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ON CONTINUUM MODELLING THE DYNAMIC BEHAVIOUR OF CERTAIN COMPOSITE LATTICE-TYPE STRUCTURES

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The aim of this contribution is to propose a new continuum model of periodic composite lattice-type elastic structures. The proposed model describes the structural length-scale effect on the dynamic behaviour of a system. The general line of approach is based on that developed by Woźniak (1993) to the refined elastodynamics of composite materials.

1. Introduction

The number of papers on continuum modelling of discrete systems is very impressive. Here we shall restrict ourselves to the periodic lattice-type structures met in engineering, which in the first approximation can be considered as systems of regularly distributed mass-points (nodal points) interconnected by the linear-elastic straight rods, transfering exclusively axial forces, Fig.1. The continuum models of such lattice-type structures were introduced and investigated by Woźniak and his collaborators in a series of contributions; for details cf Woźniak (1970) and the references therein. More sophisticated approach, based on the asymptotic homogenization method was analysed by Cioranescu and Saint J.Paulin (1991). However, in both cases the resulting equations have the form similar to those of the elasticity theory for anizotropic media and hence they are not able to describe properly the dynamic response of the system, neglecting both higher order vibration frequencies and dispersion phenomena. This drawback stands for a motivation of the research presented below, where a certain refined continuum model of the aforementioned lattice-type periodic structures will be proposed. The general idea of the approach is based on that leading to the refined elastodynamics of periodic

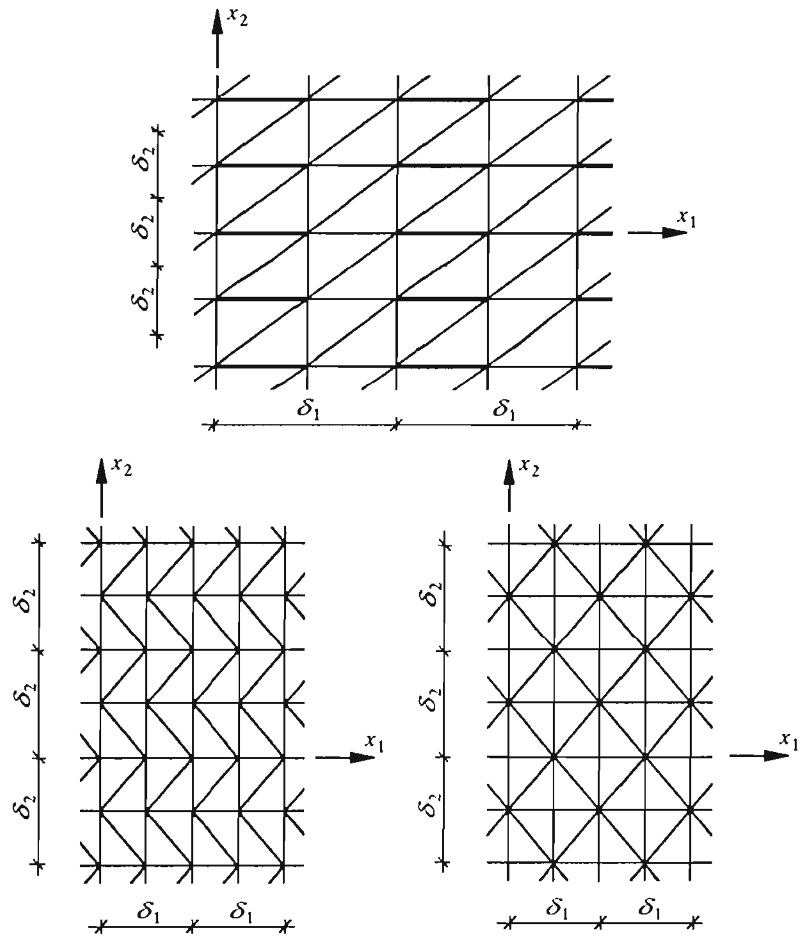

Fig. 1. Examples of plane periodic lattice-type structures

composite materials, cf Woźniak (1993). The considerations will be restricted to the linear theory of periodic systems of mass-points (nodal points of a lattice) interacting by means of the linear-elastic rods. Assuming that the material and geometric characteristics of different rods are different we shall deal with certain (discrete) composite structures. It has to be emphasized that the continuum models of these structures, introduced in the paper can be applied to engineering problems only on the condition that the structure is made of a large number of periodically repeated structural elements length dimensions of which are small enough compared to the minimum characteristic length dimension of the whole periodic structure.

Denotations. Subscripts i, j, k, l run over $1, 2, 3$ and are related to the cartesian orthogonal coordinates x_1, x_2, x_3 in the reference space. Indices a, b and A, B run over $1, ..., n$ and $1, ..., N$, respectively; indices α, β take the values $1, ..., \kappa$. Summation convention holds for all the aforementioned indices unless otherwise stated. Points in the reference space are denoted by $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ and t is time coordinate.

2. Analysis

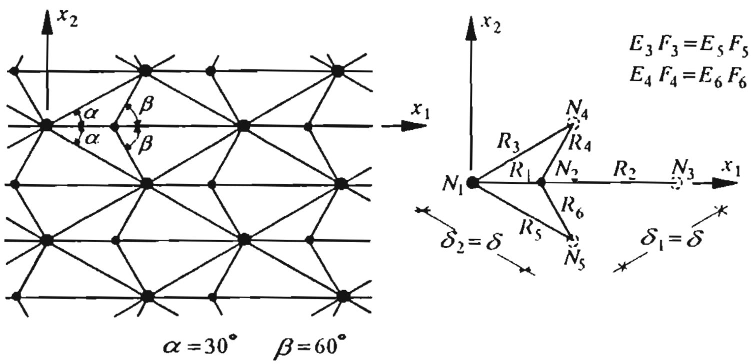

Fig. 2. Scheme of the plane lattice and its representative element

Let $V = (0, l^1) \times (0, l^2) \times (0, l^3)$ be a cell in the reference space of points $\boldsymbol{x} = (x_1, x_2, x_3)$ in which the representative structural element of the periodic lattice structure under consideration is situated. Hence the periods l^i will be treated as certain small parameters due to the assumption that the whole structure is made of a large number of structural elements. By Ω we shall denote the region composed of all the spatial cells (and their interfaces) with repetitive structural elements. Schemes of the plane lattice and the representative element are shown in Fig.2; by N_a, $a = 1, ..., n$, we denote nodes of the lattice in V, by R_A, $A = 1, ..., N$, the rods interconnecting nodal points. All rods are assumed to be prismatic and homogenous. Hence to every R_A there is assigned a pair (N_a, N_b) of nodal points, where $a < b$. Unit vectors $\boldsymbol{t}^A$, $\boldsymbol{t}^A_a$, $\boldsymbol{t}^A_b$ with components t^A_i, t^A_{ai}, t^A_{bi}, are shown in Fig.3, where also the length l_A of the rod R_A is indicated. The area of the cross section and the Young modulus of the rod R_A will be denoted by F_A and E_A, respectively. Moreover, it is assumed that mass of the whole system is assigned to the nodal points;

the mass M_a is concentrated at the node N_a. Thus, the periodic lattice-type structure is represented by a periodic system of interacting mass-points.

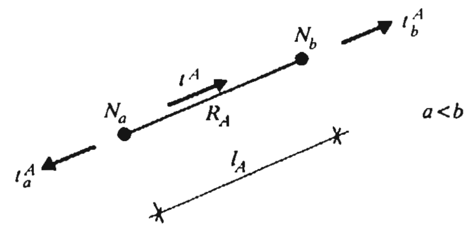

Fig. 3. Unit vectors assigned to the rod R_A bounded by nodal points N_a, N_b

Define $V(x) \equiv x + V$ and $\Omega_0 \equiv \{x \in \Omega : V(x) \subset \Omega\}$. Let us introduce the micro-shape matrix of numbers h_α^a satysfying the following conditions

$$M_a h_\alpha^a = 0 \qquad \alpha = 1, ..., \kappa \qquad h_\alpha^a \in \mathcal{O}(l) \tag{2.1}$$

where $l \equiv \sqrt{(l^1)^2 + (l^2)^2 + (l^3)^2}$ will be called the microstructure length parameter. Let $U_i(x,t)$, $Q_i^\alpha(x,t)$ for every t be sufficiently regular V-macro functions defined on Ω. It means that $x' - x'' \in V$ implies $|U_i(x',t) - U_i(x'',t)| < \lambda$, $|Q_i^\alpha(x',t) - Q_i^\alpha(x'',t)| < \lambda$, λ being the numerical accuracy parameter, and similar conditions are also imposed on the derivatives of U_i and Q_i^α, where within a framework of admissible approximations terms $\mathcal{O}(\lambda)$ can be neglegted, cf Woźniak (1993). Moreover let $\Lambda = \{x \in \Omega_0 : x = c_1 l^1 e_1 + c_2 l^2 e_2 + c_3 l^3 e_3\}$ where c_i are integers and e_i are versors of x_i-axes. The basic kinematic hypothesis which interrelates displacements $u_i^a(\bar{x},t)$ of the node N_a in the spatial element $V(\bar{x})$, $\bar{x} \in \Lambda$, with fields U_i, Q_i^α, will be assumed in the form

$$u_i^a(\bar{x},t) = U_i(x,t) + h_\alpha^a Q_i^\alpha(x,t) \qquad \bar{x} \in \Lambda \tag{2.2}$$

where x is a position vector of the node N_a in the spatial element $V(\bar{x})$. Fields $U_i(\cdot,t)$ will be called macrodisplacements; since $|U_i(x,t) - U_i(\bar{x},t)| < \lambda$ for every $x \in V(\bar{x})$, then the macrodisplacemets of nodes situated in an arbitrary fixed spatial element $V(\bar{x},t)$, $\bar{x} \in \Lambda$, can be approximately treated as equal (with the approximation $\mathcal{O}(\lambda)$). Fields $Q_i^\alpha(\cdot,t)$ are said to be correctors and describe the disturbances in displacements caused by the composite structure of the lattice under consideration; the exact meaning of this notion will be given below.

Taking into account Eq (2.2), the strain $\varepsilon_A(\bar{x},t)$ in the rod R_A belonging to the spatial element $V(\bar{x})$, will be given by (no summation over A)

$$\varepsilon_A(\bar{x},t) = U_{(i,j)}(\bar{x},t) t_i^A t_j^A + (l_A)^{-1} t_{ai}^A h_\alpha^a Q_i^\alpha(\bar{x},t) + \mathcal{O}(\lambda) \tag{2.3}$$

where we define $t^A_{ai} \equiv 0$ if the nodal point N_a is not connected with the rod R_A.

The governing equations in macrodisplacements U_i and correctors Q^α_i will be obtained from the principle of stationary action on the assumption that terms $\mathcal{O}(\lambda)$ in the action functional can be neglected. For the strain and kinetic energy functions we obtain

$$
\begin{aligned}
J &= \frac{1}{2}\sum_{\bar{\boldsymbol{x}}\in\Lambda}\sum_{A=1}^{N} \Lambda_A\Big(\varepsilon_A(\bar{\boldsymbol{x}},t)\Big)^2 \qquad\qquad \Lambda_A \equiv F_A E_A l_A \\
K &= \frac{1}{2}\sum_{\bar{\boldsymbol{x}}\in\Lambda}\sum_{a=1}^{n} M_a \dot{u}^a_i(\bar{\boldsymbol{x}},t)\dot{u}^a_i(\bar{\boldsymbol{x}},t)
\end{aligned}
$$

respectively. Taking into account Eqs (2.2) and (2.3) and conditions $M_a h^a_\alpha = 0$, after denotations

$$
\begin{aligned}
\rho &\equiv \frac{1}{\mathrm{vol}V}\sum_{a=1}^{n} M_a \\
\Pi_{\alpha\beta} &\equiv \frac{1}{\mathrm{vol}V}\sum_{a=1}^{n} M_a h^a_\alpha h^a_\beta l^{-2} \\
A_{ijkl} &\equiv \frac{1}{\mathrm{vol}V}\sum_{A=1}^{N} \Lambda_A t^A_i t^A_j t^A_k t^A_l \\
B_{\alpha ijk} &\equiv \frac{1}{\mathrm{vol}V}\sum_{A=1}^{N} \Lambda_A t^A_i t^A_j t^A_{ak}(\varphi_A)^{-1} h^a_\alpha l^{-1} \\
C_{\alpha\beta ij} &\equiv \frac{1}{\mathrm{vol}V}\sum_{A=1}^{N} \Lambda_A t^A_{ai} t^A_{bj}(\varphi_A)^{-1} h^a_\alpha h^b_\beta l^{-2} \\
\mathrm{vol}V &\equiv l^1 l^2 l^3 \qquad\qquad \varphi_A \equiv \frac{l_A}{l}
\end{aligned}
\tag{2.4}
$$

and bearing in mind that U_i, Q^α_i are regular V-macro functions, we arrive at the formulae

$$
\begin{aligned}
J &= \int\limits_{\Omega}\Big(\frac{1}{2}A_{ijkl}U_{i,j}U_{k,l} + B_{\alpha ijk}U_{i,j}Q^\alpha_k + \frac{1}{2}C_{\alpha\beta ij}Q^\alpha_i Q^\beta_j\Big)\,dx_1dx_2dx_3 + \mathcal{O}(\lambda) \\
K &= \int\limits_{\Omega}\Big(\frac{1}{2}\rho\dot{U}_i\dot{U}_i + \frac{1}{2}l^2\Pi_{\alpha\beta}\dot{Q}^\alpha_i\dot{Q}^\beta_j\Big)\,dx_1dx_2dx_3 + \mathcal{O}(\lambda)
\end{aligned}
\tag{2.5}
$$

Neglecting in the Eqs (2.5) the terms $\mathcal{O}(\lambda)$ and assuming that the external forces (loadings) are equal to zero, from the principle of stationary action we derive the following system of equations which have to be satisfied for every $\boldsymbol{x} \in \Omega$ and for every t in the given a priori time interval $[t_0, t_f]$

$$\begin{aligned} &\rho \ddot{U}_i(\boldsymbol{x},t) - A_{ijkl}U_{k,lj}(\boldsymbol{x},t) - B_{\alpha ijk}Q^{\alpha}_{k,j}(\boldsymbol{x},t) = 0 \\ &l^2 \Pi_{\alpha\beta}\ddot{Q}^{\beta}_i(\boldsymbol{x},t) + C_{\alpha\beta ij}Q^{\beta}_j(\boldsymbol{x},t) + B_{\alpha kij}U_{k,j}(\boldsymbol{x},t) = 0 \end{aligned} \tag{2.6}$$

The derived equations represent a continuum model of the discrete periodic system of interacting mass-points under consideration. The basic unknowns are macrodisplacements U_i and correctors Q^{α}_i, which are sufficiently regular functions defined for every $t \in [t_0, t_f]$ in the region Ω of the reference space. Let us observe that for Q^{α}_i we have obtained ordinary differential equations $(2.6)_2$, involving exclusively time derivatives of correctors, while the macrodisplacements U_i are governed by the partial differential equations $(2.6)_1$. Hence on the boundary $\partial\Omega$ the values of U_i have to be prescribed; the alternative formulations of boundary conditions are also possible.

The formal structure of Eqs (2.6) is similar to that obtained by Woźniak (1993) for the linear-elastic periodic composite materials. Since $h^a_\alpha \in \mathcal{O}(l)$ then the values of all coefficients in the above equations which were defined in terms of Eqs (2.4), are independent of the microstructure length parameter l. Hence for the correctors Q^{α}_i we have obtained the second one from Eqs (2.6) with the first term involving the square of the microstructure length parameter l^2. This term describes the micro-inertial properties of the system considered.

Setting $l \searrow 0$ in Eqs (2.6) and assuming that all quantities defined by Eqs (2.4) are constant we arrive at what will be called asymptotic (homogenized) model of the periodic mass-point system; in this case for correctors Q^{α}_i we obtain the system of linear algebraic equations. It can be shown that the linear transformation $\mathcal{R}^{3\kappa} \to \mathcal{R}^{3\kappa}$, given by $C_{\alpha\beta ij}$ is invertible and hence in the asymptotic model Q^{α}_i can be expressed as linear functions of $U_{(k,j)}$.

Let us observe that if $B_{\alpha kij} = 0$ then from Eqs (2.6) we obtain two independent systems of equations in U_i and Q^{α}_i. Moreover, if the initial conditions for correctors have the form $Q^{\alpha}_i(\boldsymbol{x}, t_0) = 0$, $\dot{Q}^{\alpha}_i(\boldsymbol{x}, t_0) = 0$, $\boldsymbol{x} \in \Omega$, then $Q^{\alpha}_i \equiv 0$ for every $\boldsymbol{x} \in \Omega$, $t \in [t_0, t]$. The mass-point system in the case $B_{\alpha kij} = 0$ will be referred to as the micro-homogeneous structure; in any other case the lattice structures modelled by Eqs (2.6) will be called the composite (or micro-heterogeneous) structures. Thus, for micro-homogeneous structures (within a framework of the continuum models given by Eqs (2.6)) the micro-inertial effects are caused exclusively by non-trivial initial conditions for correctors.

At the end of the above conclusions it has to be emphasized that the fields U_i, Q^{α}_i satisfying Eqs (2.6) have the physical sense only if they are V-macro functions, i.e., if their oscillations in every $V(x)$, $x \in \Omega_0$, are sufficiently small to be neglected.

3. Example

As the example let us consider the wave propagation problem related to the unbounded plane lattice structure shown in Fig.2. We shall deal with the longitudinal wave propagating along x_1-axis. Assuming Eq (2.1) in the form $u_1^a = U_1 + h_1^a Q_1^1$ and setting $U \equiv U_1$, $Q \equiv Q_1^1$ and $x \equiv x_1$, from Eqs (2.6) we get

$$\begin{aligned} &\rho\ddot{U}(x,t) - A_{1111}U_{,11}(x,t) + B_{1111}Q_{,1}(x,t) = 0 \\ &l^2\Pi_{11}\ddot{Q}(x,t) + C_{1111}Q(x,t) + B_{1111}Q_{,1}(x,t) = 0 \end{aligned} \tag{3.1}$$

The micro-shape matrix h_α^a is reduced here to the vector h_1^a given by $h_1^1 = l$, $h_1^2 = -M_1 l/M_2$, $h^3 = h^4 = h_1^5 = l$. Denoting $\psi_A \equiv E_A F_A/l$, $\tilde{\rho} \equiv (M_1 + M_2)/l$ and assuming $\psi_3 = \psi_5$, $\psi_4 = \psi_6$, we obtain

$$A_{1111} = \frac{1}{12}(8\psi_1 + 16\psi_2 + 9\sqrt{3}\psi_3 + \sqrt{3}\psi_4)$$

$$B_{1111} = \frac{\sqrt{3}}{6}(4\psi_2 - 4\psi_1 + \psi_4)\Big(1 + \frac{M_1}{M_2}\Big)$$

$$C_{1111} = \frac{1}{3}(6\psi_1 + 3\psi_2 + \sqrt{3}\psi_4)\Big(1 + \frac{M_1}{M_2}\Big)^2$$

$$\rho = \frac{2\tilde{\rho}}{\sqrt{3}} \qquad\qquad \Pi_{11} = \frac{2\tilde{\rho}}{\sqrt{3}}\frac{M_1}{M_2}$$

The solution to Eqs (3.1) will be assumed in the form

$$U = C_1 \sin kx \cos(\omega t) \qquad\qquad Q = C_2 \cos kx \cos(\omega t) \tag{3.2}$$

where C_1, C_2 are arbitrary constants. Substituting Eqs (3.2) into Eqs (3.1) we obtain non-trivial solutions for C_1, C_2 only if

$$\begin{vmatrix} Ak^2 - \rho\omega^2 & kB \\ kB & C - l^2\Pi\omega^2 \end{vmatrix} = 0 \tag{3.3}$$

where for the sake of simplicity we have denoted $A \equiv A_{1111}$, $B \equiv B_{1111}$, $C \equiv C_{1111}$, $\Pi \equiv \Pi_{11}$. After simple calculations from Eq (3.3) we obtain the following formulae for the frequencies ω^2

$$(\omega_1)^2 = \frac{AC - B^2}{\rho C} k^2 \Big[1 - \frac{\Pi B^2}{\rho C^2}(kl)^2\Big] + \mathcal{O}\Big((kl)^4\Big)$$
$$(\omega_2)^2 = \frac{C}{l^2 \Pi} + \frac{B^2}{\rho C} k^2 \Big[1 + \frac{\Pi(AC - B^2)}{\rho C^2}(kl)^2\Big] + \mathcal{O}\Big((kl)^4\Big) \tag{3.4}$$

The formulae (3.4) have the physical sense only if macrodisplacements $U(x,t)$ and corrector field $Q(x,t)$ are V-macro functions, where now $V = (0, \sqrt{3}l) \times (0, l)$. By means of Eqs (3.2) this condition holds if kl is small compared to 1. Since the terms $\mathcal{O}(l^2)$ are retained in Eqs (2.6) then the above condition can be assumed in the form $1 + \mathcal{O}\Big((kl)^4\Big) \cong 1$. Thus we arrive at the conclusion that the frequencies ω_1, ω_2 can be represented in the explicit form which under denotations $\Omega \equiv \omega l$, $q \equiv kl$ is given by

$$(\Omega_1)^2 = \frac{AC - B^2}{\rho C} q^2 - \frac{\Pi B^2(AC - B^2)}{\rho^2 C^3} q^4 + \mathcal{O}(q^6)$$
$$(\Omega_2)^2 = \frac{C}{\Pi} + \frac{B^2}{\rho C} q^2 + \frac{\Pi B^2(AC - B^2)}{\rho^2 C^3} q^4 + \mathcal{O}(q^6)$$

It can be seen that the dispersion effect in the formula for Ω_1 as well as the higher frequency Ω_2 are caused by the presence in Eqs (2.6) of terms involving the microstructure parameter l. This situation cannot be described within a framework of the asymptotic (homogenized) continuum model of the mass-point systems under consideration, which does not involve the time derivatives of correctors and can be obtained from Eqs (2.6) by rescaling the microstructure down, using the limit passage $l \searrow 0$ and assuming that ρ, A_{ijkl}, $B_{\alpha ijkl}$, $C_{\alpha\beta ij}$ are constants under this rescaling.

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Ciągłe modele dynamiki pewnych kompozytowych struktur siatkowych

Streszczenie

Celem opracowania jest przedstawienie ciągłego modelu periodycznych liniowo-sprężystych struktur siatkowych, uwzględniającego wpływ wielkości elementu strukturalnego na dynamikę układu. Proponowane podejście korzysta z założeń dotyczących rozszerzonej makro-dynamiki materiałów kompozytowych, przedstawionych w pracy Woźniak (1993).

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