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THERMOELASTICITY OF FIBRE COMPOSITES

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This paper presents the complete theory of coupled thermoelasticity of fibre composites. The same displacement and temperature fields have been accepted for both phases (i.e. for matrix and fibre phase). The continuous model of a fibre composite body has been created so, that fibre phase is ideally "fuzzy field". The principles of energy conservation and principles of macroscopic thermodynamics have been used. The constitutive equations of fibre composite have been obtained as well as displacement equations and the conductivity equation. The equations obtained, include anisotropic terms, while the parameters of anisotropy of continuum are given explicitly. The paper is supplied with an example concerning heat conductivity in a finite space reinforced with fibres, which are parallel to the axis x_1.

1. Introduction

We will consider a body consisting of a matrix and any number of fibre families. The matrix is an elastic and homogeneous body. The fibres of every family create a family of parallel straight lines, uniformly distributed in space, with such density, that they can approximately be treated as the specific continuous medium, in which only normal stresses can be trought along the fibres direction. The fibre phase, containing all fibre families, is immersed in the matrix. So, one could say, that every point of geometrical space is refered to two material points, one of which belongs to the matrix, while the other one – to the fibre phase. As the consequence of this, we have: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^I + \boldsymbol{\sigma}^{II}$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^I = \boldsymbol{\varepsilon}^{II}$, where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ is the strain tensor; upper index I or II defines the attachment to the phase I (matrix) or II (fibres). The above description is in line with the concept of biphase medium, which was presented by Holnicki-Szulc (1990). This concept created the base for the works by Świtka (1992) and (1993).

The method, which has been used in the present study differs from methods known in mechanics of composites, which are based on the substitution for the composite medium the uniform one, showing the substitute characteristics (cf Christensen (1979), Jones (1975), Vinson and Sierakowski (1986)). Presented theory takes into account the structure of fibre composite when formulating the constitutive equations and it allows one to define explicitly all the characteristics.

This work systematizes the notion and formal details, which are related to mechanics of fibre composites. The interactive relations between tensor fields of stresses and strains, a vector field of displacements and a scalar field of temperature will be defined here. The constitutive relations and the conductivity equation have been defined basing on the laws of thermodynamics (cf Fung (1965), Nowacki (1970) and (1978)).

2. Mutual relations referring to geometry and the structure of fibre composite

We will consider a fibre composite reinforced with one family of fibres. We will define the field elements on a plane, which is vertical to s_r ($|s_r| = 1$) vector, when such a vector defines fibres direction in rth family

$dA^{(s)}$ – total field element

$dA_m^{(s)}$ – part of field element, which is refered to the matrix

$dA_r^{(s)}$ – part of field element, which is refered to the fibre of rth family, perpendicular to the $dA^{(s)}$ element

A_r – single fibre cross-section area.

The index r, which is the number of fibre family and the index m, which defines attachment to the matrix, in contrast to other indexes, will be excluded from the convention of summation.

It will now be convenient to introduce a quantity denoted as the density of rth family of fibres and defined as follows

$$\mu_r = \frac{dA_r^{(s)}}{dA^{(s)}} \tag{2.1}$$

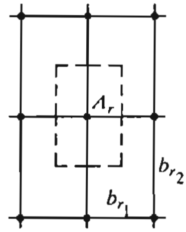

Fig. 1.

If the traces of fibres create a rectangular grid on the perpendicular plane $\boldsymbol{s}_r$ with netting dimensions being equal to b_{r_1}, b_{r_2} (see Fig.1), then

$$\mu_r = \frac{A_r}{b_{r_1} b_{r_2}} \tag{2.2}$$

So, we have

$$dA_r^{(s)} = \mu_r dA^{(s)} \qquad dA_m^{(s)} = (1 - \mu_r) dA^{(s)} \tag{2.3}$$

The number of fibres per one field element $dA^{(s)}$ can be calculated from the equation $l_r / dA^{(s)} = 1/(b_{r_1} b_{r_2})$, so

$$l_r = \frac{1}{b_{r_1} b_{r_2}} dA^{(s)} = \frac{\mu_r}{A_r} dA^{(s)} \tag{2.4}$$

We will denote the force in a single fibre as Z_r.

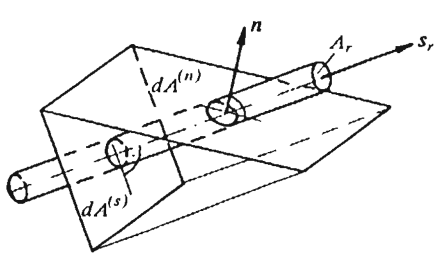

Fig. 2.

Then we will denote field elements on a plane, which is perpendicular to a vector $\boldsymbol{n}$ (cf Fig.2). Let us assume, that

$dA^{(n)}$ – total field element on a plane oriented according to the $\boldsymbol{n}$ normal

$dA_m^{(n)}$ – part of field element, which refers to the matrix

$dA_r^{(n)}$ – part of field element, which refers to the fibres of rth family

$dA_a^{(n)}$ – part of field element, which refers to the total reinforcement $dA_a^{(n)} = \sum\limits_r dA_r^{(n)}$.

The relation between $dA^{(n)}$ and $dA^{(s)}$ is the following (see Fig.2)

$$dA^{(s)} = dA^{(n)}\boldsymbol{n}\boldsymbol{s}_r = dA^{(n)}n_i s_{r_i} \tag{2.5}$$

In a similar way

$$dA_m^{(s)} = dA_m^{(n)}n_i s_{r_i} \qquad dA_r^{(s)} = dA_r^{(n)}n_i s_{r_i} \tag{2.6}$$

Now it is easy to calculate

$$dA_m^{(n)} = \frac{dA_m^{(s)}}{n_i s_{r_i}} = \frac{(1-\mu_r)dA^{(s)}}{n_i s_{r_i}} = (1-\mu_r)dA^{(n)}$$

$$dA_r^{(n)} = \mu_r dA^{(n)}$$

By generalization of the above results on all families of fibres, we have

$$dA_a^{(n)} = \mu dA^{(n)} \qquad dA_m^{(n)} = (1-\mu)dA^{(n)} \qquad \mu = \sum_r \mu_r \tag{2.7}$$

3. Cauchy's formula

Let $dF^{(n)}$ be a vector of elementary resultant force, due to the stresses in $dA^{(n)}$ field element, and let $dF_m^{(n)}$ be the resultant force due to the stresses. So

$$dF^{(n)} = dF_m^{(n)} + \sum_r Z_r l_r \boldsymbol{s}_r \tag{3.1}$$

If we take into account, that $dF^{(n)}/dA^{(n)} = \boldsymbol{T}^{(n)}$ is the vector of total stress in a fibre composite, $dF_m^{(n)}/dA_m^{(n)} = \boldsymbol{S}^{(n)}$ is the vector of total stress in a matrix and if we take into account Eqs (2.2), (2.4) and (2.7), then

$$T_i^{(n)} = (1-\mu)S_i^{(n)} + \sum_r \mu_r \sigma_r n_j s_{r_j} s_{r_i} \tag{3.2}$$

where $\sigma_r = Z_r/A_r$ is the stress in a fibre of rth family. Then, by putting $T_j^{(i)} = \tau_{ij}$ (the tensor of stress in fibre composite), $S_j^{(i)} = \sigma_{ij}$ (the tensor of stress in matrix), we obtain

$$\tau_{ij} = (1-\mu)\sigma_{ij} + \sum_r \mu_r \sigma_r s_{r_i} s_{r_j} \tag{3.3}$$

The Cauchy's formula refers to total stresses

$$T_j^{(n)} = \tau_{ij} n_i \tag{3.4}$$

*After substitution of Eqs (3.2) and (3.3) into Eq (3.4) we find, that the Cau-*chy's formula is fully transferred to the matrix

$$S_j^{(n)} = \sigma_{ij} n_i \tag{3.5}$$

When defining the boundary conditions imposed on stresses, the following should be assumed

$$\tau_{ij} n_i = p_j \tag{3.6}$$

where $\boldsymbol{p}$ is the vector field of loads, defined on the plane, which bounds a fibre composite body.

4. The principle of conservation of energy and the first law of thermodynamics

Total stresses appear in equations of equilibrium

$$\tau_{ji,j} + \rho f_i = \rho \dot{v}_i \tag{4.1}$$

where
- $\boldsymbol{f}$ – vector of mass forces
- $\boldsymbol{v}$ – velocity vector
- ρ – density of fibre composite

$$\rho = (1-\mu)\rho_m + \sum_r \mu_r \rho_r \tag{4.2}$$

- ρ_m – matrix density
- ρ_r – density of fibre material of rth family.

Eq (4.1) is multiplied by v_i and integrated over the $B_1 \subset B$ region, where B is a region occupied by fibre composite, B_1 is any subregion of B region, bounded with smooth closed surface Ω_1. The field element on Ω_1 is oriented in accordance with $\boldsymbol{n}$ normal line, that is $dA^{(n)} \equiv d\Omega$. It should be taken into account, that

$$\begin{aligned}\iiint\limits_{B_1} \tau_{ji,j} v_i \, dV &= \iiint\limits_{B_1} (\tau_{ji} v_i)_{,j} \, dV - \iiint\limits_{B_1} \tau_{ji} v_{i,j} \, dV = \\ &= \iint\limits_{\Omega_1} T_i^{(n)} v_i \, d\Omega - \iiint\limits_{B_1} \tau_{ji} \dot{\varepsilon}_{ij} \, dV\end{aligned}$$

As the result, we obtain

$$\iint\limits_{\Omega_1} T_i^{(n)} v_i \, d\Omega - \iiint\limits_{B_1} \tau_{ij} \dot{\varepsilon}_{ij} \, dV + \iiint\limits_{B_1} \rho f_i v_i \, dV = \iiint\limits_{B_1} \rho \dot{v}_i v_i \, dV \tag{4.3}$$

so

$$\mathcal{L} = \frac{d}{dt}(\mathcal{K} + \mathcal{U}) \tag{4.4}$$

where

$$\mathcal{L} = \iint\limits_{\Omega_1} T_i^{(n)} v_i \, d\Omega + \iiint\limits_{B_1} \rho f_i v_i \, dV \tag{4.5}$$

is a fibre composite strain power

$$\mathcal{K} = \frac{1}{2} \iiint\limits_{B_1} \rho v_i v_i \, dV \tag{4.6}$$

is a kinematic energy and

$$\mathcal{U} = \frac{1}{2} \iiint\limits_{B_1} \tau_{ij} \varepsilon_{ij} \, dV \tag{4.7}$$

is an elastic energy of fibre composite occupying the B_1 subregion.

The principle of mechanical energy conservation (4.4) works in every B_1 subregion, so for B region as well. In such case, one should take into account, that $B_1 = B$ and $\Omega_1 = \Omega$.

According to the first law of thermodynamics we obtain then

$$\frac{d}{dt}(\mathcal{U} + \mathcal{K}) = \mathcal{L} + \frac{dQ}{dt} \tag{4.8}$$

where Q is the heat taken from the neighbourhood.

The following specification is made to enable calculation of dQ/dt. Let $\dot{\boldsymbol{q}}$ be the vector of heat flux density in a matrix [W/m^2], while $\dot{q}_r$ is the heat flux density along fibres of rth family. The vector of fibre heat flux density will be equal to $\dot{q}_r \boldsymbol{s}_r$. So, we have

$$\begin{aligned} \frac{dQ}{dt} &= -\iint\limits_{\Omega_1} \dot{q}_i n_i \, dA_m^{(n)} - \sum_r \iint\limits_{\Omega_1} \dot{q}_r s_{r_i} n_i \, dA_r^{(n)} = \\ &= (1-\mu) \iiint\limits_{B_1} \dot{q}_{i,i} \, dV - \sum_r \mu_r \iiint\limits_{B_1} \left(\dot{q}_r s_{r_i}\right)_{,i} dV \end{aligned} \tag{4.9}$$

Let us return to the strain power. According to Eq (4.5) we obtain

$$\begin{aligned} \mathcal{L} &= \iint\limits_{\Omega_1} \tau_{ji} n_j v_i \, d\Omega + \iiint\limits_{B_1} \rho f_i v_i \, dV = \iiint\limits_{B_1} \left[(\tau_{ji} v_i)_{,j} + \rho f_i v_i\right] dV = \\ &= \iiint\limits_{B_1} (\tau_{ji,j} v_i + \tau_{ij} v_{i,j} + \rho f_i v_i) \, dV \end{aligned} \tag{4.10}$$

and in line with the energy conservation law, we can write as follows

$$\begin{aligned} &\iiint\limits_{B_1} \dot{U} \, dV + \iiint\limits_{B_1} \rho \dot{v}_i v_i \, dV = \iiint\limits_{B_1} \left[(\tau_{ji,j} + \rho f_i) v_i + \tau_{ij} \dot{\varepsilon}_{ij}\right] dV + \\ &-(1-\mu) \iiint\limits_{B_1} \dot{q}_{i,i} \, dV - \sum_r \mu_r \iiint\limits_{B_1} (\dot{q}_r s_{r_i})_{,i} \, dV \end{aligned} \tag{4.11}$$

where U is the specific elastic energy (referred to the unit of volume). As Eq (4.11) must be met for every B_1, so

$$\dot{U} = \tau_{ij} \dot{\varepsilon}_{ij} - (1-\mu) \dot{q}_{i,i} - \sum_r \mu_r (\dot{q}_r s_{r_i})_{,i} \tag{4.12}$$

or

$$\dot{U} = (1-\mu) \sigma_{ij} \dot{\varepsilon}_{ij} + \sum_r \mu_r \sigma_r \dot{\varepsilon}_r - (1-\mu) \dot{q}_{i,i} - \sum_r \mu_r (\dot{q}_r s_{r_i})_{,i} \tag{4.13}$$

5. Second law of thermodynamics

We will denote entropy of the system with S^* and the specific entropy with S, i.e.

$$S^* = \iiint\limits_{B_1} S \, dV \tag{5.1}$$

Due to the fact, that

$$T dS^* = dQ \tag{5.2}$$

where T is thermodynamic temperature, so

$$T \iiint\limits_{B_1} \frac{dS}{dt}\, dV = -(1-\mu) \iiint\limits_{B_1} \dot{q}_{i,i}\, dV - \sum_r \mu_r \iiint\limits_{B_1} (\dot{q}_r s_{r_i})_{,i}\, dV$$

and from this

$$\dot{S}T = -(1-\mu)\dot{q}_{i,i} - \sum_r \mu_r (\dot{q}_r s_{r_i})_{,i} \tag{5.3}$$

which is the equation of entropy balance.

Eq (5.3) can be rewritten in the following way

$$\begin{aligned} \dot{S} &= -(1-\mu)\frac{\dot{q}_{i,i}}{T} - \sum_r \mu_r \frac{(\dot{q}_r s_{r_i})_{,i}}{T} = \\ &= -(1-\mu)\Big[\Big(\frac{\dot{q}_i}{T}\Big)_{,i} + \frac{\dot{q}_i T_{,i}}{T^2}\Big] - \sum_r \mu_r \Big[\Big(\frac{\dot{q}_r s_{r_i}}{T}\Big)_{,i} + \frac{\dot{q}_r s_{r_i} T_{,i}}{T^2}\Big] \end{aligned}$$

The above expression is subject to integration over B_1 region to enable calculation of the velocity of entropy variations for any subregion. After employing the Gauss-Ostrogradzki theorem, the following is obtained

$$\begin{aligned} \frac{dS^*}{dt} &= -\iint\limits_{\Omega_1} \frac{(1-\mu)\dot{q}_i n_i + \sum_r \mu_r \dot{q}_r s_{r_i} n_i}{T}\, d\Omega + \\ &- \iiint\limits_{B_1} \frac{(1-\mu)\dot{q}_i T_{,i} + \sum_r \mu_r \dot{q}_r s_{r_i} T_{,i}}{T^2}\, dV = S^*_{\rm ext} + S^*_{\rm int} \end{aligned} \tag{5.4}$$

The first term stands for the exchange of entropy with the neighbourhood (change of entropy by incoming heat). The second term stands for the change of entropy in the closed system and its value is non-negative

$$\dot{S}_{\rm int} = -(1-\mu)\frac{\dot{q}_i T_{,i}}{T^2} - \sum_r \mu_r \frac{\dot{q}_r s_{r_i} T_{,i}}{T^2} \geq 0 \tag{5.5}$$

Denoting

$$\Phi_i = -\frac{T_{,i}}{T^2} \qquad\qquad \Phi_r = -\frac{T_{,i} s_{r_i}}{T^2} \tag{5.6}$$

Eq (5.5) can be written

$$\dot{S}_{\rm int} = (1-\mu)\Phi_i \dot{q}_i + \sum_r \mu_r \Phi_r \dot{q}_r \tag{5.7}$$

The objects Φ_i and Φ_r are the so called thermodynamical incentives and they are related to the heat flux density by the following linear relationships

$$\dot{q}_i = L_{ij}\Phi_j \qquad\qquad \dot{q}_r = L_r\Phi_r \tag{5.8}$$

According to the Onsager principle, $L_{ij} = L_{ji}$.

Now, the Eq (5.5) can be written in the following way

$$\dot{S}_{\rm int} = (1-\mu)L_{ij}\Phi_i\Phi_j + \sum_r \mu_r L_r \Phi_r^2 \tag{5.9}$$

so

$$\dot{S}_{\rm int} = (1-\mu)L_{ij}\frac{T_{,i}}{T^2}\frac{T_{,j}}{T^2} + \sum_r \mu_r L_r \Big(\frac{T_{,i}s_{r_i}}{T^2}\Big)^2 \tag{5.10}$$

By denoting

$$\lambda_{ij} = \frac{L_{ij}}{T^2} \qquad\qquad \lambda_r = \frac{L_r}{T^2} \tag{5.11}$$

Eq (5.10) can be presented in the form

$$\dot{S}_{\rm int} = (1-\mu)\frac{\lambda_{ij}}{T^2}T_{,i}T_{,j} + \sum_r \mu_r \frac{\lambda_r}{T^2}(T_{,i}s_{r_i})^2 \tag{5.12}$$

The Fourier's laws appear after Eqs (5.5) and (5.12) are compared for matrix and fibre phase (of rth family)

$$\dot{q}_i = -\lambda_{ij}T_{,j} \qquad\qquad \dot{q}_r = -\lambda_r s_{r_i} T_{,i} \tag{5.13}$$

Eq $(5.13)_2$ can also be written in the following form

$$\dot{q}_r = -\lambda_r\frac{\partial T}{\partial s} \tag{5.14}$$

After taking Eqs (5.13) and (5.3) account, the following is obtained

$$\dot{S}T = (1-\mu)\lambda_{ij}T_{,ij} + \sum_r \mu_r\lambda_r\Big(s_{r_i}s_{r_j}T_{,j}\Big)_{,i} + \dot{q}_v \tag{5.15}$$

Eq (5.3) has been supplemented with the term $\dot{q}_v$ which represents the volumetric output of heat sources [W/m^3].

6. Constitutive equations

A thermodynamic function F, which is called the free energy is introduced

$$F = U - ST \tag{6.1}$$

After differentiation with respect to time and after Eqs (4.13) and (5.3) are used, the following is obtained

$$\dot{F} = (1-\mu)\sigma_{ij}\dot{\varepsilon}_{ij} + \sum_r \mu_r\sigma_r\dot{\varepsilon}_r - S\dot{T} \tag{6.2}$$

It can be assumed, that the specific entropy of the composite is the sum of entropies of individual phases, according to the formula

$$S = (1-\mu)S_m + \sum_r \mu_r S_r \tag{6.3}$$

It appears, that after substitution of Eq (6.3) into Eq (6.2), one can write

$$\dot{F} = (1-\mu)\dot{F}_m + \sum_r \mu_r\dot{F}_r \tag{6.4}$$

where

$$\dot{F}_m = \sigma_{ij}\dot{\varepsilon}_{ij} - S_m\dot{T} \qquad \dot{F}_r = \sigma_r\dot{\varepsilon}_r - S_r\dot{T} \tag{6.5}$$

As $F_m = F_m(\varepsilon_{ij}, T)$ and $F_r = F_r(\varepsilon_r, T)$ and dF is the total differential, then

$$\begin{aligned}\dot{F}_m &= \frac{\partial F_m}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial F_m}{\partial T}\dot{T}\\ \dot{F}_r &= \frac{\partial F_r}{\partial\varepsilon_r}\dot{\varepsilon}_r + \frac{\partial F_r}{\partial T}\dot{T}\end{aligned} \tag{6.6}$$

After comparing (6.5) and (6.6) it appears, that

$$\begin{aligned}\sigma_{ij} &= \frac{\partial F_m}{\partial\varepsilon_{ij}} \qquad & S_m &= -\frac{\partial F_m}{\partial T}\\ \sigma_r &= \frac{\partial F_r}{\partial\varepsilon_r} \qquad & S_r &= -\frac{\partial F_r}{\partial T}\end{aligned} \tag{6.7}$$

The functions F_m and F_r are expanded into the Taylor's series in a neighbourhood of the point of thermodynamical equilibrium (natural conditions)

$$
\begin{aligned}
F_m(\varepsilon_{ij},T) &= F_m(0,T_0) + \frac{\partial F_m(0,T_0)}{\partial \varepsilon_{ij}}\varepsilon_{ij} + \frac{\partial F_m(0,T_0)}{\partial T}(T-T_0) + \\
&+ \frac{1}{2}\Big[\frac{\partial^2 F_m(0,T_0)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}}\varepsilon_{ij}\varepsilon_{kl} + 2\frac{\partial^2 F_m(0,T_0)}{\partial \varepsilon_{ij}\partial T}\varepsilon_{ij}(T-T_0) + \\
&+ \frac{\partial^2 F_m(0,T_0)}{\partial T^2}(T-T_0)^2\Big] + \dots \\
F_r(\varepsilon_r,T) &= F_r(0,T_0) + \frac{\partial F_r(0,T_0)}{\partial \varepsilon_r}\varepsilon_r + \frac{\partial F_r(0,T_0)}{\partial T}(T-T_0) + \\
&+ \frac{1}{2}\Big[\frac{\partial^2 F_r(0,T_0)}{\partial \varepsilon_r^2}\varepsilon_r^2 + 2\frac{\partial^2 F_r(0,T_0)}{\partial \varepsilon_r\partial T}\varepsilon_r(T-T_0) + \\
&+ \frac{\partial^2 F_r(0,T_0)}{\partial T^2}(T-T_0)^2\Big] + \dots
\end{aligned}
\tag{6.8}
$$

It should be taken into account, that $F_m(0,T_0)$ =const., $F_r(0,T_0)$ =const. (free energy in natural conditions); $\partial F_m(0,T_0)/\partial\varepsilon_{ij} = \sigma_{ij}(0,T_0) = 0$, $\partial F_r(0,T_0)/\partial\varepsilon_r = \sigma_r(0,T_0) = 0$, $\partial F_m(0,T_0)/\partial T = -S_m(0,T_0)$ =const., $\partial F_r(0,T_0)/\partial T = -S_r(0,T_0)$ =const. (entropy in natural conditions). After introduction of the following notations

$$
\begin{aligned}
&\frac{\partial^2 F_m(0,T_0)}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = C_{ijkl} \qquad && \frac{\partial^2 F_m(0,T_0)}{\partial \varepsilon_{ij}\partial T} = -\beta_{ij} \\
&\frac{\partial^2 F_r(0,T_0)}{\partial \varepsilon_r^2} = E_r && \frac{\partial^2 F_r(0,T_0)}{\partial \varepsilon_r\partial T} = -\beta_r
\end{aligned}
\tag{6.9}
$$

and having in mind Eq $(6.7)_{1,3}$ the from Eq (6.8) one can obtain

$$
\begin{aligned}
&\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\Theta \\
&\sigma_r = E_r\varepsilon_r - \beta_r\Theta \qquad\qquad \Theta = T - T_0
\end{aligned}
\tag{6.10}
$$

We will formulate the constitutive equations of fibre composite for the isotropic matrix

$$
\begin{aligned}
&C_{ijkl} = \lambda_L\delta_{ij}\delta_{kl} + \mu_L(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \\
&\beta_{ij} = \beta\delta_{ij}
\end{aligned}
\tag{6.11}
$$

where μ_L and λ_L are the Lame's elasticity constants referring to the matrix. When we include Eq (3.3), we finally have

$$\begin{aligned}\tau_{ij} &= \Big[2(1-\mu)\mu_L\delta_{ik}\delta_{jl} + \sum_r \mu_r E_r s_{r_i} s_{r_j} s_{r_k} s_{r_l}\Big]\varepsilon_{kl} + \\ &+ (1-\mu)\lambda_L e\delta_{ij} - \Big[(1-\mu)\beta\delta_{ij} + \sum_r \mu_r\beta_r s_{r_i} s_{r_j}\Big]\Theta\end{aligned} \tag{6.12}$$

where $e = \varepsilon_{kk}$ stands for the dilatation.

7. Conductivity equations

The following equations are obtained from Eqs $(6.7)_{2,4}$ and (6.8)

$$\begin{aligned}S_m &= \beta_{ij}\varepsilon_{ij} - \frac{\partial^2 F_m(0,T_0)}{\partial T^2}(T-T_0) + \text{const} \\ S_r &= \beta_r\varepsilon_r - \frac{\partial^2 F_r(0,T_0)}{\partial T^2}(T-T_0) + \text{const}\end{aligned} \tag{7.1}$$

and then from Eq (6.3)

$$\dot{S} = (1-\mu)\beta_{ij}\dot{\varepsilon}_{ij} + \sum_r \mu_r\beta_r\dot{\varepsilon}_r - \Big[(1-\mu)\frac{\partial^2 F_m(0,T_0)}{\partial T^2} + \sum_r \mu_r\frac{\partial^2 F_r(0,T_0)}{\partial T^2}\Big]\dot{T} \tag{7.2}$$

On the other hand, dS is a total differential, so

$$\dot{S} = (1-\mu)\Big[\Big(\frac{\partial S_m}{\partial\varepsilon_{ij}}\Big)_T\dot{\varepsilon}_{ij} + \Big(\frac{\partial S_m}{\partial T}\Big)_{\varepsilon_{ij}}\dot{T}\Big] + \sum_r \mu_r\Big[\Big(\frac{\partial S_r}{\partial\varepsilon_r}\Big)_T\dot{\varepsilon}_r + \Big(\frac{\partial S_r}{\partial T}\Big)_\varepsilon\dot{T}\Big] \tag{7.3}$$

$T(\partial S/\partial T)_\varepsilon$ represents the heat quantity in the unit volume, which is produced due to the unit change of temperature at constant strain. So it can be written, that

$$T\Big(\frac{\partial S_m}{\partial T}\Big)_{\varepsilon_{ij}} = c_{m_\varepsilon}\rho_m \qquad\qquad T\Big(\frac{\partial S_r}{\partial T}\Big)_\varepsilon = c_{r_\varepsilon}\rho_r$$

where c_{m_ε} and c_{r_ε} is the specific heat at constant strain, for matrix and rth family fibres materials, respectively.

Comparing Eqs (7.2) and (7.3) allows one to write the formula for $(\partial S/\partial\varepsilon)_T$

$$\Big(\frac{\partial S_m}{\partial\varepsilon_{ij}}\Big)_T = \beta_{ij} \qquad\qquad \Big(\frac{\partial S_r}{\partial\varepsilon_r}\Big)_T = \beta_r$$

The following result is obtained

$$\dot{S}T = (1-\mu)\beta_{ij}\dot{\varepsilon}_{ij}T + \sum_r \mu_r\beta_r\dot{\varepsilon}_r T + c_\varepsilon\rho\dot{T} \tag{7.4}$$

where

$$c_\varepsilon\rho = (1-\mu)c_{m_\varepsilon}\rho_m + \sum_r \mu_r c_{r_\varepsilon}\rho_r \tag{7.5}$$

If an isotropic matrix material is assumed ($\lambda_{ij} = \lambda\delta_{ij}$, $\beta_{ij} = \beta\delta_{ij}$) as well as a straight line fibre paths $\Big[(s_{r_i}s_{r_j})_{,i} = 0\Big]$, then after Eqs (5.15) and (7.4) are compared, the fibre composite conductivity equation is obtained

$$\Big[(1-\mu)\lambda\delta_{ij} + \sum_r \mu_r\lambda_r s_{r_i}s_{r_j}\Big]T_{,ij} - c_\varepsilon\rho\dot{T} - \Big[(1-\mu)\beta\delta_{ij} + \sum_r \mu_r\beta_r s_{r_i}s_{r_j}\Big]\dot{\varepsilon}_{ij}T = -\dot{q}_v \tag{7.6}$$

For very small variations of temperature, such that $|\Theta/T_0| \ll 1$, the following can be assumed

$$T = T_0 + \Theta = T_0\Big(1 + \frac{\Theta}{T_0}\Big) \sim T_0$$

and Eq (7.6) is linearized

$$\begin{aligned} &\Big[(1-\mu)\lambda\delta_{ij} + \sum_r \mu_r\lambda_r s_{r_i}s_{r_j}\Big]\Theta_{,ij} - c_\varepsilon\rho\dot{\Theta} + \\ &-\Big[(1-\mu)\beta\delta_{ij} + \sum_r \mu_r\beta_r s_{r_i}s_{r_j}\Big]\dot{\varepsilon}_{ij}T_0 = -\dot{q}_v \end{aligned} \tag{7.7}$$

8. Displacement equations

The following displacement equations are obtained from the equilibrium equations (4.1), the geometrical equations $\varepsilon_{ij} = (u_{i,j}+u_{j,i})/2$ and the physical equations (6.12)

$$\begin{aligned} &(1-\mu)\Big[\mu_L\nabla^2 u_i + (\lambda_L + \mu_L)e_{,i} - \beta\Theta_{,i}\Big] + \\ &+\sum_r \mu_r\Big(E_r s_{r_i}s_{r_j}s_{r_k}s_{r_l}u_{j,kl} - \beta_r s_{r_i}s_{r_j}\Theta_{,j}\Big) + \rho f_i = \rho\ddot{u}_i \end{aligned} \tag{8.1}$$

where $e = u_{k,k}$.

9. Example

The example given below illustrates the influence of fibre phase on the heat conductivity in a solid body (in the fibre composite). Let us assume a solid body, which occupies unlimited space, and is reinforced with the fibres being parallel to the x_1 axis, and distributed uniformly with the density equal to μ_1. In the origin of coordinate system there is the point-type heat source, activated immediately at $t = 0$ and then operating constantly in time. The coupling with the field of strain velocities is neglected in Eq (7.7). The problem stated above can be described by the following equation

$$\lambda_1\Theta_{,11} + \lambda_2\Theta_{,22} + \lambda_2\Theta_{,33} - c_\varepsilon\rho\dot{\Theta} = -\dot{q}_0\delta(x_1)\delta(x_2)\delta(x_3)H(t) \tag{9.1}$$

where

$$\lambda_1 = (1-\mu_1)\lambda + \mu_1\lambda_1 \qquad \lambda_2 = (1-\mu_1)\lambda \qquad \lambda_1 > \lambda_2 \tag{9.2}$$

and

$\dot{q}_0$ – output of the source, [W]

$\delta(x)$ – Dirac function

$H(t)$ – Heaviside function.

We will transform the Cartesian coordinate system (x_1, x_2, x_3) to the Cartesian system (y_1, y_2, y_3) according to the formulas

$$y_1 = x_1 \qquad y_2 = \sqrt{\frac{\lambda_1}{\lambda_2}}x_2 \qquad y_3 = \sqrt{\frac{\lambda_1}{\lambda_2}}x_3 \tag{9.3}$$

Eq (9.1) is transformed into the following form

$$\lambda_1\Big(\frac{\partial^2\Theta}{\partial y_1^2} + \frac{\partial^2\Theta}{\partial y_2^2} + \frac{\partial^2\Theta}{\partial y_3^2}\Big) - c_\varepsilon\rho\frac{\partial\Theta}{\partial t} = -\dot{q}_0\delta(y_1)\delta(y_2)\delta(y_3)H(t)$$

which is characteristic of the symmetry with the reference to point. So, it is convenient to make change into the spatial polar coordinates

$$\nabla^2\Theta - \frac{1}{a}\dot{\Theta} = -\frac{\dot{q}_0}{\lambda_1}\delta(R)H(t) \tag{9.4}$$

where

$$a = \frac{\lambda}{c_\varepsilon\rho} \tag{9.5}$$

is a coefficient of temperature equalization (cf Gdula (1984)) expressed in $[\mathrm{m}^2/\mathrm{s}]$ and the unit of λ_1 is [W/mK].

The solution to Eq (9.4) with the initial condition $\Theta(R,0) = 0$ is known (cf Nowacki (1960))

$$\Theta = \frac{\dot{q}_0}{4\pi\lambda_1 R}\mathrm{erfc}\Big(\frac{R}{2\sqrt{at}}\Big) \tag{9.6}$$

erfc(z) is an error cofunction

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$$

and

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\, dt$$

$$R = \sqrt{y_1^2 + y_2^2 + y_3^2} = \sqrt{x_1^2 + \frac{\lambda_1}{\lambda_2}x_2^2 + \frac{\lambda_1}{\lambda_2}x_3^2} \tag{9.7}$$

With $t \to \infty$ the stationary temperature field is obtained

$$\Theta(R,\infty) = \frac{\dot{q}_0}{4\pi\lambda_1}\frac{1}{R} \tag{9.8}$$

for $\mathrm{erfc}(0) = 1$.

Isothermal surface for $\Theta = \Theta_1$ is described by the following equation

$$\frac{\lambda_2}{\lambda_1}x_1^2 + x_2^2 + x_3^2 = \frac{\lambda_2}{\lambda_1}\Big(\frac{\dot{q}_0}{4\pi\lambda_1\Theta_1}\Big)^2 \tag{9.9}$$

The isothermal surface is an ellipsoid, which cuts-off the

$$a = \sqrt{\frac{\lambda_2}{\lambda_1}\frac{\dot{q}_0}{4\pi\lambda_1\Theta_1}}$$

segments on the axes x_2 and x_3, while it cuts-off the $a\sqrt{\lambda_1/\lambda_2}$ segment on the x_1 axis, where $(\lambda_1/\lambda_2) > 1$.

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Termosprężystość włóknokompozytów

Streszczenie

Praca zawiera zamkniętą teorię sprzężonej termosprężystości włóknokompozytów. Dla obu faz (tzn. dla matrycy i fazy włóknistej) przyjęto wspólne pola przemieszczeń i temperatury. Zbudowano ciągły model ośrodka idealnie "rozmywając" fazę włóknistą. Wykorzystano prawa zachowania energii oraz prawa termodynamiki fenomenologicznej. Otrzymano równania konstytutywne włóknokompozytu, równania przemieszczeniowe i równanie przewodnictwa. Otrzymane równania zawierają rozbudowane człony anizotropowe, przy czym parametry anizotropii ośrodka dane są explicite. Pracę kończy przykład dotyczący przewodnictwa ciepła w przestrzeni nieograniczonej zbrojonej włóknami równoległymi do osi x_1.

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