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LENGTH-SCALE EFFECTS IN WAVE PROPAGATION AND STABILITY OF ELASTIC COMPOSITES UNDER FINITE DEFORMATIONS

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A new micromodelling approach to micro-periodic highly-elastic inhomogeneous materials is proposed. The model equations obtained describe an effect of the microstructure length paramater on dynamic behaviour of the body. Within linearized approximation they reduce to a certain generalization of the refined macrodynamics equations proposed by Woźniak (1993). The main aim of the contribution is to show how the obtained model can be applied to the analysis of wave propagation and stability within the range of finite deformations of elastic composite materials under consideration.

1. Introduction

The matter under consideration consists in composite materials which in a certain configuration (e.g., in a natural state) have a micro-periodic structure, i.e., are composed of many small repeated cells which form a certain new material. As it is known, the overall (apparent) properties of a composite material are quite different than those of its constituents. Following Jones (1987) by macromechanics we shall understand the study of composite materials where the effects of constituents are detected only as averaged apparent properties of the composite. On the other hand, the study of composites wherein the interactions of constituents are examined in detail is referred to as micromechanics. In this contribution the emphasis is placed mainly on macromechanics of highly-elastic composite materials and structures. The derivation of overall properties from the material structure of a single representative volume element (unit cell) is known as *the micromechanical approach* to macromechanics.

In contemporary research into composite mechanics a number of micromechanical approaches were formulated, leading to different mathematical models of periodic heterogeneous materials and structures. Models of this kind are often referred to as *the macro-models.* In order to interrelate results of a macro-modelling procedure proposed in this paper with the existing macro-models, some main trends in formulation of averaged theories for macromechanics of periodic composites will be outlined below.

Generally speaking, the known micromechanical approaches to macromechanics can be separated into main groups:

- *General approaches*, in which there are no *a priori* restrictions imposed on distributions of constituents within the representative volume element of a micro-periodic material structure
- *Special approaches*, developed independently for various types of composite materials such as laminated composites, composites with long and short fibers and for solids with periodic distribution of inclusions or cavities of various shapes (i.e., for particulate composites).

Obviously, the general micromechanical approaches have the practical meaning provided that the resulting macro-models can be applied to the investigations of special types of composites. We have stated above that every micromechanical approach leads to a certain macro-model of composite. In this contribution two kinds of macro-models will be discussed:

- *Length-scale macro-models*, where the above effect of length dimensions of the representative volume element on overall properties of a composite is taken into account. Following Woźniak (1993), the length-scale macro-models investigated in this paper can be referred to as the *refined macro-models*
- *Local macro-models*, which are invariant under arbitrary rescaling of the unit cell, i.e., where the above effect is neglected.

So far, the main efforts to construct the macro-models for periodic composites were concentrated on special micromechanical approaches to macromechanics. The overview of different local models can be found in the textbook by Jones (1976); for composites with long and short fibers (cf Aboudi (1991)). The special length-scale macro-models were developed for laminated elastic composites by Sun et al. (1968), Achenbach and Hermann (1968), Grot and Achenbach (1970); for investigations into dynamics of fiber-reinforced composites (cf Aboudi (1981); Tolf (1983)) and for media with voids (cf Nunciato and Cowin (1979)). The list of references on this subject is rather extensive and will be not discussed here.

Among the general micromechanical approaches to macromechanics we can mention those based on the asymptotic homogenization approach (cf Bensoussan et al. (1987); Sanchez-Palencia and Zaoui (1985)) and the extensive list of references therein. The resulting macro-models are described by equations involving constant coefficients (called the effective modulae) and time dependent functions (for nonstationary processes). These mathematical objects have to be determined independently for every periodic structure by obtaining solutions to certain variational problems posed on the periodicity cell as well as certain initial value problems for materials with memory (e.g., for visco-elastic materials). Hence, the formulation of macro-models by the asymptotic approach, as a rule, is restricted only to the first approximation. Within this approximation we deal with the local macro-models, in which the effect of size of the periodicity cell on the body behaviour is neglected. An alternative general micromodelling approach, also leading to the local macro-models, was discussed by Nemat-Nasser and Hori (1993) resulting in a concept of homogeneous equivalent body. To describe length-scale phenomena (i.e., to formulate the length-scale macro-models) by the asymptotic homogenization approach, the higher steps in the formal asymptotic procedure have to be considered. Due to serious difficulties at the stage of formulation of governing equations of macromechanics for a selected composite body the above line of procedure is not accepted by most of researchers interested mainly in engineering applications of the resulting theories. Free from this drawback are general macro-modelling methods, based on theories of material continua with microstructure suggested by Mindlin (1964), Eringen and Suhubi (1964) and others. Models of this kind are called the microstructural models and belong to the length-scale macro-models. They can be formulated without any reference to the boundary-value problems on the unit cell. The pertinent modelling procedures are specified by certain *a priori* assumptions related to the expected class of micro-deformations and certain smoothing operations. The obtained macro-models for elastic materials are governed by the systems of second-order partial differential equations for a number of fields representing macro-kinematics of a composite. Many unknown independent kinematic variables appearing result in serious difficulties dye to complicated forms of the boundary-value problems. Moreover, in microstructural models we can also deal with considerable discrepancies between the number of boundary conditions required by the mathematical structure of the theory and the number of these conditions describing the boundary interactions for composite materials from the viewpoint of engineering applications of the theory. Hence the microstructural models were successfully applied mainly to the investigations of the wave propagation in unbounded media. A certain alternative to microstructural models constitute macro-models based on the mixture and interacting continuum theories, developed by Green and Naghdi (1965), (1966), (1967), Green and Steel (1966), Steel (1967) and (1968), Bedford and Stern (1971), (1972), Hegemeier (1972), Tiersten and Jahanmir (1977) and others. They are the length-scale macro-models being often oriented towards different investigations of selected dynamic problems.

In this contribution a new general micromechanical approach to the formation of length-scale macro-models for highly-elastic periodic inhomogeneous materials is given. The idea of the proposed method is a certain alternative to that expressed by Wierzbicki et al. (1995) for nonlinear problems and previously by Woźniak (1993) within the framework of what was called the refined macrodynamics for the linear-elastic composites. The main feature of the refined macrodynamics of composite periodic materials is that a micro-inhomogeneity effect on the global dynamic behaviour of the body is described in terms of certain macro-internal variables (independent of the boundary conditions), satisfying the system of ordinary differential equations. This fact makes the refined macrodynamics a convenient tool in applications to many engineering problems since the corresponding boundary conditions have the form similar to that used in elasticity theory. Different applications of the refined macrodynamics, developed by Baron and Woźniak (1995), Matysiak and Nagórko (1995), Michalak et al. (1995), Mielczarek and Woźniak (1995), Wierzbicki (1995), Wierzbicki et al. (1995), Woźniak (1993), Woźniak et al. (1993) and (1995), were restricted mainly to the linear problems. From the formal viewpoint the field equations derived in this paper coincide with those obtained from the general procedure outlined by Woźniak (1995). However, in the author opinion, the introduced dynamic modelling assumptions and the new procedure applied here yield more evident physical interpretation of the concepts involved and the scope of their applicability. The main aim of the paper is to show how the resulting model equations can be applied to the analysis of wave propagation and stability for elastic micro-periodic inhomogeneous materials under finite deformations. The attention will be given to the microstructure length-scale effect on the composite body behaviour in investigations of the aforementioned problems. The results obtained constitute the basis for possible further attempts in this domain of composite mechanics.

Throughout the paper sub- and superscripts $\alpha, \beta, \gamma, \ldots$ run over $1,2,3$ and are related to the material coordinates of a body. Subscripts $i, j, k, \ldots$ also run over $1,2,3$ but are related to the Cartesian orthogonal coordinate system in the physical space. Non-tensorial indices $A, B, \ldots$ run over $1,2,\ldots,N$, respectively. The summation convention holds for all the aforementioned indices unless stated otherwise. Points in the reference space R^3 are denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$, t stands for a time coordinate and (x_1, x_2, x_3, t) are assumed to be inertial coordinates in the Gallilean space-time. In investigations of finite deformations, $\boldsymbol{X}$ stands for an arbitrary material point which coincides with its material coordinates $\boldsymbol{X} = (X_1, X_2, X_3)$ and a superscript R is used in order to distinguish entities related to the reference configuration.

2. Preliminaries

Let Ω_R be a regular region in the physical 3D space $\boldsymbol{R}^3$, occupied by a hyperelastic body B in a certain reference configuration $\kappa_R : B \to \boldsymbol{R}^3$. Define by $\rho_R = \rho_R(\boldsymbol{X})$, $\sigma_R = \sigma_R(\boldsymbol{X}, \boldsymbol{D})$ (where $\boldsymbol{X} \in \Omega_R$ and $\boldsymbol{D}$ is a displacement gradient) the mass density and the strain energy function, respectively, related to this configuration. Also define by $V_R = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ a representative volume element of a body, where the length parameter $l \equiv \sqrt{(l_1)^2 + (l_2)^2 + (l_3)^2}$ is assumed to be negligibly small as compared to the smallest characteristic length dimension L of Ω_R. Throughout the paper $\rho_R = \rho_R(\cdot)$ and $\sigma_R = \sigma_R(\cdot, \boldsymbol{D})$ are non-constant V_R-periodic functions; hence B is said to be a micro-periodic elastic body. Moreover, l will be called the microstructure length parameter and $\lambda := l/L$ will be referred to as the nondimensional macro-accuracy parameter. Since the subject of this paper is a certain micro-periodic elastic body then all entities mentioned above and related to this body are assumed to be known a priori. To be more exact, by a micro periodic body we shall understand the fourtuple $(B, \kappa_R, \rho_R, \sigma_R)$, where ρ_R, σ_R, satisfy the aforementioned mentioned conditions. If $\rho_R(\cdot)$, $\sigma_R(\cdot, \boldsymbol{D})$ are V_R-periodic and piecewise constant than this body is made of a certain composite material.

The general motivation to formulate different averaged models (macro-models) for micro-periodic composites was outlined in the Section 1 and will be not repeated here. The properties of the aforementioned bodies within the framework of their homogenized models are described by the functions which are explicitly independent of $\boldsymbol{X}$, $\boldsymbol{X} \in \Omega_R$; some of them can reduce to certain material constants. At the same time the governing equations defining macro-models involve certain unknown fields, which describe the process realized by a body under the known external agents and the known initial conditions. Following the terminology of the refined macrodynamics, components of these fields will be referred to as the macro-functions. The general idea of this notion was given by Woźniak (1993); here we begin considerations with more rigorous definition of a macro-function.

Let (t_0, t_f), $t_0 < t_f$, be the known time interval and $F : \Omega_R \times (t_0, t_f) \ni (\boldsymbol{X}, t) \to F(\boldsymbol{X}, t) \in \boldsymbol{R}$ be an arbitrary real-valued function which will be related to the macro-description of a certain process in a micro-periodic body. Let us assume that the values $F(\boldsymbol{X}, t)$ have to be measured or calculated up to a certain accuracy ε_F which is assumed to be known in every special problem under consideration. Denoting by F_0 a certain expected value of F (mind that in general F is not known a priori) we assume $\varepsilon_F = \lambda F_0$ and refer ε_F to as macro-accuracy of F. Moreover let $\|\boldsymbol{X} - \boldsymbol{Z}\|$ be a distance between points $\boldsymbol{X}, \boldsymbol{Z} \in \boldsymbol{R}^3$.

- *Definition 1.* A triple $(F; l, \varepsilon_F)$ is called a macro-function if condition

$$\forall \boldsymbol{Z}, \boldsymbol{X} \in \Omega_R \qquad \|\boldsymbol{Z} - \boldsymbol{X}\| < l \rightarrow |F(\boldsymbol{Z}, t) - F(\boldsymbol{X}, t)| < \varepsilon_F$$

holds for every $t \in (t_0, t_f)$.

Now assume that F is continuous and has continuous derivatives $\nabla F, \dot{F}, \ldots$, up to a certain order. Assume that the macro-accuracies: $\varepsilon_F, \varepsilon_{\nabla F}, \varepsilon_{\dot{F}}, \ldots$, related to $F, \nabla F, \dot{F}, \ldots$, respectively, are known.

- *Definition 2.* A sequence $(F; l, \varepsilon_F, \varepsilon_{\nabla F}, \varepsilon_{\dot{F}}, \ldots)$ is called a regular macro-function if for each $\Phi \in \{F, \nabla F, \dot{F}, \ldots\}$ condition

$$\forall \boldsymbol{Z}, \boldsymbol{X} \in \Omega_R \qquad \|\boldsymbol{Z} - \boldsymbol{X}\| < l \rightarrow |\Phi(\boldsymbol{Z}, t) - \Phi(\boldsymbol{X}, t)| < \varepsilon_\Phi$$

holds for every $t \in (t_0, t_f)$.

- *Corollary.* For every regular macro-function $(F; l, \varepsilon_F, \varepsilon_{\nabla_F}, \varepsilon_{\dot{F}}, \ldots)$ the condition

$$|lF_{,\alpha}(\boldsymbol{X}, t)| \leq \varepsilon_F + l\varepsilon_{\nabla F}$$

holds for every $\boldsymbol{X} \in \Omega_R$.

In the sequel, terms "macro-function F" and "regular macro-function F" will be used since the parameters $l, \varepsilon_F, \varepsilon_{\nabla F}, \varepsilon_{\dot{F}}, \ldots$ are assumed to be known in every problem under consideration.

The modelling procedure applied to the refined macrodynamics takes into account a certain kinematic hypothesis which has to be formulated for every class of motions we are going to investigate. To formulate this hypothesis the concept of a micro-shape functions system will be recalled. It is the system $h^A(\cdot)$, $A = 1,...,N$, of linear independent, V_R-periodic continuous functions, having piecewise continuous first derivatives, such that $\sup|h^A(\cdot)| = l$, $\sup|h^A{}_{,\alpha}(\cdot)| = 1$ and $\langle h^A \rangle = 0$ for $A = 1,...,N$; symbol $\langle h^A \rangle$ stands here for the averaged value of a V_R-periodic function h^A. The kinematic hypothesis specifying the class of expected displacement fields $u_i(\cdot,t)$ (related to the reference configuration) is assumed in the form

$$u_i(X,t) = U_i(X,t) + h^A(X)V_i^A(X,t) \qquad X \in \Omega_R \qquad t \in (t_0,t_f) \tag{2.1}$$

where $U_i(\cdot,t)$, $V_i^A(\cdot,t)$ are sufficiently regular macro-functions and $h^A(\cdot)$, $A = 1,...,N$, constitute a micro-shape function system, posulated in every special problem. Functions U_i and V_i^A represent the basic kinematic unknowns of the proposed macro-model and are called macro-displacements and macro-internal variables, respectively. The above formula is based on the heuristic assumption that the micro-periodic material structure of the medium under consideration leads to certain macro-disturbances in its motion and these disturbances have the form $h^A(X)V_i^A(X,t)$, $X \in \Omega_R$, for every instant t. It follows that in kinematics of micro-periodic media we are to investigate only some classes of the aforementioned disturbances. More detailed discussion of the kinematic hypothesis (2.1) is given by Woźniak (1993) and can be also found in the series of papers on the refined macrodynamics, mentioned in Section 1.

At the end of these preliminaries some auxiliary concepts will be introduced. Setting $V_R(Z) := Z + V_R$ for every $Z \in R^3$, let us define $\Omega_R^0 := \{X \in \Omega_R : \ V_R(X) \subset \Omega_R\}$ as a macro-interior of Ω_R. In the sequel the attention will be focused on an arbitrary but fixed element $V_R(X)$, $X \in \Omega_R^0$, of the region Ω_R. The averaging operator related to this element will be denoted by $\langle\cdot\rangle_X$, where

$$\langle f(Z)\rangle_X = \frac{1}{l_1 l_2 l_3} \int\limits_{V_R(X)} f(Z)\, dV_R(Z) \tag{2.2}$$

and where $f(\cdot)$ is an arbitrary integrable function. Obviously if $f(\cdot)$ is a V_R-periodic function then $\langle f\rangle_X = \langle f\rangle$ for every X; in this case instead of $\langle f(Z)\rangle$ the symbol $\langle f\rangle$ will be used.

3. Modelling procedure

The macro-modelling approach proposed in this contribution will be based on the kinematic hypothesis (2.1) and on two modelling assumptions formulated below. In order to formulate the first one let us observe that due to the kinematic restrictions of the form (2.1) the well known local equations of motion and stress continuity conditions for the micro-periodic elastic body may be not satisfied. Hence, for every motion restricted by a condition (2.1) define the residuals

$$
\begin{aligned}
r_i(X,t) &:= -\left(\frac{\partial\sigma_R\big(X,\nabla u(X,t)\big)}{\partial u_{i,\alpha}}\right)_{,\alpha} + \rho_R(X)\ddot{u}_i(X,t) - \rho_R(X)b_i \\
& \qquad\qquad\qquad\qquad\qquad X \in \Omega_R \setminus \Gamma_R \\
s_i(X,t) &:= -\left\|\left[\frac{\partial\sigma_R\big(X,\nabla u(X,t)\big)}{\partial u_{i,\alpha}}\right]\right\| n_\alpha \qquad X \in \Gamma_R
\end{aligned}
\tag{3.1}
$$

where Γ_R is a sum of interfaces in Ω_R on which $\sigma_R(\cdot,D)$ and $\nabla h^A(\cdot)$ suffer jump discontinuities. Define $\Gamma_R(X) := \Gamma_R \cap V_R(X)$. Here and in the sequel, for the sake of simplicity, body forces b_i are assumed to be constant. The first modelling assumption is related to the dynamics of a body.

- *Macro-Averaging Assumption.* For every $X \in \Omega_R^0$, $t \in (t_0,t_f)$, residuals defined by Eq (3.1) are assumed to satisfy the conditions

$$
\begin{aligned}
&\langle r_i(Z,t)\rangle_X + \frac{1}{l_1l_2l_3}\int\limits_{\Gamma_R(X)} s_i(Z,t)\,dA_R(Z) = 0 \\
&\langle r_i(Z,t)h^A(Z)\rangle_X + \frac{1}{l_1l_2l_3}\int\limits_{\Gamma_R(X)} s_i(Z,t)h^A(Z)\,dA_R(Z) = 0
\end{aligned}
\tag{3.2}
$$

  provided that the displacement field $u(\cdot)$ in Eq (3.1) is restricted by a condition (2.1).

Hence the known equations of motion for a micro-periodic elastic body under consideration have to be met in the form averaged over every element $V_R(X)$, $X \in \Omega_R^0$. Since $X \in \Omega_R^0$ then the aforementioned averaging conditions hold only in the macro-interior of Ω_R. However, bearing in mind that $\lambda = l/L \ll 1$, it can be seen that from the computational viewpoint regions Ω_R^0 and Ω_R nearly coincide.

The second modelling assumption is strictly related to the fact that $U_i(\cdot,t)$, $V_i^A(\cdot,t)$ and their derivatives are macro-functions.

- *Macro-Approximation Assumption.* When calculating the averages in Eqs (3.2) terms $\mathcal{O}(\varepsilon_F)$ will be neglected as compared to terms involving F, where F stands for an arbitrary regular macro-function.

Let us denote by F a certain macro-function defined on Ω_R and by $\varphi\Big(Z, F(Z)\Big)$, $Z \in \Omega_R$, an arbitrary continuous function. By means of

$$\varphi\Big(Z, F(Z)\Big) = \varphi\Big(Z, F(X)\Big) + \mathcal{O}(\varepsilon_F) \qquad Z \in V_R(X) \tag{3.3}$$

the macro-approximation assumption yields

$$\Big\langle\varphi\Big(Z, F(Z)\Big) + \mathcal{O}(\varepsilon_F)\Big\rangle_X \cong \Big\langle\varphi\Big(Z, F(X)\Big)\Big\rangle_X \tag{3.4}$$

for every $X \in \Omega_R^0$. The approximation formulae (3.4) will be used in the course of a macro-modelling. It has to be emphasized that the modelling approach, proposed in this contribution, which is based on the aforementioned macro-approximation assumption, does not involve any asymptotic approximation procedure. It has to be remembered that the terms $\mathcal{O}(\varepsilon_F)$ are neglected only in formulas $\varphi\Big(Z, F(X)\Big) + \mathcal{O}(\varepsilon_F)$ subjected to an averaging procedure over some $V_R(X)$, $X \in \Omega_R$.

Applying Eqs (3.2) to the second and the third term on the right-hand side of Eqs (3.1) and using Eq (2.1) we obtain

$$\begin{aligned}
&-\langle\rho_R(Z)\ddot{u}_i(Z,t)\rangle_X + \langle\rho_R(Z)\rangle_X b_i = -\langle\rho_R(Z)\rangle_X \ddot{U}_i(X,t) + \\
&-\langle\rho_R h^A\rangle \ddot{V}_i^A(X,t) + \langle\rho_R\rangle b_i + \langle\mathcal{O}(\varepsilon_U) + \mathcal{O}(\varepsilon_V)\rangle_X \\
&-\langle\rho_R(Z)\ddot{u}_i(Z,t)h^A(Z)\rangle_X + \langle\rho_R(Z)h^A(Z)\rangle_X b_i = \\
&= -\langle\rho_R(Z)h^A h^B\rangle_X \ddot{V}_i^B(X,t) + -\langle\rho_R h^A\rangle \ddot{U}_i(X,t) + \langle\rho_R h^A\rangle b_i + \\
&+\langle\mathcal{O}(\varepsilon_U) + \mathcal{O}(\varepsilon_V)\rangle_X
\end{aligned} \tag{3.5}$$

Here and in the sequel the simplified notation $\langle\cdot\rangle$ for the averages of V_R-periodic functions is used, see the remark at the end of Section 2. It can be seen that by the virtue of Eq (2.1) and after setting $g^A(\cdot) := h^A(\cdot)l^{-1}$, where now $\sup|g^A(\cdot)| = 1$, we obtain

$$\begin{aligned}
&\sigma_R\Big(Z, \nabla U(Z,t) + \nabla h^A(Z)V^A(Z,t) + g^A(Z)l\nabla V^A(Z,t)\Big) = \\
&\sigma_R\Big(Z, \nabla U(Z,t) + \nabla h^A(Z)V^A(Z,t)\Big) + \mathcal{O}(\varepsilon_V) + l\mathcal{O}(\varepsilon_{\nabla V})
\end{aligned} \tag{3.6}$$

where we have taken into account that $|lF_{,\alpha}| < \varepsilon_F + l\varepsilon_{\nabla F}$. Moreover, for every $Z \in V_R(X)$

$$\begin{gathered}\sigma_R\Big(Z, \nabla U(Z,t) + \nabla h^A(Z)V^A(Z,t)\Big) = \\ \sigma_R\Big(Z, \nabla U(X,t) + \nabla h^A V^A(X,t)\Big) + \mathcal{O}(\varepsilon_{\nabla U}) + \mathcal{O}(\varepsilon_V)\end{gathered} \tag{3.7}$$

Setting $\boldsymbol{V} := (V^1, ..., V^N)$ and denoting

$$\pi_R(Z, \nabla U, \boldsymbol{V}) := \sigma_R\Big(Z, \nabla U + \nabla h^A(Z)V^A\Big) \tag{3.8}$$

where $\pi_R(\cdot, \nabla U, \boldsymbol{V})$ is a V_R-periodic function, from Eqs (3.6), (3.7) it follows

$$\begin{aligned}\Big\langle\Big(\frac{\partial \sigma_R}{\partial u_{i,\alpha}}\Big)_{,\alpha}\Big\rangle_X &= \Big\langle\frac{\partial^2 \pi_R}{\partial U_{i,\alpha}\,\partial U_{j,\beta}}\Big\rangle_X U_{j,\beta\alpha} + \Big\langle\frac{\partial^2 \pi_R}{\partial U_{i,\alpha}\,\partial V^A}\Big\rangle_X V^A{}_{,\alpha} + \\ &+ \mathcal{O}(\varepsilon_{\nabla U}) + \mathcal{O}(\varepsilon_V) + \mathcal{O}(\varepsilon_{\nabla V})\end{aligned} \tag{3.9}$$

because by means of a V_R-periodicity of $\pi_R(\cdot, \nabla U, \boldsymbol{V})$

$$\Big\langle\frac{\partial^2 \pi_R}{\partial U_{i,\alpha}\,\partial Z^\alpha}\Big\rangle_X = \frac{1}{l_1 l_2 l_3}\Big(\oint\limits_{\partial V_R(X)} \frac{\partial \sigma_R}{\partial U_{i,\alpha}} n_\alpha\, da + \int\limits_{\Gamma_R(X)} \Big[\!\Big[\frac{\partial \sigma_R}{\partial U_{i,\alpha}}\Big]\!\Big] n_\alpha\, da\Big)$$

Similarly, using

$$\Big(\frac{\partial \sigma_R}{\partial u_{i,\alpha}}\Big)_{,\alpha} h^A = \Big(\frac{\partial \sigma_R}{\partial u_{i,\alpha}} h^A\Big)_{,\alpha} - \Big(\frac{\sigma_R}{\partial u_{i,\alpha}}\Big) h^A{}_{,\alpha}$$

we obtain

$$\begin{gathered}\Big\langle\Big(\frac{\partial \sigma_R}{\partial u_{i,\alpha}}\Big)_{,\alpha} h^A\Big\rangle_X = \frac{1}{l_1 l_2 l_3} \int\limits_{\Gamma_R(X)} \Big[\!\Big[\frac{\partial \sigma_R}{\partial U_{i,\alpha}}\Big]\!\Big] n_\alpha h^A\, da + \\ -\Big\langle\Big(\frac{\partial \pi_R}{\partial V^A}\Big)_{,\alpha}\Big(Z, \nabla U(X,t), V(X,t)\Big) + \mathcal{O}(\varepsilon_{\nabla U}) + \mathcal{O}(\varepsilon_V) + \mathcal{O}(\varepsilon_{\nabla V})\Big\rangle_X\end{gathered} \tag{3.10}$$

Let us define the averaged strain energy function, setting

$$\Sigma_R(\nabla U, \boldsymbol{V}) := \Big\langle \pi_R\Big(Z, \nabla U(X,t), V(X,t)\Big)\Big\rangle \tag{3.11}$$

and for an arbitrary motion satisfying Eq (2.1) introduce the fields $S_R^{i\alpha}$, H_R^{Ai} by means of relations

$$
\begin{aligned}
S_R^{i\alpha}(X,t) &= \frac{\partial \Sigma_R\Big(\nabla U(X,t), V(X,t)\Big)}{\partial U_{i,\alpha}} \\
H_R^{Ai} &= \frac{\partial \Sigma_R\Big(\nabla U(X,t), V(X,t)\Big)}{\partial V_i^A}
\end{aligned}
\tag{3.12}
$$

Applying macro-approximation assumption in the form given by Eq (3.4) to Eqs (3.5), (3.9) and (3.10)[1] the averaged equations of motion (3.2) yield

$$
\begin{aligned}
& S_{Ri,\alpha}^{\alpha}(X,t) - \langle \rho_R \rangle \ddot{U}_i(X,t) - \langle \rho_R h^A \rangle \ddot{V}_i^A(X,t) + \langle \rho_R \rangle b_i = 0 \\
& \langle \rho_R h^A h^B \rangle \ddot{V}_i^B(X,t) + \langle \rho_R h^A \rangle \ddot{U}_i(X,t) + H_{Ri}^A(X,t) - \langle \rho_R h^A \rangle b_i = 0
\end{aligned}
\tag{3.13}
$$

Let us observe that Eqs (3.12), (3.13) have been derived indepedently for every $X \in \Omega_R^0$, $t \in (t_0, t_f)$. However, due to the definition (3.11), all entities in the above equations will be assumed to hold for every $X \in \Omega_R$. Eqs (3.13) constitute a certain generalization of equations derived by Wierzbicki et al. (1995), involving also terms $\langle \rho_R h^A \rangle$. However in many problems the micro-shape functions h^A are assumed in the form satisfying extra conditions $\langle \rho_R h^A \rangle = 0$, $A = 1, ..., N$. In this case Eqs (3.13) reduce to the form

$$
\begin{aligned}
& S_{Ri,\alpha}^{\alpha}(X,t) - \langle \rho_R \rangle \ddot{U}_i(X,t) + \langle \rho_R \rangle b_i = 0 \\
& \langle \rho_R h^A h^B \rangle \ddot{V}_i^B(X,t) + H_{Ri}^A(X,t) = 0
\end{aligned}
\tag{3.14}
$$

which will be used in the subsequent sections.

After Woźniak (1993), fields $S_R^{i\alpha}$, H_R^{Ai} will be called macro-stresses and microdynamic forces, respectively. Function Σ_R will be referred to as the macro-strain energy function. At the same time Eqs (3.13) and (3.14) represent the equations of motion and Eqs (3.12) are the constitutive equations for a macro-periodic elastic body under consideration. The form of the model depends on the choice of a micro-shape function system $h^A(\cdot)$, $A = 1, ..., N$, in Eq (2.1), i.e., on a class of micro-disturbances of motion we are going to investigate using this model. Since $h^A(Z) \in \mathcal{O}(l)$ then the coefficients $\langle \rho_R h^A \rangle$, $\langle \rho_R h^A h^B \rangle$ in Eqs (3.13) depend on the microstructure length parameter l.

[1]Terms $\mathcal{O}(\varepsilon_V)$ in Eqs (3.9), (3.10) depend on $V_i^A{}_{,\alpha}$ and hence $l\mathcal{O}(\varepsilon_{\nabla V})$ can be neglected as compared to $\mathcal{O}(\varepsilon_V)$

Hence the obtained model describes the length-scale effect on the global behaviour of a composite body. Using equations of motion in the form (3.14) it can be seen that this effect appears only in dynamic problems. It has to be emphasized that the unknowns V_i^A are governed by the system of ordinary differential equations

$$\langle\rho_R h^A h^B\rangle \ddot{V}_i^B(X,t) + \frac{\partial \Sigma_R\Big(\nabla U(X,t), V(X,t)\Big)}{\partial V_i^A} = 0$$

and hence the fields V_i^A are independent on the boundary conditions. That is why these fields are referred to as the macro-internal variables. In applications of the theory it has to be remembered that the solutions to the initial-boundary problems for Eqs (3.12), (3.14) have a physical meaning only if $U_i(\cdot,t)$, $V_i^A(\cdot,t)$ are sufficiently regular macro-functions.

The detailed discussion of the linearized version of Eqs (3.12), (3.14) can be found in papers on the "refined macrodynamics" mentioned in Section 1. Alternative forms and special cases of the derived equations were also discussed by Wierzbicki et al. (1995) and Wierzbicki (1995).

4. Local models

The formulations of local macro-models on a basis of the obtained in Section 4 length-scale models can be easily done by applying the limit passage $l \searrow 0$. To this end note that the constitutive equations (3.12) are independent of the microstructure length parameter l and equations of motion (3.13), after substituting $h^A = g^A l$, take the form

$$\begin{aligned} &S^{\alpha}_{Ri,\alpha}(X,t) - \langle\rho_R\rangle \ddot{U}_i(X,t) + l\langle\rho_R g^A\rangle \ddot{V}_i^A(X,t) + \langle\rho_R\rangle b_i = 0 \\ &l^2\langle\rho_R g^A g^B\rangle \ddot{V}_i^B(X,t) + l\langle\rho_R g^A\rangle \ddot{U}_i(X,t) + \Pi^A_{Ri}(X,t) - l\langle\rho_R g^A\rangle b_i = 0 \end{aligned} \tag{4.1}$$

which depends on l in the explicit form. Restricting consideration to the problems in which the length-scale effect on the overall body behaviour can be neglected (e.g., for quasi-stationary problems, provided that $\langle\rho_R g^A\rangle = 0$) one can neglect terms involving l in Eqs (4.1). Hence from Eqs (4.1) we obtain the equations of motion

$$S^{\alpha}_{Ri,\alpha}(X,t) - \langle\rho_R\rangle \ddot{U}_i(X,t) + \langle\rho_R\rangle b_i = 0 \tag{4.2}$$

and the algebraic interrelation between macro-internal parameters V_i^A and macrodisplacement gradients $U_{i,\alpha}$

$$\frac{\partial \Sigma_R(\nabla U, V)}{\partial V_i^A} = 0 \tag{4.3}$$

Equations of motion (4.1), constitutive equations for macro-stresses

$$S_{Ri}^{\alpha}(X,t) = \frac{\partial \Sigma_R\Big(\nabla U(X,t), V(X,t)\Big)}{\partial U^i{}_{,\alpha}} \tag{4.4}$$

and algebraic equations (4.3) for macro-internal parameters, constitute a local model of the micro-periodic elastic body under consideration. Now assume that Eqs (3.1) have solutions of the form $V_i^A = \Psi_i^A(\nabla U)$, where $\Psi_i^A(\cdot)$ are known functions. In this case the macro-internal parameters can be eliminated from the constitutive equations (4.4). Setting

$$\Sigma_R^{eff}(\nabla U) := \Sigma_R\Big(\nabla U, \Psi(\nabla U)\Big) \tag{4.5}$$

and bearing in mind Eq (4.3), we obtain

$$S_{Ri}^{\alpha}(X,t) = \frac{\partial \Sigma_R^{eff}(\nabla U)}{\partial U^i{}_{,\alpha}} \tag{4.6}$$

It follows that in the case considered the governing equations (4.2), (4.6) representing the local model are similar to the equations of certain elastic homogeneous bodies, subjected to finite deformation, with the energy function given by Eq (4.5).

Taking into account definitions (3.8), (3.11) it can be seen that Eq (4.3) can be also written down in the form

$$\frac{\partial\Big\langle \sigma_R\Big(Z, \nabla U + \nabla h^B(Z) V^B\Big) h^A{}_{,\alpha}(Z)\Big\rangle}{\partial U_{i,\alpha}} = 0$$

For a homogeneous body we have $\sigma_R = \sigma_R\Big(\nabla U + \nabla h^B(Z) V^B\Big)$. Since $\langle \sigma_R(\nabla U) h^A{}_{,\alpha}(Z)\rangle = \sigma_R(\nabla U)\langle h^A{}_{,\alpha}(Z)\rangle = 0$ then it is easy to conclude that for homogeneous elastic media subjected to finite deformations Eqs (4.3) have always trivial solutions $V_i^A = 0$. It follows that the macro-internal variables and hence also disturbances $h^A V_i^A$ in the formula (2.1), for homogeneous bodies can be taken as equal to zero.

5. Wave propagation

In order to apply the obtained macro model to the analysis of wave propagation and stability problems, an alternative form of Eqs (3.12), (3.14) will be derived. To this end define

$$\begin{aligned} A_R^{i\alpha j\beta}(\nabla U, V) &:= \frac{\partial^2 \Sigma_R(\nabla U, V)}{\partial U_{i,\alpha}\, \partial U_{j,\beta}} \\ B_R^{Aji\alpha}(\nabla U, V) &:= \frac{\partial^2 \Sigma_R(\nabla U, V)}{\partial U_{i,\alpha}\, \partial V_j^A} \\ G_{Ri}^A(\nabla U, V) &:= \frac{\partial \Sigma_R(\nabla U, V)}{\partial V_i^A} \end{aligned} \tag{5.1}$$

Under denotations (5.1) and setting $g^A = h^A l^{-1}$ Eqs (3.12), (3.14) can be written down in the form

$$\begin{aligned} &A_R^{i\alpha j\beta}(\nabla U, V) U_{j,\alpha\beta} + B_R^{i\alpha j\beta}(\nabla U, V) V_j^A{}_{,\alpha} - \langle \rho_R \rangle \ddot{U}^i + \langle \rho_R \rangle b^i = 0 \\ &l^2 \langle \rho_R g^A g^B \rangle \ddot{V}_i^B + G_{Ri}^A(\nabla U, V) = 0 \end{aligned} \tag{5.2}$$

which depends explicitly on the microstructure length parameter l. Since the coefficients in Eqs (5.2) are known functions of $U_{i,\alpha}$, V_i^A, then Eqs (5.2) represent the system of 3 quasi-linear second order partial differential equations in macro-displacements U_i, coupled with the system of $3N$ second order ordinary differential equations in macro-inertial parameters V_i^A. In this section Eqs (5.2) will be used to the analysis of a wave propagation problem for a micro-periodic elastic medium. Since the problem is investigated within the framework of macro-model then the term "wave" will be understood in a certain averaged manner related to this model. Considerations will be restricted to acceleration waves described by the motion $t \to S_R^t$ of a certain smooth discontinuity surface S_R^t in Ω_R. Let every S_R^t be oriented by a unit normal vector field with components $N_\alpha(X)$, $X \in S_R^t$. Let $f(\cdot, t)$ stand for a continuous field, defined on Ω_R for every $t \in (t_0, t_f)$ having continuous first order time derivatives and continuous material derivatives up to the $(n-1)$th order. As it is known, the following compatibility conditions hold on every oriented discontinuity surface S_R^t

$$[\![f_{,\alpha_1,\dots,\alpha_n}]\!](X,t) = v_R^2 a_f N_{\alpha_1}(X) \dots N_{\alpha_2}(X) \qquad X \in S_R^t$$

where $[\![\cdot]\!]$ stands for a jump across S_R^t, v_R is the propagation velocity of the surface S_R^t and a_f is the amplitude of a jump of the field $f(\cdot, t)$ across

S^t_R. It has to be remembered that all entities in the problem considered are related to the reference configuration of a medium; the passage to the actual configuration can be realized by using Eq (2.1).

Since displacement gradients in the definition (3.8) are given by $U_{i,\alpha} + h^A{}_{,\alpha} V_i^A$ then discontinuities of the second derivatives in the displacement field u_i can be described, within the framework of macro-model under consideration, by discontinuities of the second order derivatives $U_{i,\alpha\beta}$, $\ddot{U}_i$ of the macro-displacement field $U_i(\cdot)$ and discontinuities of the first order derivatives $V_i^A{}_{,\alpha}$ of the macro-internal variables $V_i^A(\cdot)$. Thus, the acceleration macro-wave of the second order will be defined as the discontinuity surface S^t_R in Ω_R across which U_i, V_i^A, $\dot{U}_i$, $U_{i,\alpha}$ are continuous and $U_{i,\alpha\beta}$, $\dot{U}_{i,\alpha}$, $\ddot{U}_i$, $V_i^A{}_{,\alpha}$, suffer jump discontinuities. Taking the jumps $[\![\cdot]\!]$ across S^t_R of the left-hand sides of Eqs (5.2) we obtain

$$
\begin{aligned}
&A_R^{i\alpha j\beta}(\nabla U, V)[\![U_{j,\alpha}]\!] - \langle\rho_R\rangle[\![\ddot{U}_i]\!] + B_R^{Aij\alpha}[\![V_j^A{}_{,\alpha}]\!] = 0 \\
&\langle\rho_R g^A g^B\rangle[\![\ddot{V}_i^B]\!] = 0
\end{aligned}
$$

Since $\langle\rho_R g^A g^B\rangle$ represent elements of a non-singular $N \times N$ matrix then $[\![\ddot{V}_i^B]\!] = 0$, i.e., fields $\ddot{V}_i^B$ are continuous across S^t_R. Taking the derivative with respect to X^α of the left-hand sides from the second one of Eqs (5.2) and denoting

$$
C_R^{ABij} := \frac{\partial^2 \Sigma_R(\nabla U, V)}{\partial V_i^A \partial V_j^B} \tag{5.3}
$$

we arrive at the system of equations

$$
\begin{aligned}
&A_R^{i\alpha j\beta} U_{j,\alpha\beta} - \langle\rho_R\rangle \ddot{U}^i + B_R^{Aij\alpha} V_j^A{}_{,\beta} + \langle\rho_R\rangle b^i = 0 \\
&C_R^{ABij} V_j^B{}_{,\alpha} + l^2 \langle\rho_R g^A g^B\rangle \ddot{V}^{Bi}{}_{,\alpha} + B_R^{Aij\beta} U_{,\beta\alpha} = 0
\end{aligned} \tag{5.4}
$$

Let us take the jumps $[\![\cdot]\!]$ across S^t_R of the left-hand sides of the above equations; after denotations

$$
A_R^{ij} = A_R^{i\alpha j\beta} N_\alpha N_\beta \qquad B_R^{Aij} = B_R^{Aij\alpha} N_\beta
$$

the aforementioned compatibility conditions on S^t_R yield the system of linear algebraic equations in the amplitudes a_i, a_i^A related to jumps of fields U_i and V_i^A, respectively

$$
\begin{aligned}
&(A_R^{ij} - \langle\rho_R\rangle v_R^2 \delta^{ij}) a_j + B_R^{Aij} a_j^A = 0 \\
&B_R^{Aij} N_\alpha a_j + (C_R^{ABij} + l^2 \langle\rho_R g^A g^B\rangle v_R^2 \delta^{ij}) N_\alpha a_j^B = 0
\end{aligned} \tag{5.5}
$$

The condition of existence of non-trivial solution a_j, a_j^A to the above system of linear algebraic equations yields the propagation condition. It can be seen that the form of this condition depends on the microstructure length parameter l, i.e., it takes into account the microstructure length-scale effect on the dynamic behaviour of the medium.

Treating l^2 in Eqs (5.5) as a small parameter one can verify that the linear transformation $R^{3n} \to R^{3n}$ given by $C_R^{ABij} + l^2\langle\rho_R g^A g^B\rangle\delta^{ij}v_R^2$, is invertible. Denoting by D_R^{ABij} the elements of the linear transformation inverse to that given by C_R^{ABij} (which is also assumed to be invertible) we obtain from the second one of Eqs (5.5) the following asymptotic formula

$$a^{Bi} = -(D_R^{BCij} - l^2 v_R^2 D_R^{BDik}\langle\rho_R g^D g^E\rangle D_R^{ECjk})B_R^{Cjl}a_l + o(l^2)$$

It follows that the amplitudes a_i^B can be eliminated from Eqs (5.5). Substituting the right-hand side of the above formulae into the first one from Eqs (5.5), after introducing the denotations

$$\begin{aligned} Q_R^{ij} &:= A_R^{ij} - B_R^{Bik}D_R^{BCkl}B_R^{Clj} \\ M_R^{ij} &:= B_R^{Bik}D_R^{BDkl}\langle\rho_R g^D g^E\rangle D_R^{EClm}B_R^{Cmj}\langle\rho_R\rangle^{-1} \end{aligned} \tag{5.6}$$

we arrive at the system of equations in the amplitudes a_i

$$[Q_R^{ij} - \langle\rho_R\rangle(\delta^{ij} - l^2 M_R^{ij})v_R^2 + o(l^2)]a_j = 0$$

Neglecting terms $o(l^2)$ as small ones when compared to $\langle\rho_R\rangle v_R^2$, we obtain finally the propagation condition

$$\det[Q_R^{ij} - \langle\rho_R\rangle(\delta^{ij} - l^2 M_R^{ij})v_R^2] = 0 \tag{5.7}$$

Since l^2 is treated in Eq (5.7) as a small parameter then the 3×3 matrix with components $\delta^{ij} - l^2 M_R^{ij}$ is positive definite. A matrix with components Q_R^{ij} will be called the macro-acoustic matrix. Hence the final conclusion is that the squares of propagation velocities v_R for the macro-wave under consideration are generalized eigenvalues of the macro-acoustic matrix.

Let us observe that after neglecting the length-scale effects, i.e., after using the limit passage $l \searrow 0$ in Eq (5.6), we obtain

$$\det(Q_R^{ij} - \langle\rho_R\rangle\delta^{ij}v_R^2) = 0$$

The above condition has a similar form to that for a homogeneous elastic body.

6. Stability

Let us assume that the micro-periodic elastic body under consideration is subjected to a certain finite deformation, which due to Eqs (2.1) is given by the fields $U_i = U_i(X)$, $V_i^A = V_i^A(X)$, $X \in \Omega_R$. Neglecting the body forces, from Eqs (6.2) we obtain for U_i, V_i^A the system of equations

$$A_R^{i\alpha j\beta}(\nabla U, V) U_{j,\alpha\beta} + B_R^{Aij\alpha}(\nabla U, V) V_j^A{}_{,\alpha} = 0$$

$$G_{Ri}^A(\nabla U, V) = 0$$

with the functional coefficients defined by formulae (5.1), which has to be considered together with the boundary conditions for macro-displacements U_i. It can be seen that the macro-internal parameters V_i^A are governed by a system of $3N$ nonlinear algebraic equations. Let us assume that the solution to this boundary-value problem has been found and hence the functions $U_i(X)$, $V_i^A(X)$ are known and depending on the boundary surface tractions related to $\partial\Omega_R$. The model involved is local and hence the solution obtained is independent of l.

Now substitute the fields $U_i(X) + \varepsilon\, 'U_i(X,t)$, $V_i^A(X) + \varepsilon\, 'V_i^A(X,t)$ into Eqs (5.2), where ε is a small parameter and U_i, V_i^A determine a certain small motion superimposed on the known finite deformation given by $U_i(X)$, $V_i^A(X)$. In this case Eqs (5.2), after linearization with respect to ε, assume the form

$$\begin{aligned} &A_R^{i\alpha j\beta} U_{j,\alpha\beta} - \langle \rho_R \rangle \ddot{U}^i + B_R^{Aij\alpha} V_j^A{}_{,\alpha} = 0 \\ &l^2 \langle \rho_R g^A g^B \rangle \ddot{V}_i^B + C_R^{ABij} V_j^B + B_R^{Aij\alpha} U_j^A{}_{,\alpha} = 0 \end{aligned} \tag{6.1}$$

Bearing in mind Eqs (5.1), (5.3) it is easy to conclude that the functions $A_R^{i\alpha j\beta}$, $B_R^{Aij\alpha}$, C_R^{ABij} are known and depending on the surface tractions acting on the body. That is why the stability after the known finite deformation of the micro-periodic elastic body (within the framework of macro-model proposed in the paper) can be analysed using the approach similar to that used in the finite deformation of homogeneous bodies. Setting

$$'U_i(X,t) = \widehat{U}_i(X) e^{\omega t} \qquad 'V_i^A(X,t) = \widehat{V}_i^A(X) e^{\omega t} \tag{6.2}$$

where $\omega = \alpha + i\beta$ and substituting solutions (6.2) into Eqs (6.1), we obtain

$$\begin{aligned} &A_R^{i\alpha j\beta} \widehat{U}_{j,\alpha\beta} + B_R^{Aij\alpha} \widehat{V}_j^A{}_{,\alpha} - \langle \rho_R \rangle \omega^2 \widehat{U}^i = 0 \\ &(C_R^{ABij} + l^2 \langle \rho_R g^A g^B \rangle \delta^{ij} \omega^2) \widehat{V}_j^B + B_R^{Aij\beta} \widehat{U}_{j,\beta} = 0 \end{aligned} \tag{6.3}$$

Define

$$E_R^{ABij} := D_R^{ADik}\langle \rho_R g^D g^E\rangle D_R^{EBkj}\langle \rho_R\rangle^{-1} \tag{6.4}$$

where D_R^{ABij} were defined in Section 5. From the second of Eqs (6.3) it follows that

$$\widehat{V}_i^A = -(\widehat{D}_R^{ABij} - l^2\langle \rho_R\rangle \omega^2 E_R^{ABij})B_R^{Bjk\beta}\widehat{U}_{k,\beta} + o(l^2)$$

where l^2 is treated as a small parameter. After neglecting terms $o(l^2)$ in the above formulae, one eliminate the functions $\widehat{V}_i^A$ from Eqs (6.3). Under denotations

$$N_R^{i\alpha j\beta} := A_R^{i\alpha j\beta} - B_R^{Aik\alpha} D_R^{ABkl} B_R^{Blj\beta} \tag{6.5}$$

we obtain finally the following system of equations in $\widehat{U}_i = \widehat{U}_i(X)$, $X \in \Omega_R$

$$\begin{aligned} N_R^{i\alpha j\beta}\widehat{U}_{i,\alpha\beta} - B_R^{Aik\alpha}(D_R^{ABkl}B_R^{Blj\beta})_{,\beta}\widehat{U}_{j,\alpha} &= \\ &= \langle \rho_R\rangle \omega^2[\widehat{U}_i + l^2 B_R^{Aij\alpha}(E_R^{ABjk}\widehat{U}_{l,\beta})_{,\alpha}] \end{aligned} \tag{6.6}$$

Since $e^{\omega t} = e^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t)$ then for $t \to \infty$ the superimposed motion, given by $U_i(X,t)$, $V_i^A(X,t)$, $X \in \Omega_R$, tends to zero if $\alpha < 0$. Taking into account the boundary value problem for $\widehat{U}_i$ determined by Eqs (6.6) and the homogeneous boundary conditions $\widehat{U}_i(X) = 0$, $X \in \partial\Omega_R$, we obtain non-trivial solutions only for certain values of $\omega = \alpha + \mathrm{i}\beta$. The loss of stability takes place where α is passing from negative to positive values together with a continuous increment of boundary surface tractions which are involved in the coefficients $N_R^{i\alpha j\beta}$, $B_R^{Aik\alpha}$, D_R^{ABkl}, E_R^{ABjk} of Eqs (6.6). It can be seen that the solutions to the boundary-value problem under consideration also depend on the microstructure length parameter since Eqs (6.6) depend explicitly on l^2.

7. Final remarks

The objective of this contribution was twofold. First, it has been shown that the field equations representing length-scale macro-model of a composite can be obtained by simple averaging of certain residuals related to the equations of motion for an elastic micro-periodic body under finite deformations. The procedure proposed does not involve any asymptotic approximation and hence describes the effect of the microstructure length parameter on the global behaviour of the body. Second, on the basis of the obtained model new equations were proposed for the analysis of wave propagation and stability problems in finite deformations of elastic micro-periodic composites. The general conclusion is that solutions to special problems, both for the wave propagation and stability, performed within the framework of the macro-model proposed, can be obtained using the procedures similar to those met in the corresponding problems for elastic homogeneous bodies under finite deformations.

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Efekt skali w problemach propagacji fal i stabilności dla kompozytów sprężystych poddanych skończonym deformacjom

Streszczenie

W pracy zaproponowano nowe podejście do modelowania mikro-periodycznych kompozytów sprężystych. W otrzymanych równaniach uwzględniono wpływ liniowego parametru mikrostruktury na dynamikę ciała. W przybliżeniu liniowym redukują się one do pewnego uogólnienia równań "wzbogaconej makrodynamiki", zaproponowanej przez Woźniaka (1993). Celem pracy jest zastosowanie otrzymanych równań do analizy propagacji fal i stateczności kompozytów sprężystych poddanych skończonym deformacjom.

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