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BELLMAN'S EQUATION OF OPTIMIZATION WITH THE PERIODIC CONTROL [1]

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In this paper the dynamic programming approach is used, and the necessary optimality conditions under a periodic control constraint in a finite interval have been found. On the basis of dynamic programming theory, the optimality conditions were formulated in the form of the modified Bellman's functional equation, which was derived using the method given by Piekarski (1992). The independent variable interval $[0, L]$, is divided into N identical subintervals of the length $T = L/N$. Let us assume that the control is the same shape within each subintervals. The initial optimization problem within $[0, L]$ is replaced by an equivalent optimization problem valid within one subinterval. Consequently, it is enough to analyze the control function inside a single subinterval, i.e., $0 \leq t \leq T$. It is worth noting that a modified optimality criterion, on the basis of Pontriagin's maximum principle was derived by Piekarski (1992) and Leitman (1966) (for the first time).

1. Introduction

The aim of the previous paper by Piekarski (1992) was to analyze the optimization problem on the assumption of periodicity of control variables in a finite interval. Solution to the problem or the necessary optimality conditions have been obtained by using Pontriagin's maximum principle, e.g., Leitman (1966). The results obtained can be applied to the optimization problems in discrete-continuous systems. For example, Kuhta and Kravchenko (1976) considered boundary-value problems under piece-wise analytical conditions, in vibrating elastic systems subject to discrete and discrete-continuous excitations, respectively. Optimization under periodic control over a given, finite

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range seems to be advantageous in some industrial and economics problems. This approach could be used if a system consists of many elementary models, e.g., vibrating beems or rotating shafts with concentrated masses. In such problems the periodic optimal control may be used in a natural way. To illustrate this modified optimization problem and apply the theorem obtained, Gajewski and Piekarski (1994) considered the optimal control problem of minimizing the volume of a vibrating elastic beam a constant frequency constraint and periodical optimal shape. In the paper by Piekarski (1994), the necessary optimality conditions for periodic control have been found, taking into account certain constraints imposed on the state variables. The objective of this paper is to generalize the results obtained previously. We shall formulate a theorem formulating relationships of dynamic programming (cf Bellman (1957)). The necessary optimality conditions are specified in terms of Bellman's functional equation with the periodic control in the limited interval.

2. Formulation and solution to the problem

The interval $[0, L]$ of independent variable t is divided into N subintervals $[(q-1)T, qT]$ of the variables

$$t_q = t_1 + (q-1)T \qquad q = 1, 2, ..., N < \infty \tag{2.1}$$

where

$$0 \leq t_1 \leq T \qquad NT = L \tag{2.2}$$

The problem of optimization in the interval $[0, L]$ is to be formulated as follows: determine a minimum of the cost functional

$$J = \int_0^L f_0(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, dt \tag{2.3}$$

with constraints

— the state vector

$$\boldsymbol{x}(t) = [x_1(t), ..., x_n(t)]^\top \tag{2.4}$$

satisfies the equations

$$\frac{d}{dt}x_i(t) = f_i(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \qquad i = 1, 2, ..., n \tag{2.5}$$

with the initial conditions

$$x(0) = x_0 \tag{2.6}$$

— the vector of control

$$u(T) = [u_1(t), u_2(t), ..., u_m(t)]^\top \tag{2.7}$$

is periodical in a limited interval

$$u(t_1) = u(t_q) \tag{2.8}$$

and belongs to a given set of determined and piece-wise continuous functions

$$u(t_1) \in U \tag{2.9}$$

We introduce the following vector functions

$$x^q(t_1) \equiv x(t_q) = [x_1(t_q), x_2(t_q), ..., x_n(t_q)]^\top \tag{2.10}$$

$$u^q(t_1) \equiv u(t_q) = [u_1(t_q), u_2(t_q), ..., u_m(t_q)]^\top \tag{2.11}$$

for every value of t_q from Eq (2.1). Dimensions of the state and control spaces are $n \times n$ and $m \times n$, respectively. Upon substituting Eqs (2.8), (2.10) and (2.11) into Eqs (2.3) and (2.5) we get

$$J = \int_0^T \sum_{q=1}^{N} f_0(x^q(t_1), u(t_1), t_q)\, dt_1 \tag{2.12}$$

and

$$\frac{d}{dt_1} x_i^q(t_1) = f_i(x^q(t_1), u(t_1), t_q) \tag{2.13}$$

under the conditions

$$x^1(0) = x_0 \tag{2.14}$$

The problem of optimization, we started with, defined in the interval $[0, l]$ with the periodic control is equivalent to the problem in the subinterval $[0, T]$ given by Eqs (2.12), (2.13) and (2.9) with non periodic control $u(t_1)$. Now, using the method of dynamic programming, cf Bellman (1957), we can find the necessary conditions of optimality. The minimum of cost functional (2.12) can be presented in the form $S[x^1(t_0), x^2(t_0), ..., x^N(t_0), t_0]$ for $t_0 = 0$. Applying to the optimality principle to the interval $[t_1, T]$ from Eq (2.12) we have

$$S[x^1(t_0), x^2(t_0), ..., x^N(t_0), t_0] = \min_{u(t_1)\in U} \int_0^T \sum_{q=1}^{N} f_0(x^q(t_1), u(t_1), t_q)\, dt_1 \tag{2.15}$$

The integration interval $[t_1, T]$ was divided into two intervals: $[t_1, t_1 + \Delta t]$ and $[t_1 + \Delta t, T]$. From Eq (2.15) we obtain

$$S[\boldsymbol{x}^1(t_1), \boldsymbol{x}^2(t_1), ..., \boldsymbol{x}^N(t_1), t_1] = \tag{2.16}$$
$$= \min_{\boldsymbol{u}(t_1)\in U} \Big(\int\limits_{t_1}^{t_1+\Delta t} \sum_{q=1}^{N} f_0(\boldsymbol{x}^q(t_1), \boldsymbol{u}(t_1), t_q)\ dt_q + \int\limits_{t_1+\Delta t}^{T} \sum_{q=1}^{N} f_0(\boldsymbol{x}^q(t_1), \boldsymbol{u}(t_1), t_q)\ dt_1 \Big)$$

The condition (2.16) by virtue of Eq (2.15) may be rewritten as a functional equation of the form

$$S[\boldsymbol{x}^1(t_1), \boldsymbol{x}^2(t_1), ..., \boldsymbol{x}^N(t_1), t_1] = \min_{\boldsymbol{u}(t_1)\in U} \Big(\sum_{q=1}^{N} f_0(\boldsymbol{x}^q(t_1), \boldsymbol{u}(t_1), t_q)\Delta t + \tag{2.17}$$
$$+ S[\boldsymbol{x}^1(t_1 + \Delta t), \boldsymbol{x}^2(t_1 + \Delta t), ..., \boldsymbol{x}^N(t_1 + \Delta t), t_1 + \Delta t] \Big) + \mathcal{O}_1(\Delta t)$$

The function S at the point $(t_1 + \Delta t)$ is expanded into the Taylor series

$$S[\boldsymbol{x}^1(t_1 + \Delta t), \boldsymbol{x}^2(t_1 + \Delta t), ..., \boldsymbol{x}^N(t_1 + \Delta t), t_1 + \Delta t] =$$
$$= S[\boldsymbol{x}^1(t_1), \boldsymbol{x}^2(t_1), ..., \boldsymbol{x}^N(t_1), t_1] + \tag{2.18}$$
$$+ \sum_{q=1}^{N} \sum_{i=1}^{n} \frac{\partial S}{\partial x_i^q(t_1)} f_i(\boldsymbol{x}^q(t_1), \boldsymbol{u}(t_1), t_q)\Delta t + \frac{\partial S}{\partial t_1}\Delta t + \mathcal{O}_2(\Delta t)$$

Since the function S is independent of the vector control $\boldsymbol{u}(t_1)$, by virtue of Eq (2.18) we get from Eq (2.17)

$$-\frac{\partial}{\partial t_1} S[\boldsymbol{x}^1(t_1), \boldsymbol{x}^2(t_1), ..., \boldsymbol{x}^N(t_1), t_1] = \min_{\boldsymbol{u}(t_1)} \sum_{q=1}^{N} \Big(f_0 + \sum_{i=1}^{n} \frac{\partial S}{\partial x_i^q(t_1)} f_i \Big) \tag{2.19}$$

From Eq (2.19) the form of function S follows

$$S[\boldsymbol{x}^1(t_1), \boldsymbol{x}^2(t_1), ..., \boldsymbol{x}^N(t_1), t_1] = \sum_{q=1}^{N} S[\boldsymbol{x}^q(t_1), t_1)] \tag{2.20}$$

When the functions f_0 and f_i are independent of t (an autonomous problem) we have

$$\frac{\partial}{\partial t_1} S[\boldsymbol{x}^q(t_1), t_1] = 0 \tag{2.21}$$

From Eqs (2.19) and (2.20) we obtain

$$\min_{\boldsymbol{u}(t_1)\in U} \sum_{q=1}^{N} P_q(t_1) = 0 \tag{2.22}$$

in which the partial functions P_q are given by the formula

$$\begin{aligned} P_q(t_1) &= \frac{\partial}{\partial t_1} S[\boldsymbol{x}^q(t_1), t_1] + f_0(\boldsymbol{x}^q(t_1), \boldsymbol{u}(t_1), t_q) + \\ &+ \sum_{i=1}^{n} f_i(\boldsymbol{x}^q(t_1), \boldsymbol{u}(t_1), t_q) \frac{\partial}{\partial x_i^q(t_1)} S[\boldsymbol{x}^q(t_1), t_1] \end{aligned} \tag{2.23}$$

Eq (2.22) when combined with Eq (2.23) represents Bellman's functional equation of dynamic programming for the periodic control in a limited interval.

3. Example

The following example, due to its simplicity, can be solved analytically. We shall be seeking the minimum of cost functional

$$J = \int_0^L \left[a_1 x(t) + \frac{1}{6} u^2(t)\right] dt \tag{3.1}$$

under the conditions

$$\frac{d}{dt} x(t) = b u(t) \qquad\qquad x(0) = 1 \tag{3.2}$$

The control constrains are given by Eq (2.8)

$$u(t_1) = u(t_q) \qquad\qquad q = 1, 2, 3 \qquad N = 3 \tag{3.3}$$

where t_q is given by Eqs (2.1) and (2.2). Eqs (3.1) ÷ (3.3) given in the interval $[0, L]$ can be presented in an equivalent form in the subinterval $[0, T]$

$$J = \int_0^T \left[a_1 \sum_{q=1}^{3} x^q(t_1) + \frac{1}{2} u^2(t_1)\right] dt_1 \tag{3.4}$$

and

$$\frac{d}{dt_1}x^q(t_1) = bu(t_1) \qquad\qquad x^1(0) = 1 \tag{3.5}$$

Eqs (2.20) and (2.21) in view of Eq (2.22) take the following form

$$0 = \min_{u(t_1)}\Big[a_1\sum_q x^q(t_1) + \frac{1}{2}u^2(t_1) + bu(t_1)\sum_q \frac{\partial S}{\partial x^q(t_1)}\Big] \tag{3.6}$$

We can see that in view of Eq (3.6) one obtains the following two equations

$$\begin{aligned}
&\sum_q \frac{\partial S}{\partial x^q(t_1)} = -\frac{1}{b}\widehat{u}(t_1)\\
&a_1\sum_q x^q(t_1) + \frac{1}{2}\widehat{u}^2(t_1) + b\widehat{u}(t_1)\sum_q \frac{\partial S}{\partial x^q(t_1)} = 0
\end{aligned} \tag{3.7}$$

By virtue of Eq (3.7) it is seen that the optimal control $\widehat{u}(t_1)$ takes the form

$$\widehat{u}(t_1) = \sqrt{2a_1\sum_q x^q(t_1)} \tag{3.8}$$

It is assumed that the new state variables $z(t)$ fulfill the following conditions

$$\begin{aligned}
z_1(t_1) &= x(t_1) + x(t_1+T) + x(t_1+2T)\\
z_2(t_1) &= x(t_1) - x(t_1+T)\\
z_3(t_1) &= x(t_1) - x(t_1+2T)
\end{aligned} \tag{3.9}$$

Eqs (3.5), (2.10), (3.8) and (3.9) can be presented in as follows

$$\begin{aligned}
&\frac{d}{dt_1}z_1(t_1) = 3b\sqrt{2a_1 z_1(t_1)}\\
&\frac{d}{dt_1}z_2 = 0 \qquad\qquad \frac{d}{dt_1}z_3 = 0
\end{aligned} \tag{3.10}$$

The solutions of these equations are provided by the following functions

$$\begin{aligned}
z_1(t_1) &= \Big(\frac{3}{2}b\sqrt{2a_1}t_1 + \frac{1}{2}c_1\Big)^2\\
z_i(t_1) &= \text{const} = c_i \qquad\qquad i = 2,3
\end{aligned} \tag{3.11}$$

From Eqs (3.9) and (3.11), it follows that

$$
\begin{aligned}
x^1(t_1) &= \frac{3}{2}b^2 a_1 t_1^2 + \frac{b}{2}\sqrt{2a_1}c_1 t_1 + \frac{1}{12}c_1^2 + \frac{1}{3}c_2 + \frac{1}{3}c_3 \\
x^i(t_1) &= \frac{3}{2}b^2 a_1 t_1^2 + \frac{b}{2}\sqrt{2a_1}c_1 t_1 + 1 - c_i \qquad i = 1,2
\end{aligned}
\tag{3.12}
$$

) It is assumed that $T = 2$, $b = a_1 = 1$. From Eqs (3.5) and the conditions of continuity (3.12) specified at the points T and $2T$, we get

$$
c_1^2 + 12c_2 - 12 = 0 \qquad c_2 = -6 - \sqrt{2}c_1 \qquad c_3 = 2c_2 \tag{3.13}
$$

The solutions of Eqs (3.13) are

$$
c_1 = -4.0047 \qquad c_2 = -0.3365 \qquad c_3 = -0.6730 \tag{3.14}
$$

From (3.12) and (3.14) we have

$$
\begin{aligned}
x(t_1) &= 1.5t_1^2 - 2.8317_1 + 1 \\
x(t_1 + T) &= 1.5t_1^2 - 2.8317t_1 + 1.3365 \\
x(t_1 + 2T) &= 1.5t_1^2 - 2.8317t_1 + 1.6730
\end{aligned}
\tag{3.15}
$$

After substituting Eqs (3.9) and (3.11) into Eq (3.8) we obtain

$$
\widehat{u}(t_1) = \sqrt{2a_1}\Big(\frac{3}{2}b\sqrt{2a_1}t_1 + \frac{1}{2}c_1\Big) \tag{3.16}
$$

from which it follows that

$$
\widehat{u}(t_1) = 3t_1 - 2.83175 \tag{3.17}
$$

The control (3.17) and state variables (3.15), respectively, are shown in Fig.1.

For the case of additional constraints imposed on state variables the graphical representation of control is shown by Gajewski and Piekarski (1964), in which the necessary optimality conditions of periodic control on the basis of Pontryagin's maximum principle have been found.

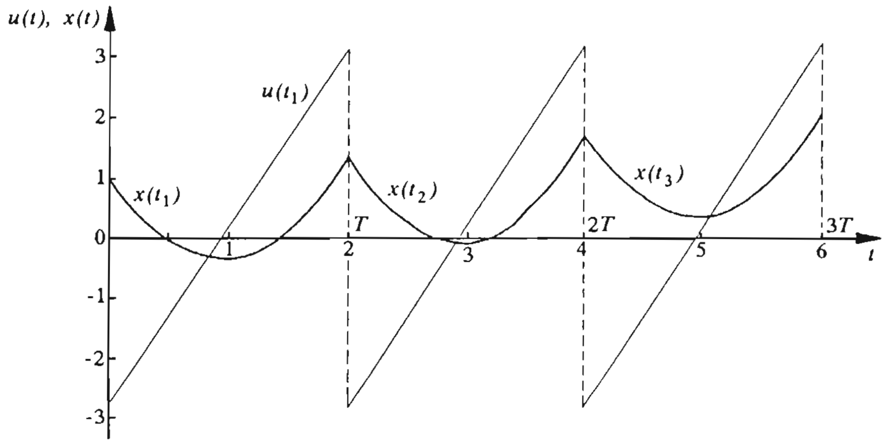

Fig. 1.

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Równanie Bellmana optymalizacji z okresowym sterowaniem

Streszczenie

W pracy rozpatrzony został problem optymalizacji układów ze sterowaniem okresowym w skończonym przedziale zmiennej niezależnej. Warunki konieczne optymalności otrzymane zostały na bazie programowania dynamicznego. Dla układów ciągłych warunki te zapisano w formie zmodyfikowanego równania funkcyjnego Bellmana. Równanie to otrzymane zostało metodą zastosowaną w pracy Piekarskiego

(1992). Przedział $[0, L]$ zmiennej niezależnej t dzielimy na N równych podprzedziałów o długości $T = L/N$. Przyjmujemy, że w każdym z nich optymalne sterowanie ma taki sam kształt. Problem optymalizacji zadany początkowo w przedziale $[0, L]$ sprowadzamy do optymalizacji równoważnej określonej w jednym z podprzedziałów. Pozwala to sprowadzić badanie funkcji sterowania układu do badania jej w jednym okresie np. dla $0 \leq t \leq T$. Do tak otrzymanego problemu stosujemy następnie jedną z teorii sterowania. Na bazie zasady maksimum Pontriagina zmodyfikowany warunek optymalności po raz pierwszy podany został w pracy Piekarskiego (1992).

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