtained after the homogenization.

In this paper the homogenization is performed by the two-scale expansion method as described by Sanchez-Palencia (1980) (cf also Galka et al. (1992) and (1994), Wojnar (1992) and (1993)).

2. Basic equations

We consider a thermoelastic body occupying a volume V and composed of the identical elementary cells such that physical properties of the body change periodically over the body and the period is equal to the length dimensions of the elementary cell.

Let T be the absolute temperature of the body, s – its entropy and ε_{ij} the strain tensor related to the displacement u_i by the relation

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{2.1}$$

The relation between strain ε_{ij}, stress σ_{ij} and temperature T is assumed in the linear form of the Duhamel-Neumann equations

$$\sigma_{ij} = c_{ijmn}\varepsilon_{mn} - \gamma_{ij}(T - T_0) \tag{2.2}$$

where c_{ijmn} and γ_{ij} are the elastic moduli and the stress-temperature moduli tensors, respectively; γ_{ij} are related to the thermal expansion moduli α_{ij} by the equality $\gamma_{ij} = c_{ijmn}\alpha_{mn}$; T_0 is the reference temperature related to the *natural state*, where ε_{ij} and σ_{ij} vanish. It is also assumed that in this natural state treated as the reference state both the internal energy and the entropy vanish.

The nonlinear temperature – entropy – strain relation is postulated in the form

$$T = T_0 e^{(s-\gamma_{ij}\varepsilon_{ij})C_\varepsilon^{-1}} \tag{2.3}$$

where C_ε denotes the specific heat at a constant deformation. The problem under consideration will be governed by the conservation laws of momentum and energy, which in the local form read

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j}\sigma_{ij}$$
$$T\dot{s} = -\frac{\partial}{\partial x_i}q_i \tag{2.4}$$

The heat flux is given by the Fourier law

$$q_i = -K_{ij}\frac{\partial}{\partial x_j}T$$

Hence

$$T\dot{s} = \frac{\partial}{\partial x_i}\Big(K_{ij}\frac{\partial}{\partial x_j}T\Big) \tag{2.5}$$

Here ρ is the mass density and K_{ij} is the heat conductivity tensor. The coefficients ρ, c_{ijmn} and K_{ij} satisfy the inequalities

$$\rho > 0 \tag{2.6}$$
$$c_{ijmn}\xi_{ij}\xi_{ij} > 0 \qquad \text{for every} \quad \xi_{ij} \in \mathcal{E}_s^3 \tag{2.7}$$
$$K_{ij}\eta_i\eta_j > 0 \qquad \text{for every} \quad \eta_i \in \mathcal{R}^3 \tag{2.8}$$

3. Entropy – displacement field equations

Eliminating ε_{ij}, σ_{ij} and q_i from Eqs (2.1) ÷ (2.5) and denoting

$$S = \Big(s - \gamma_{ab}\frac{\partial u_a}{\partial x_b}\Big)C_\varepsilon^{-1}$$

we obtain the displacement – entropy field equations

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j}\Big[c_{ijmn}\frac{\partial u_m}{\partial x_n} - \gamma_{ij}T_0(\mathrm{e}^S - 1)\Big] \tag{3.1}$$

$$\dot{s} = K_{ij}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j} + \frac{\partial}{\partial x_i}\Big(K_{ij}\frac{\partial S}{\partial x_j}\Big) \tag{3.2}$$

These are two field equations which will be discussed below.

4. Homogenization: an outline of the procedure

Since the composite under consideration can be treated as a periodic repetition of the elementary εY-cells, then using the two-scale asymptotic expansion method the following form of u_i and s is postulated

$$u_i^\varepsilon = u_i^0(\boldsymbol{x},\boldsymbol{y}) + \varepsilon u_i^{(1)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 u_i^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \dots \tag{4.1}$$

$$s^\varepsilon = s^{(0)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon s^{(1)}(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 s^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \dots \tag{4.2}$$

The functions u_i^ε and s^ε are εY-periodic with respect to $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$.

Substituting Eqs (4.1) and (4.2) into Eqs (3.1) and (3.2) and keeping in mind that for the function $f(\boldsymbol{x},\boldsymbol{y})$ the space differentiation $\partial/\partial x_i$ should be replaced by $(\partial/\partial x_i + \varepsilon^{-1}\partial/\partial y_i)$ we get

$$\rho \ddot{u}_i^\varepsilon = \Big(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\Big)\Big[c_{ijmn}\Big(\frac{\partial}{\partial x_n} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\Big)u_m^\varepsilon - \gamma_{ij}T_0\Big(e^{S^\varepsilon} - 1\Big)\Big] \tag{4.3}$$

$$\begin{aligned}\dot{s}^\varepsilon = {} & K_{ij}\Big[\Big(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\Big)S^\varepsilon\Big]\Big(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\Big)S^\varepsilon + \\ & + \Big(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\Big)\Big[K_{ij}\Big(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\Big)S^\varepsilon\Big]\end{aligned} \tag{4.4}$$

where

$$S^\varepsilon \equiv \Big[s^\varepsilon - \gamma_{mn}\Big(\frac{\partial}{\partial x_n} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\Big)u_m^\varepsilon\Big]C_\varepsilon^{-1}$$

The material coefficients ρ, c_{ijmn}, K_{ij} etc. in Eqs (4.3) and (4.4) are assumed to be the εY-periodic functions of $\boldsymbol{y}$ coordinate.

5. Homogenization of the energy equation

(i) Equating to zero the coefficient at ε^{-4} in Eq (4.4) and denoting

$$f \equiv -\gamma_{ab}\frac{\partial u_a^{(0)}}{\partial y_b}C_\varepsilon^{-1} \qquad\qquad f_i \equiv \frac{\partial f}{\partial y_i} \tag{5.1}$$

one obtains

$$K_{ij}f_if_j = 0$$

By the positive definiteness of K_{ij}, the last equation yields

$$f_i = 0$$

Integrating over the cell Y we find

$$\Big(-\gamma_{ab}\frac{\partial u_a^{(0)}}{\partial y_b}\Big)C_\varepsilon^{-1} = \omega(\boldsymbol{x},t) \tag{5.2}$$

where the function $\omega(\boldsymbol{x},t)$ plays role of an integration constant (for variable $\boldsymbol{y}$) and is yet unknown.

(ii) Equating to zero the coefficient at ε^{-3} and using Eq $(5.1)_1$ we obtain

$$\begin{aligned} 0 = K_{ij}\Big[\Big(\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial y_i}\{[s^{(0)} - \gamma_{ab}(\frac{\partial u_a^{(0)}}{\partial x_b} + \frac{\partial u_a^{(1)}}{\partial y_b})]C_\varepsilon^{-1}\}\Big)\frac{\partial f}{\partial y_j} + \\ + \frac{\partial f}{\partial y_i}\Big(\frac{\partial f}{\partial x_j} + \frac{\partial}{\partial y_i}\{[s^{(0)} - \gamma_{pq}(\frac{\partial u_p^{(0)}}{\partial x_q} + \frac{\partial u_p^{(1)}}{\partial y_q})]C_\varepsilon^{-1}\}\Big)\Big] + \frac{\partial}{\partial y_i}\Big(K_{ij}\frac{\partial f}{\partial y_j}\Big) \end{aligned} \tag{5.3}$$

Keeping in mind (5.2), the above equation is identically satisfied.

(iii) Denoting

$$S^{(0)} = \Big[s^{(0)} - \gamma_{ab}\Big(\frac{\partial u_a^{(0)}}{\partial x_b} + \frac{\partial u_a^{(1)}}{\partial y_b}\Big)\Big]C_\varepsilon^{-1}$$

and equating to zero the coefficient at ε^{-2}, we have

$$0 = K_{ij}\frac{\partial S^{(0)}}{\partial y_i}\frac{\partial S^{(0)}}{\partial y_j} + \frac{\partial}{\partial x_i}\Big(K_{ij}\frac{\partial f}{\partial y_j}\Big) + \frac{\partial}{\partial y_i}\Big(K_{ij}\frac{\partial S^{(0)}}{\partial y_j}\Big) \tag{5.4}$$

and after use of Eq (5.2) we arrive at

$$\frac{\partial}{\partial y_i}\Big[K_{ij}\frac{\partial}{\partial y_j}\Big(\mathrm{e}^{[s^{(0)} - \gamma_{mn}(\partial u_m^{(0)}/\partial x_n + \partial u_m^{(1)}/\partial y_n)]C_\varepsilon^{-1}}\Big)\Big] = 0 \tag{5.5}$$

Bearing in mind Eq (2.8), the above equation implies

$$\mathrm{e}^{[s^{(0)} - \gamma_{mn}(\partial u_m^{(0)}/\partial x_n + \partial u_m^{(1)}/\partial y_n)]C_\varepsilon^{-1}} = C_T(\boldsymbol{x},t) \tag{5.6}$$

where the function $C_T(\boldsymbol{x},t)$ plays a role of an integration constant (for the variable $\boldsymbol{y}$). Taking the logarithms of both sides of Eq (5.6) we get

$$\Big[s^{(0)} - \gamma_{pq}\Big(\frac{\partial u_p^{(0)}}{\partial x_q} + \frac{\partial u_p^{(1)}}{\partial y_q}\Big)\Big]C_\varepsilon^{-1} = \ln C_T(\boldsymbol{x},t) \tag{5.7}$$

(iv) Equating to zero the coefficient at ε^{-1}, after some transformation one gets

$$0 = K_{ij}\frac{\partial^2\omega(\boldsymbol{x},t)}{\partial x_i\partial x_j} + \frac{\partial}{\partial y_i}\Big[K_{ij}\frac{\partial}{\partial x_j}\Big(\ln C_T(\boldsymbol{x},t)\Big)\Big] + \\ + \frac{\partial}{\partial y_i}\Big(K_{ij}\frac{\partial}{\partial y_j}\{[s^{(1)} - \gamma_{pq}(\frac{\partial u_p^{(1)}}{\partial x_q} + \frac{\partial u_p^{(2)}}{\partial y_q})]C_\varepsilon^{-1}\}\Big) \tag{5.8}$$

Introducing the function $\vartheta_k(\boldsymbol{y})$ which satisfies in Y the following local equation

$$\frac{\partial}{\partial y_i}\Big(K_{ik} + K_{ij}\frac{\partial\vartheta_k(\boldsymbol{y})}{\partial y_j}\Big) = 0 \tag{5.9}$$

one obtains

$$\frac{\partial K_{ik}}{\partial y_i} = -\frac{\partial}{\partial y_i}\Big(K_{ij}\frac{\partial\vartheta_k}{\partial y_j}\Big) \tag{5.10}$$

Hence Eq (5.8) takes the form

$$0 = K_{ij}\frac{\partial^2\omega(\boldsymbol{x},t)}{\partial x_i\partial x_j} + \\ + \frac{\partial}{\partial y_i}\Big[K_{ij}\frac{\partial}{\partial y_j}\Big(-\vartheta_k\frac{\partial\ln C_T(\boldsymbol{x},t)}{\partial x_k} + \{[s^{(1)} - \gamma_{pq}(\frac{\partial u_p^{(1)}}{\partial x_q} + \frac{\partial u_p^{(2)}}{\partial y_q})]C_\varepsilon^{-1}\}\Big)\Big] \tag{5.11}$$

In the subsequent section it will be shown that $\omega(\boldsymbol{x},t) = 0$. The positive definiteness of K_{ij} implies that

$$s^{(1)} - \gamma_{pq}\Big(\frac{\partial u_p^{(1)}}{\partial x_q} + \frac{\partial u_p^{(2)}}{\partial y_q}\Big) = C_\varepsilon\Big(k(\boldsymbol{x},t) + \vartheta_k\frac{\partial\ln C_T(\boldsymbol{x},t)}{\partial x_k}\Big) \tag{5.12}$$

where k is a new function independent of $\boldsymbol{y}$.

Now, we are ready to analyse the last term of the energy equation, by equating to zero coefficient at ε^0 in Eq (4.4). The result is given by

$$\dot{s}^{(0)} = K_{ij}\Big[L_iL_j + \frac{\partial}{\partial y_i}(\vartheta_kL_k)L_j + L_i\frac{\partial}{\partial y_j}(\vartheta_qL_q) + \frac{\partial}{\partial y_i}(-\vartheta_kL_k)\frac{\partial}{\partial y_j}(\vartheta_qL_q)\Big] + \\ + \frac{\partial}{\partial x_i}\Big\{K_{ij}\Big[L_j + \frac{\partial}{\partial y_j}(-\vartheta_kL_k)\Big]\Big\} + \frac{\partial}{\partial y_i}\Big\{\ \Big\} \tag{5.13}$$

where

$$L_i = \frac{\partial \ln C_T(\boldsymbol{x}, t)}{\partial x_i}$$

and the last term { } is irrelevant for the subsequent analysis since its contribution to the cell-averaging operation vanishes.

Let us introduce the averaging operator over the elementary cell Y

$$<(\cdot)>= \frac{1}{|Y|} \int\limits_Y (\cdot)\, dy$$

Averaging of Eq (5.13) yields

$$\begin{aligned} < \dot{s}^{(0)} > &= < K_{ij} > L_i L_j + < K_{ij} \frac{\partial \vartheta_k}{\partial y_i} > L_k L_j + L_i < K_{ij} \frac{\partial \vartheta_k}{\partial y_j} > L_k + \\ &+ < K_{ij} \frac{\partial \vartheta_k}{\partial y_i} \frac{\partial \vartheta_q}{\partial y_j} > L_k L_q + \frac{\partial}{\partial x_i} < K_{ik} + K_{ij} \frac{\partial \vartheta_k}{\partial y_j} > L_k \end{aligned} \tag{5.14}$$

By integrating by parts and using periodic boundary conditions imposed on ϑ_k we obtain

$$< K_{ij} \frac{\partial \vartheta_k}{\partial y_i} \frac{\partial \vartheta_q}{\partial y_j} >=< -K_{iq} \frac{\partial \vartheta_k}{\partial y_i} > \tag{5.15}$$

Reducing remaining terms in Eq (5.14) we arrive at the result

$$< \dot{s}^{(0)} >= K^H_{ik} \frac{\partial \ln C_T(\boldsymbol{x}, t)}{\partial x_i} \frac{\partial \ln C_T(\boldsymbol{x}, t)}{\partial x_k} + K^H_{ik} \frac{\partial^2 \ln C_T(\boldsymbol{x}, t)}{\partial x_i \partial x_k} \tag{5.16}$$

where

$$K^H_{ik} =< K_{ik} + K_{ij} \frac{\partial \vartheta_k}{\partial y_j} > \tag{5.17}$$

6. Homogenization of the displacement – entropy equation and the final results

Let us observe that the highest singularity on the right-hand side Eq (4.3) is due to the exponent factor at the temperature term

$$\exp\Big(-\frac{1}{\varepsilon}\gamma_{ab}\frac{\partial u_a^{(0)}}{\partial y_b} C_\varepsilon^{-1}\Big) = \exp\Big(\frac{1}{\varepsilon}\omega(\boldsymbol{x}, t)\Big) \tag{6.1}$$

In order to avoid this singularity which does not appear on the left-hand side of Eq (4.3), assume

$$\omega(\boldsymbol{x},t)=0 \tag{6.2}$$

Equating to zero the coefficient at ε^{-2} in Eq (4.3) one obtains

$$\frac{\partial}{\partial y_j}\Big(c_{ijmn}\frac{\partial u_m^{(0)}}{\partial y_n}\Big)=0$$

By the positive definiteness of c_{ijmn} this equation yields

$$u_a^{(0)}=u_a^{(0)}(\boldsymbol{x},t) \tag{6.3}$$

In order to find the coefficients at ε^{-1} and ε^{0} in Eqs (4.3) let us begin with the analysis of ε-order of terms produced by the exponential component on the right-hand side of Eq (4.3), which we denote by g_i

$$\begin{aligned} g_i &= -\frac{1}{\varepsilon}T_0\frac{\partial}{\partial y_j}\Big[\gamma_{ij}\Big(\exp\{[s^{(0)}+\varepsilon s^{(1)}+\varepsilon^2 s^{(2)}+\ldots+ \\ &- \gamma_{mn}(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n})(u_m^{(0)}+\varepsilon u_m^{(1)}+\varepsilon^2 u_m^{(2)}+\ldots)]C_\varepsilon\}-1\Big)\Big] \end{aligned} \tag{6.4}$$

With the use of Eqs (6.3), (5.7) and (5.12) the exponential term is transformed as follows

$$\begin{aligned} &\exp\Big\{\Big[s^{(0)}+\varepsilon s^{(1)}+\varepsilon^2 s^{(2)}+\ldots+ \\ &-\gamma_{mn}\Big(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\Big)(u_m^{(0)}+\varepsilon u_m^{(1)}+\varepsilon^2 u_m^{(2)}+\ldots)\Big]C_\varepsilon^{-1}\Big\}= \\ &\exp\Big\{\Big[s^{(0)}+\varepsilon s^{(1)}+\varepsilon^2 s^{(2)}+\ldots-\gamma_{mn}\Big(\frac{\partial u_m^{(0)}}{\partial x_n}+\frac{\partial u_m^{(1)}}{\partial y_n}\Big)+ \\ &-\varepsilon\gamma_{mn}\Big(\frac{\partial u_m^{(1)}}{\partial x_n}+\frac{\partial u_m^{(2)}}{\partial y_n}\Big)+\varepsilon^2\gamma_{mn}(\ldots)+\ldots\Big]C_\varepsilon^{-1}\Big\}= \\ &=C_T(\boldsymbol{x},t)\exp\Big[\varepsilon\Big(k(\boldsymbol{x},t)+\vartheta_k\frac{\partial\ln C_T(\boldsymbol{x},t)}{\partial x_k}\Big)+\varepsilon^2(\ldots)+\ldots\Big] \end{aligned}$$

Hence denoting

$$E=\varepsilon\Big(k(\boldsymbol{x},t)+\vartheta_k\frac{\partial\ln C_T(\boldsymbol{x},t)}{\partial x_k}\Big)+\varepsilon^2(\ldots)+\ldots$$

we get

$$g_i = -\frac{1}{\varepsilon} T_0 \Big[\frac{\partial \gamma_{ij}}{\partial y_j} \Big(C_T(\boldsymbol{x}, t) \mathrm{e}^E - 1 \Big) + \gamma_{ij} C_T(\boldsymbol{x}, t) \mathrm{e}^E \frac{\partial E}{\partial y_j} \Big] \tag{6.5}$$

and observe that g_i produces terms of an order ε^{-1} (the first member) and ε^0 (the second member) as $\varepsilon \to 0$. Using this result we find:

(i) Equating to zero the coefficient at ε^{-1}

$$\frac{\partial}{\partial y_j} \Big[c_{ijmn} \Big(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n} \Big) - \gamma_{ij} \Theta \Big] = 0 \tag{6.6}$$

where

$$\Theta = \Theta(\boldsymbol{x}, t) = T_0 \Big(C_T(\boldsymbol{x}, t) - 1 \Big) \tag{6.7}$$

Let

$$u_m^{(1)} = \chi_{mpq}(\boldsymbol{y}) \frac{\partial u_p^{(0)}}{\partial x_q} + \Gamma_m(\boldsymbol{y}) \Theta \tag{6.8}$$

and observe that Eq (6.6) is satisfied if local functions χ_{mpq} and Γ_m satisfy the equations

$$\frac{\partial}{\partial y_j} \Big(c_{ijpq} + c_{ijmn} \frac{\partial \chi_{mpq}}{\partial y_n} \Big) = 0 \tag{6.9}$$

$$\frac{\partial}{\partial y_j} \Big(-\gamma_{ij} + c_{ijmn} \frac{\partial \Gamma_m}{\partial y_n} \Big) = 0 \tag{6.10}$$

Eqs (6.8) ÷ (6.10) are identical with those appearing during the homogenization of equations of linear thermoelasticity (cf Francfort (1982) and (1983)).

(ii) Equating to zero the coefficient at ε^0 we obtain

$$\begin{aligned} \rho \ddot{u}_i^{(0)} &= \frac{\partial}{\partial x_j} \Big[c_{ijmn} \Big(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n} \Big) - \gamma_{ij} \Theta - \gamma_{iq} \frac{\partial \vartheta_j}{\partial y_q} \Theta \Big] + \\ &+ \frac{\partial}{\partial y_j} \Big[c_{ijmn} \Big(\frac{\partial u_m^{(1)}}{\partial x_n} + \frac{\partial u_m^{(2)}}{\partial y_n} \Big) \Big] \end{aligned} \tag{6.11}$$

where definition (6.7) of Θ was also used. Averaging (6.11) over the cell Y yields

$$<\rho> \ddot{u}_i^{(0)} = \frac{\partial}{\partial x_j} < c_{ijmn} \Big(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n} \Big) - \gamma_{ij} \Theta - \gamma_{iq} \frac{\partial \vartheta_j}{\partial y_q} \Theta > \tag{6.12}$$

After using of Eq (6.8) we arrive at

$$<\rho> \ddot{u}_i^{(0)} = c_{ijpq}^H \frac{\partial^2 u_p^{(0)}}{\partial x_j \partial x_q} - \gamma_{ij}^H \frac{\partial \Theta}{\partial x_j} \tag{6.13}$$

where

$$c_{ijpq}^H = < c_{ijpq} + c_{ijmn} \frac{\partial \chi_{mpq}}{\partial y_n} > \tag{6.14}$$

$$\gamma_{ij}^H = < \gamma_{ij} - c_{ijmn} \frac{\partial \Gamma_m}{\partial y_n} + \gamma_{iq} \frac{\partial \vartheta_j}{\partial y_q} > \tag{6.15}$$

The formula (6.14) for c_{ijpq}^H is identical with that derived in the linear theory while the result (6.15) is characteristic for the quasi-linear theory: we observe that γ_{ij}^H is composed of two terms

$$\gamma_{ij}^H = \gamma_{ij}^{HL} + < \gamma_{iq} \frac{\partial \vartheta_j}{\partial y_q} > \tag{6.16}$$

with

$$\gamma_{ij}^{HL} = < \gamma_{ij} - c_{ijmn} \frac{\partial \Gamma_m}{\partial y_n} > = < \gamma_{ij} + \gamma_{mn} \frac{\partial \chi_{mij}}{\partial y_n} > \tag{6.17}$$

being the homogenized thermoelastic coefficient of the linear theory.

Comparing Eq (6.13) and Eqs (2.4) and (2.2) we see that the term

$$\Theta = T_0 \Big(C_T(\boldsymbol{x}, t) - 1 \Big)$$

in Eq (6.13) is equivalent to the temperature difference $(T^H - T_0)$ for a homogenized body, T^H being a temperature of such a body; therefore

$$\Theta = T^H - T_0 \qquad C_T(\boldsymbol{x}, t) = \frac{T^H}{T_0} \tag{6.18}$$

The same result is obtained by comparison of Eq (5.16) with Eqs (3.2) and (2.3).

Finally, substituting Eq (6.8) into Eq (5.7) we get

$$s^{(0)} - \Big(\gamma_{pq} + \gamma_{mn} \frac{\partial \chi_{mpq}}{\partial y_n} \Big) \frac{\partial u_p^{(0)}}{\partial x_q} - \gamma_{mn} \frac{\partial \Gamma_m}{\partial y_n} \Theta = C_\varepsilon \ln C_T(\boldsymbol{x}, t) \tag{6.19}$$

and by taking mean value of the result, we obtain

$$< s^{(0)} > -\gamma_{mn}^{HL}\frac{\partial u_m^{(0)}}{\partial x_n} - \sigma(T^H - T_0) =< C_\varepsilon > \ln\frac{T^H}{T_0} \tag{6.20}$$

where

$$\sigma =< \gamma_{mn}\frac{\partial \Gamma_m}{\partial y_n} > \tag{6.21}$$

and γ_{pq}^{HL} is given by (6.17) and the relations (6.18) have been used. Eq (6.20) is a transcendental equation with respect to T^H; it can be solved formally as

$$< s^{(0)} > -\gamma_{mn}^{HL}\frac{\partial u_m^{(0)}}{\partial x_n} = C_\varepsilon^H \ln\frac{T^H}{T_0} \tag{6.22}$$

with the following "homogenized" specific heat

$$C_\varepsilon^H =< C_\varepsilon > +\sigma\frac{T^H - T_0}{\ln(T^H/T_0)} \tag{6.23}$$

Also after use of Eq $(6.18)_2$, the averaged entropy production equation (5.16) takes the form

$$< \dot{s}^{(0)} >= K_{ik}^H\frac{\partial \ln T^H(x,t)}{\partial x_i}\frac{\partial \ln T^H(x,t)}{\partial x_k} + K_{ik}^H\frac{\partial^2 \ln T^H(x,t)}{\partial x_i \partial x_k} \tag{6.24}$$

or

$$T^H(x,t) < \dot{s}^{(0)} >= K_{ik}^H\frac{\partial^2 T^H(x,t)}{\partial x_i \partial x_k} \tag{6.25}$$

7. Shift of the initial condition for temperature

Let the initial conditions for our problem be

$$u_i(x,0) = U_i(x) \qquad\qquad T(x,0) = \mathcal{T}(x) \tag{7.1}$$

Then, after making calculations analogous to that of Francfort (1982) and (1983), cf also Galka et al. (1994), we arrive at the following initial condition for the homogenized temperature field

$$T^H(x,0) = T_0\Big(\frac{\mathcal{T}}{T_0}\Big)^{(<C_\varepsilon>/C_\varepsilon^H)} \exp\Big[(< \gamma_{ij} > -\gamma_{ij}^H)\frac{\partial U_i}{\partial x_j}\frac{1}{C_\varepsilon^H}\Big] \tag{7.2}$$

It is the desired results for a shift of the initial temperature of a homogenized body.

8. Conclusions

The homogenized field equations (6.13) and (6.24) and the effective coefficients for a homogenized quasi-linear thermoelastic body can be obtained if, similarly to the linear case the three local problems for the functions ϑ_k, χ_{mpq} and Γ_m are solved, Eqs (5.9), (6.9) and (6.10).

The homogenized coefficients K_{ik}^H and c_{ijpq}^H are the same as in the linear theory, (5.17) and (6.14) while γ_{ij}^H is different, given by (6.15) or (6.16) being the homogenized γ_{ij} coefficient from the linear theory; the second term on the right-hand side of Eq (6.16) represents the influence of heat conduction on the stress-temperature coefficient as it comprises the function ϑ_j, being a solution of the local problem (5.9). The "homogenized" counterpart of specific heat for the quasi-linear theory is given by the function C_ε^H, Eq (6.23).

For the linearized case the coefficient of σ in Eq (6.23) is equal to 1 and C_ε^H is constant (cf Ffrancfort (1982) and (1983)).

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Termosprężystość i homogenizacja

Streszczenie

Rozważamy zadanie homogenizacji niejednorodnego ośrodka termosprężystego, o zadanej w przestrzeni położeń komórce okresowości współczynników materiałowych. Przyjmujemy liniowe prawo termosprężystości i liniowe prawo przewodnictwa ciepła, natomiast związek termodynamiczny między entropią, temperaturą i odkształceniem nie jest linearyzowany. Korzystając z metody rozwinięć dwuskalowych wyprowadzamy efektywne współczynniki materiałowe, różne w ogólności od danych przez teorię zlinearyzowaną. Jednak tzw. zadania lokalne (na komórce) są u nas te same co w teorii zlinearyzowanej. Zachodzi również zmiana warunku początkowego, podobna do zauważonej przez Franckforta dla ośrodka zlinearyzowanego.

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