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EQUATIONS OF MOTION OF A SPIN-STABILIZED PROJECTILE FOR FLIGHT STABILITY TESTING

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The paper presents a mathematical model of motion of a balanced spin-stabilized projectile considered as a rigid body with 6 degrees of freedom. The modeling uses coordinate systems conforming to Polish and International Standard ISO 1151. The design of kinematic equations describing motion around the center of mass uses the system of Tait-Bryan angles or Euler parameters. The total angle of attack and aerodynamic roll angle express aerodynamic forces and moments.

Key words: spin-stabilized projectile, flight stability, exterior ballistics, equations of motion of projectile

1. Introduction

Now, ballistic computations use mathematical models of projectile motion of varying degrees of simplification depending on the purpose of the model. One of the following two models is used for developing firing tables: the point-mass model (with 2 degrees of freedom) describing motion of the center of projectile mass with an underlying assumption that the projectile becomes ideally stabilized on its trajectory and the effect of aerodynamic forces can be substituted with the effect of drag force or the modified point-mass model (with 4 degrees of freedom) with one of its implementation contained in STANAG 4355.

A model representing the projectile as a rigid body is one of the most often used for testing dynamic properties of the projectile. In this model, aviation angles (Ψ, Θ, Φ) are used to determine angular position of the projectile relative to the ground-fixed system, and the angle of attack α and angle of sideslip β to determine angular position of the projectile relative to air flow.

Flight stability testing requires a projectile motion model in which the projectile is represented as a rigid body with 6 degrees of freedom, addressing the effect of full aerodynamic force, including specifically Magnus force and moment, to enable stimulation of actual atmospheric flight, particularly for large quadrant elevation QE. This relates to the fact that the inclination angle of the projectile Θ often comes up to $90°$ in the final flight phase and the total angle of attack α_t (contained between the projectile axis and the relative velocity vector) can become large (40 or more degrees) near the vertex.

To develop such a mathematical model, the work uses standard coordinate systems conforming to Polish and International Standard ISO 1151, provided that the transformation matrix between the ground-fixed system $Ox_gy_gz_g$ and the body-fixed system $Oxyz$ uses the new system of Tait-Bryan angles $(\Theta_n, \Psi_n, \Phi_n)$ instead of the conventional aviation angles (Ψ, Θ, Φ) for the avoidance of singularities in kinematic equations for projectile motion around the center of mass. In addition, the paper proposes kinematic equations of motion of the projectile as a rigid body based on Euler parameters.

To eliminate the error from computation of components of the aerodynamic force and moment in the case when the projectile axis deviates significantly from the relative velocity vector, the paper proposes expressing the aerodynamic forces and moments with the total angle of attack α_t and the aerodynamic roll angle φ rather than with the conventional angles: angle of attack α and angle of sideslip β.

2. Using the new system of Tait-Bryan angles in designing kinematic equations for motion of the projectile as a rigid body

To avoid singularities in kinematic equations of motion around the center of projectile mass in a simulation of firing at maximum quadrant elevations (where the inclination angle of the projectile Θ often comes up to $90°$ in the final flight phase), the transformation matrix between the ground-fixed system $Ox_gy_gz_g$ and the body-fixed system $Oxyz$ was derived using the new rotation system (system of Tait-Bryan angles) shown in Fig. 1 (Roberson and Shwertassek, 1988; Wittenburg, 2008) instead of the conventional aviation angles (ISO 1151, 1988): azimuth angle Ψ, inclination angle Θ and bank angle Φ.

The first rotation is around the horizontal axis of the ground-fixed system Oy_g by the new angle of inclination Θ_n, the second rotation is around instantaneous axis Oz_g' by the new angle of azimuth Ψ_n and the third rotation is around the axis Oz_g'' by the new angle of bank Φ_n. The transformation matrix between the ground-fixed system $Ox_gy_gz_g$ and the body-fixed system $Oxyz$ using the new system of Tait-Bryan angles can be obtained from the following dependence

$$\mathbf{L}_{\Phi_n\Psi_n\Theta_n} = \mathbf{L}_{\Phi_n}\mathbf{L}_{\Psi_n}\mathbf{L}_{\Theta_n} \tag{2.1}$$

Using the formulas for elementary matrices (Fig. 1) will provide the following

$$\mathbf{L}_{\Phi_n\Psi_n\Theta_n} = \begin{bmatrix} \cos\Theta_n\cos\Psi_n & \sin\Psi_n & -\sin\Theta_n\cos\Psi_n \\ \sin\Theta_n\sin\Phi_n - \cos\Theta_n\sin\Psi_n\cos\Phi_n & \cos\Psi_n\cos\Phi_n & \cos\Theta_n\sin\Phi_n + \sin\Theta_n\sin\Psi_n\cos\Phi_n \\ \sin\Theta_n\cos\Phi_n + \cos\Theta_n\sin\Psi_n\sin\Phi_n & -\cos\Psi_n\sin\Phi_n & \cos\Theta_n\cos\Phi_n - \sin\Theta_n\sin\Psi_n\sin\Phi_n \end{bmatrix} \tag{2.2}$$

The angular velocity of the body-fixed system $Oxyz$ relative to the ground-fixed system (see Fig. 1) can be expressed with vectors of angular velocities of the new system of Tait-Bryan angles as $\mathbf{\Omega} = \dot{\mathbf{\Psi}}_n + \dot{\mathbf{\Theta}}_n + \dot{\mathbf{\Phi}}_n$, and its components along the axis of the body-fixed system $Oxyz$ can be expressed with the following dependence

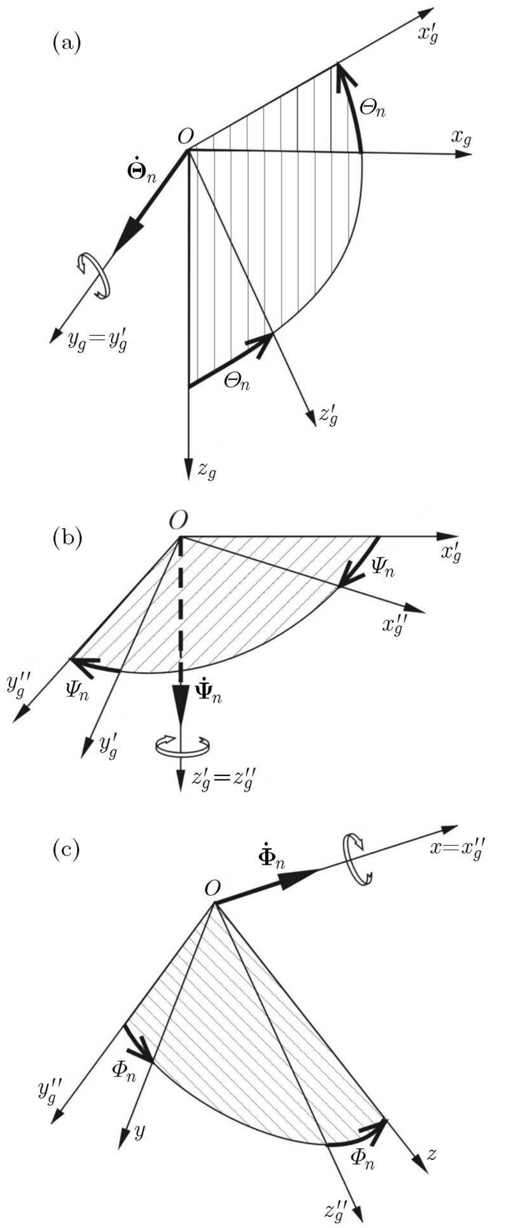
$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_{\Phi_n\Psi_n\Theta_n}\begin{bmatrix} 0 \\ \dot{\Theta}_n \\ 0 \end{bmatrix} + \mathbf{L}_{\Phi_n\Psi_n}\begin{bmatrix} 0 \\ 0 \\ \dot{\Psi}_n \end{bmatrix} + \mathbf{L}_{\Phi_n}\begin{bmatrix} \dot{\Phi}_n \\ 0 \\ 0 \end{bmatrix} \tag{2.3}$$

Resolving appropriate matrix multiplications in equation (2.3) will provide the following

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 & \sin\Psi_n & 1 \\ \sin\Phi_n & \cos\Psi_n\cos\Phi_n & 0 \\ \cos\Phi_n & -\cos\Psi_n\sin\Phi_n & 0 \end{bmatrix}\begin{bmatrix} \dot{\Psi}_n \\ \dot{\Theta}_n \\ \dot{\Phi}_n \end{bmatrix} \tag{2.4}$$

Using the concept of inverse matrix, the equation for derivatives of transformation angles can be expressed as follows

$$\begin{bmatrix} \dot{\Psi}_n \\ \dot{\Theta}_n \\ \dot{\Phi}_n \end{bmatrix} = \begin{bmatrix} 0 & \sin\Psi_n & 1 \\ \sin\Phi_n & \cos\Psi_n\cos\Phi_n & 0 \\ \cos\Phi_n & -\cos\Psi_n\sin\Phi_n & 0 \end{bmatrix}^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{2.5}$$



$$\begin{bmatrix} x'_g \\ y'_g \\ z'_g \end{bmatrix} = \mathbf{L}_{\Theta_n} \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix}$$

where

$$\mathbf{L}_{\Theta_n} = \begin{bmatrix} \cos\Theta_n & 0 & -\sin\Theta_n \\ 0 & 1 & 0 \\ \sin\Theta_n & 0 & \cos\Theta_n \end{bmatrix}$$

$$\begin{bmatrix} x''_g \\ y''_g \\ z''_g \end{bmatrix} = \mathbf{L}_{\Psi_n} \begin{bmatrix} x'_g \\ y'_g \\ z'_g \end{bmatrix}$$

where

$$\mathbf{L}_{\Psi_n} = \begin{bmatrix} \cos\Psi_n & \sin\Psi_n & 0 \\ -\sin\Psi_n & \cos\Psi_n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{L}_{\Phi_n} \begin{bmatrix} x''_g \\ y''_g \\ z''_g \end{bmatrix}$$

where

$$\mathbf{L}_{\Phi_n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi_n & \sin\Phi_n \\ 0 & -\sin\Phi_n & \cos\Phi_n \end{bmatrix}$$

Fig. 1. (a) Rotation of ground-fixed system $Ox_gy_gz_g$ around axis Oy_g by angle Θ_n, (b) rotation of instantaneous system $Ox'_gy'_gz'_g$ around axis Oz'_g by angle Ψ_n, (c) rotation of instantaneous system $Ox''_gy''_gz''_g$ around axis Ox''_g by angle Φ_n

3. Alternative method of determining projectile position relative to air flow for computation of aerodynamic forces and moments

Determination of aerodynamic forces and moments affecting the projectile in flight requires computation of the angular position of the projectile relative to air flow (or projectile velocity vector with respect to the air $\mathbf{V}$).

The most popular method of computing aerodynamic force components consists of determining the angle of attack α and angle of sideslip β. For axial-symmetric artillery projectiles however, it is not the most convenient one because large spin produces continuous change of the angles even if the angles that the forces and moments really depend on do not change so fast.

Aerodynamic forces affecting spinning projectiles operate in the plane of drag and perpendicularly to the plane of drag (Magnus force) independently of the projectile bank angle.

Accordingly, for axial-symmetric flying objects, it is better (for the determination of the angular position of the projectile relative to the vector of velocity $\mathbf{V}$) to use angles that are independent of the projectile spin, such as the total angle of attack α_t and aerodynamic roll angle φ (Baranowski, 2006). The values of the angles, shown in Fig. 2, can be computed from the following equations

$$\alpha_t = \arctan\frac{\sqrt{(w_K - w_W)^2 + (v_K - v_W)^2}}{u_K - u_W} \qquad \varphi = \arctan\frac{w_K - w_W}{v_K - v_W} \tag{3.1}$$

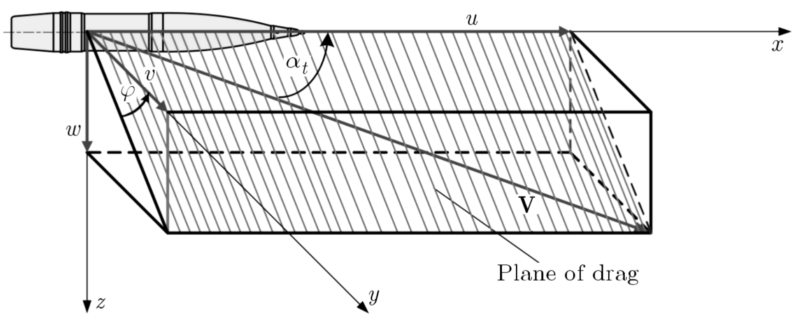


Fig. 2. Illustration of spatial position of angles α_t and φ

The total aerodynamic force $\mathbf{R}^A$ and total aerodynamic moment $\mathbf{M}_O^A$ acting on axial-symmetric spinning projectiles can be presented as follows (Fig. 3)

$$\mathbf{R}^A = \mathbf{R}_\alpha^A + \mathbf{R}_\Omega^A \qquad \mathbf{M}_O^A = \mathbf{M}_{O\alpha}^A + \mathbf{M}_{O\Omega}^A \tag{3.2}$$

where

$\mathbf{R}_\alpha^A$ – aerodynamic force operating in the plane of drag, resulting from the effect of air on non-spinning projectile, the longitudinal axis of which is inclined from the air flow direction by the angle α_t

$\mathbf{R}_\Omega^A$ – aerodynamic force acting perpendicularly to the plane of drag, resulting from the projectile spin and angle α_t (Magnus force)

$\mathbf{M}_{O\alpha}^A$ – aerodynamic moment acting on a non-spinning projectile

$\mathbf{M}_{O\Omega}^A$ – aerodynamic moment resulting from the projectile spin and angle α_t.

To facilitate the determination of components of the aerodynamic force and moment acting on the spinning projectile, the body-fixed system $Oxyz$ uses the splitting of aerodynamic force $\mathbf{R}_\alpha^A$ acting in the plane of drag into a component following the longitudinal axis of the projectile $X^A = C_X^A S\rho V^2/2$ and a component perpendicular to the longitudinal axis of the projectile $P^A = C_N^A(M,\alpha_t)S\rho V^2/2$ (M – Mach number), Fig. 3.

The aerodynamic moment $\mathbf{M}_{O\alpha}^A$ produced by the force P^A is referred to as:

- overturning moment, for spin-stabilized projectiles;
- or stabilizing moment, for fin-stabilized projectiles.

For artillery projectiles, it can be expressed with the coefficient of overturning moment $C_m(M,\alpha_t)$ as follows

$$M_{O\alpha}^A = C_m(M,\alpha_t)\frac{\rho V^2}{2}Sl \tag{3.3}$$

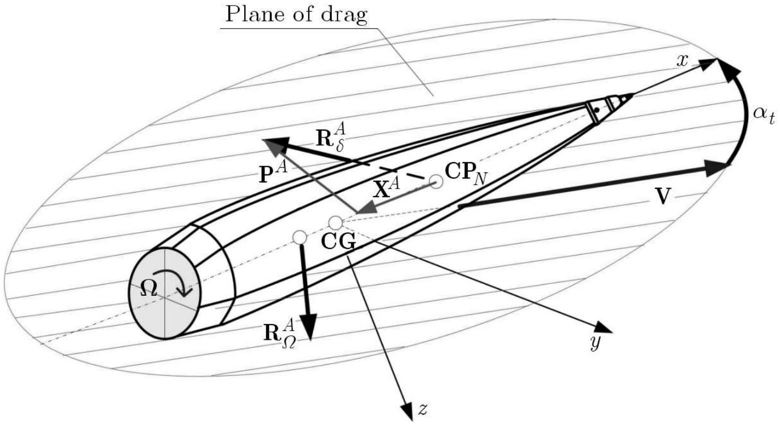


Fig. 3. Components of aerodynamic force acting on the projectile in flight

Using the angle φ, projections of the force P^A and moment $M_{O\alpha}^A$ on the axes Oy and Oz of the body-fixed system $Oxyz$ can be expressed as follows

$$\begin{aligned} Y_\alpha^A &= C_N^A(M,\alpha_t)\frac{\rho V^2}{2}S\cos\varphi & \qquad Z_\alpha^A &= C_N^A(M,\alpha_t)\frac{\rho V^2}{2}S\sin\varphi \\ M_\alpha^A &= C_m(M,\alpha_t)\frac{\rho V^2}{2}Sl\sin\varphi & \qquad N_\alpha^A &= -C_m(M,\alpha_t)\frac{\rho V^2}{2}Sl\cos\varphi \end{aligned} \tag{3.4}$$

Using notations conforming to ISO 1151 (1988), components of the total aerodynamic force $\mathbf{R}^A$ in the body-fixed system $Oxyz$ take the following form:
— axial force

$$X^A = -[C_{X_0}^A(M) + C_{X\alpha^2}^A(M)\alpha_t^2]\frac{\rho V^2}{2}S \tag{3.5}$$

— transverse force

$$Y^A = [-C_{Z\alpha}^A(M)\alpha_t\cos\varphi + C_{Yp\alpha}^A(M)p^*\alpha_t\sin\varphi]\frac{\rho V^2}{2}S \tag{3.6}$$

— normal force

$$Z^A = [-C_{Z\alpha}^A(M)\alpha_t\sin\varphi - C_{Yp\alpha}^A(M)p^*\alpha_t\cos\varphi]\frac{\rho V^2}{2}S \tag{3.7}$$

In turn, components of the total aerodynamic moment $\mathbf{M}_O^A$ in the body-fixed system $Oxyz$ can be expressed as follows

$$\begin{aligned} L^A &= C_{lp}^A(M)\frac{\rho V^2}{2}p^*Sl \\ M^A &= [C_{m\alpha}^A(M)\alpha_t\sin\varphi + C_{mq}^A(M)q^* + C_{np\alpha}^A(M)p^*\alpha_t\cos\varphi]\frac{\rho V^2}{2}Sl \\ N^A &= [-C_{m\alpha}^A(M)\alpha_t\cos\varphi + C_{mq}^A(M)r^* + C_{np\alpha}^A(M)p^*\alpha_t\sin\varphi]\frac{\rho V^2}{2}Sl \end{aligned} \tag{3.8}$$

In accordance with equations (3.5)-(3.8), determination of the main aerodynamic properties of ground artillery projectiles consists of computing the following quantities:

- axial force coefficient for $\alpha_t = 0$ – $C_{X_0}^A(M)$
- derivative of the axial force coefficient – $C_{X\alpha^2}^A(M)$

- derivative of the normal force coefficient – $C^A_{Z\alpha}(M)$
- derivative of the Magnus force coefficient – $C^A_{Yp\alpha}(M)$
- derivative of the spin damping moment coefficient – $C^A_{lp}(M)$
- derivative of the overturning moment coefficient – $C^A_{m\alpha}(M)$
- derivative of the pitch damping moment coefficient – $C^A_{mq}(M)$
- derivative of the Magnus moment coefficient – $C^A_{np\alpha}(M)$

where

$$
\begin{aligned}
&C^A_{X\alpha^2} = \frac{\partial^2 C^A_X}{\partial \alpha^2} \qquad C^A_{Z\alpha} = \frac{\partial C^A_Z}{\partial \alpha} \qquad \dots \qquad C^A_{m\alpha} = \frac{\partial C^A_m}{\partial \alpha} \\
&C^A_{lp} = \frac{\partial C^A_l}{\partial p^*} \qquad C^A_{mq} = \frac{\partial C^A_m}{\partial q^*} \qquad \dots \qquad C^A_{Yp\alpha} = \frac{\partial^2 C^A_Y}{\partial p^* \alpha} \qquad C^A_{np\alpha} = \frac{\partial^2 C^A_n}{\partial p^* \alpha} \qquad C^A_{np} = \frac{\partial C^A_n}{\partial p^*} \\
&p^* = \frac{pd}{2V} \qquad q^* = \frac{qd}{2V} \qquad r^* = \frac{rd}{2V}
\end{aligned}
\tag{3.9}
$$

In the case when the Magnus moment coefficient shows strong non-linear reliance on the total angle of attack, the following equation can be used

$$
C^A_{np\alpha}(M) p^* \alpha_t = C^A_{np}(M, \alpha_t) p^* \tag{3.10}
$$

4. Mathematical model of motion of the projectile as a rigid body

There are two groups of methods for the development of mathematical models of motion of flying objects based on the principles of classical and analytical mechanics. In the group of methods of analytical mechanics, one can distinguish methods based on inertial generalized coordinates and referring directly to the Hamilton principle and Lagrange equations (Koruba *et al.*, 2010) and the methods consisting in applying the equations of analytical mechanics in quasi-coordinates, e.g. Boltzman-Hamel equations (Ładyżyńska-Kozdraś and Koruba, 2012). Classical mechanics uses the law of change of the momentum and angular momentum of a rigid body (Gacek, 1997; Kowaleczko and Żyluk, 2009).

Based on the principles of classical mechanics, spatial motion of the projectile as a rigid body in the frame moving together with the projectile, with the origin of coordinates located in the center of mass of the projectile, can be described with the following vector equations:

— vector dynamic equations of motion

$$
m\Big(\frac{\delta \mathbf{V}_K}{dt} + \mathbf{\Omega}_r \times \mathbf{V}_K\Big) = \mathbf{R}^A + \mathbf{G} + \mathbf{F}_c \qquad \frac{\delta \mathbf{K}_O}{dt} + \mathbf{\Omega}_r \times \mathbf{K}_O = \mathbf{M}^A_O \tag{4.1}
$$

— vector kinematic equations of motion

$$
\frac{d\mathbf{r}_K}{dt} = \mathbf{V}_K \qquad \mathbf{\Omega} = \dot{\mathbf{\Psi}}_n + \dot{\mathbf{\Theta}}_n + \dot{\mathbf{\Phi}}_n \tag{4.2}
$$

where

$\mathbf{V}_K$	–	vector of the projectile velocity with respect to the ground
$\mathbf{K}_O$	–	vector of the projectile angular momentum relative to its center of mass
$\mathbf{\Omega}$	–	vector of the projectile angular velocity
$\mathbf{\Omega}_r$	–	vector of the angular velocity of the frame moving together with the projectile respect to the ground-fixed system $Ox_gy_gz_g$
$\mathbf{\Omega}_Z$	–	vector of the angular velocity of the Earth

$\mathbf{r}_K = [x_g, y_g, z_g]$	–	initial position vector of the center of mass of the projectile and its components in the ground-fixed system $Ox_gy_gz_g$
$\mathbf{R}_A = [X^A, Y^A, Z^A]$	–	vector of the aerodynamic force and its components in the body-fixed system $Oxyz$
$\mathbf{G} = [G_{x_g}, G_{y_g}, G_{z_g}]$	–	vector of the gravity force and its components in the ground-fixed system $Ox_gy_gz_g$
$\mathbf{F}_c = -2m(\mathbf{\Omega}_Z \times \mathbf{V}_K)$	–	vector of the Coriolis force due to rotation of the Earth
$\mathbf{M}_O^A = [L^A, M^A, N^A]$	–	vector of the aerodynamic moment and its components in the body-fixed system $Oxyz$.

The scalar form of the foregoing vector – dynamic and kinematic – equations (in appropriate coordinate systems), together with the complementary equations, represents a mathematical model of motion of the projectile as a rigid body.

4.1. Scalar form of equations of motion in the body-fixed system $Oxyz$

In its final vector-matrix form, the mathematical model of motion of ground artillery projectiles as rigid bodies contains the following groups of equations:
— dynamic differential equations of motion of the projectile center of mass in the body-fixed system $Oxyz$

$$\begin{bmatrix} \dot{u}_K \\ \dot{v}_K \\ \dot{w}_K \end{bmatrix} = \begin{bmatrix} X^A/m \\ Y^A/m \\ Z^A/m \end{bmatrix} + \mathbf{L}_{\Phi_n\Psi_n\Theta_n} \begin{bmatrix} g_{x_g} + F_{Cx_g}/m \\ g_{y_g} + F_{Cy_g}/m \\ g_{z_g} + F_{Cz_g}/m \end{bmatrix} + \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix} \tag{4.3}$$

where components of the Coriolis force in the ground-fixed system $Ox_gy_gz_g$ have the following form

$$\begin{bmatrix} F_{Cx_g} \\ F_{Cy_g} \\ F_{Cz_g} \end{bmatrix} = \begin{bmatrix} 2\Omega(\cos(lat)\sin(AZ)w_{Kg} - \sin(lat)v_{Kg}) \\ 2\Omega(\cos(lat)\cos(AZ)w_{Kg} + \sin(lat)u_{Kg}) \\ -2\Omega(\cos(lat)\cos(AZ)v_{Kg} + \cos(lat)\sin(AZ)u_{Kg}) \end{bmatrix} \tag{4.4}$$

for a spherical model of the Earth, components of the gravitational acceleration in the ground-fixed system $Ox_gy_gz_g$ can be expressed as follows (STANAG 4355 Ed.3, 2009)

$$\begin{bmatrix} g_{x_g} \\ g_{y_g} \\ g_{z_g} \end{bmatrix} = g_n \begin{bmatrix} -x_g/R_z \\ -y_g/R_z \\ 1 + 2z_g/R_z \end{bmatrix} \tag{4.5}$$

and $\Omega = 7.292115 \cdot 10^{-5}\,\mathrm{rad/s}$ – angular speed of the Earth, $g_n = 9.80665(1 - 0.0026\cos(2\,lat))\,\mathrm{m/s^2}$ – magnitude of acceleration due to gravity at the mean sea level, $R_z = 6356766\,\mathrm{m}$ – radius of the sphere, locally approximating the geoid, lat – latitude of the launch point, for the southern hemisphere lat is negative [deg], AZ – azimuth (bearing) of the x_g axis measured clockwise from true North [mil].
— kinematic differential equations of motion of the projectile center of mass

$$\begin{bmatrix} \dot{x}_g \\ \dot{y}_g \\ \dot{z}_g \end{bmatrix} = \begin{bmatrix} u_{Kg} \\ v_{Kg} \\ w_{Kg} \end{bmatrix} = \mathbf{L}^{\mathrm{T}}_{\Phi_n\Psi_n\Theta_n} \begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix} \tag{4.6}$$

— dynamic differential equations of rotational motion about the projectile center of mass in the body-fixed system $Oxyz$ overlapping with the principle central axes of inertia

$$\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L^A \\ M^A \\ N^A \end{bmatrix} + \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{4.7}$$

— kinematic differential equations of rotational motion about the projectile center of mass

$$\begin{bmatrix} \dot{\Psi}_n \\ \dot{\Theta}_n \\ \dot{\Phi}_n \end{bmatrix} = \begin{bmatrix} 0 & \sin\Phi_n & \cos\Phi_n \\ 0 & \cos\Phi_n/\cos\Psi_n & -\sin\Phi_n/\cos\Psi_n \\ 1 & -\cos\Phi_n\tan\Psi_n & \sin\Phi_n\tan\Psi_n \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{4.8}$$

— equation for the total angle of attack α_t

$$\alpha_t = \begin{cases} \dfrac{\pi}{2} & \text{if} \quad u_K - u_W = 0 \\ \arctan \dfrac{\sqrt{(w_K - w_W)^2 + (v_K - v_W)^2}}{u_K - u_W} & \text{otherwise} \end{cases} \tag{4.9}$$

— equation for the aerodynamic roll angle φ

$$\sin\varphi = \begin{cases} 0 & \text{if} \quad \sqrt{(w_K - w_W)^2 + (v_K - v_W)^2} \\ \dfrac{w_K - w_W}{\sqrt{(w_K - w_W)^2 + (v_K - v_W)^2}} & \text{otherwise} \end{cases} \tag{4.10}$$

and

$$\cos\varphi = \begin{cases} 1 & \text{if} \quad \sqrt{(w_K - w_W)^2 + (v_K - v_W)^2} \\ \dfrac{v_K - v_W}{\sqrt{(w_K - w_W)^2 + (v_K - v_W)^2}} & \text{otherwise} \end{cases} \tag{4.11}$$

— complementary equations

$$\begin{array}{lll} \gamma = \arcsin \dfrac{w_{Kg}}{V_K} & \chi = \arctan \frac{v_{Kg}}{u_{Kg}} & \\ u = u_K - u_W & v = v_K - v_W & w = w_K - w_W \\ V = \sqrt{u^2 + v^2 + w^2} & V_K = \sqrt{u_{Kg}^2 + v_{Kg}^2 + w_{Kg}^2} & \end{array} \tag{4.12}$$

where:

u, v, w	–	components of the vector of projectile velocity with respect to the air $\mathbf{V}$ in the body-fixed system $Oxyz$
u_K, v_K, w_K	–	components of the vector of projectile velocity with respect to the ground $\mathbf{V}_K$ in the body-fixed system $Oxyz$
u_{Kg}, v_{Kg}, w_{Kg}	–	components of the vector of projectile velocity with respect to the ground $\mathbf{V}_K$ in the ground-fixed system $Ox_gy_gz_g$
u_W, v_W, w_W	–	components of the vector of wind velocity with respect to the ground $\mathbf{V}_K$ in the body-fixed system $Oxyz$
γ, χ	–	path inclination angle and path azimuth angle, respectively.

A comparison of components of the matrix $\mathbf{L}_{\Phi_n\Psi_n\Theta_n}$, Eq. (2.2), with the matrix $\mathbf{L}_{\Phi\Theta\Psi}$ (Baranowski, 1998; Gacek, 1997) reveals relations between the aviation angles and the new Tait-Bryan angles

$$\sin\Theta = \sin\Theta_n \cos\Psi_n \qquad \sin\Psi = \frac{\sin\Psi_n}{\cos\Theta} \qquad \sin\Phi = \frac{\cos\Theta_n \sin\Phi_n}{\cos\Theta} \tag{4.13}$$

4.2. Using Euler parameters in designing kinematic equations of motion for the projectile as a rigid body

Also Euler parameters (Gajda, 1990; Roberson and Shwertassek, 1988) in the form of quaternions can be used for the avoidance of singularities in kinematic equations of motion around the center of mass. According to Eulers theorem, an object placed in 3D space can be moved from any starting position to any target position with a single rotation around a single axis.

So, to define spatial orientation of a movable axis system (e.g. the body-fixed system $Oxyz$) relative to a fixed system (e.g. the ground-fixed system), it is enough to specify three direction cosines of the axis of rotation (for instance, using Tait-Bryan angles: α_E, β_E,γ_E as parameters describing the instantaneous position of the axis of rotation) and the value of the angle of rotation around the axis δ_E.

These four numbers (3 direction cosines and the angle of rotation) are known as Euler parameters and can be written in the form of quaternions $\boldsymbol{\lambda}$ (Gajda, 1990; Gosiewski and Ortyl, 1995; Wittenburg, 2008).

Quaternions are defined as vector quantities with 4 degrees of freedom

$$\boldsymbol{\lambda} = \lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k \tag{4.14}$$

where i, j, k as imaginary numbers meet the following conditions

$$i^2 = j^2 = k^2 = -1 \qquad ij = -ji = k \qquad jk = -kj = i \qquad ki = -ik = j$$

and

$$\lambda_0 = \cos\frac{\delta_E}{2} \qquad \lambda_1 = \cos\alpha_E \sin\frac{\delta_E}{2} \qquad \lambda_2 = \cos\beta_E \sin\frac{\delta_E}{2} \qquad \lambda_3 = \cos\gamma_E \sin\frac{\delta_E}{2}$$

Quaternion components have to meet an additional combining equation (requirement for orthogonality)

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \tag{4.15}$$

The transformation matrix $\mathbf{T}$ between the body-fixed system $Oxyz$ and the ground-fixed system $Ox_gy_gz_g$ can be presented in two ways:
— using aviation angles: $\varPsi$, $\varTheta$, $\varPhi$ (Baranowski, 1998; Gacek, 1997)

$$\mathbf{T} = \mathbf{L}_{\varPhi\varTheta\varPsi}^{-1} = \begin{bmatrix} \cos\varTheta\cos\varPsi & -\cos\varPhi\sin\varPsi + \sin\varPhi\sin\varTheta\cos\varPsi & \sin\varPhi\sin\varPsi + \cos\varPhi\sin\varTheta\cos\varPsi \\ \cos\varTheta\sin\varPsi & \cos\varPhi\cos\varPsi + \sin\varPhi\sin\varTheta\sin\varPsi & -\sin\varPhi\cos\varPsi + \cos\varPhi\sin\varTheta\sin\varPsi \\ -\sin\varTheta & \sin\varPhi\cos\varTheta & \cos\varPhi\cos\varTheta \end{bmatrix} \tag{4.16}$$

— using quaternions (Gosiewski and Ortyl, 1995)

$$\mathbf{T} = 2\begin{bmatrix} (\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2)/2 & \lambda_1\lambda_2 - \lambda_0\lambda_3 & \lambda_1\lambda_3 + \lambda_0\lambda_2 \\ \lambda_1\lambda_2 + \lambda_0\lambda_3 & (\lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2)/2 & \lambda_2\lambda_3 - \lambda_0\lambda_1 \\ \lambda_1\lambda_3 - \lambda_0\lambda_2 & \lambda_2\lambda_3 + \lambda_0\lambda_1 & (\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2)/2 \end{bmatrix} \tag{4.17}$$

Using quaternions, in the case of deriving equations of motion for the projectile as a rigid body in the body-fixed system $Oxyz$:
— the kinematic differential equations of motion of the projectile center of mass (4.6) are as follows

$$\begin{bmatrix} \dot{x}_g \\ \dot{y}_g \\ \dot{z}_g \end{bmatrix} = \begin{bmatrix} u_{Kg} \\ v_{Kg} \\ w_{Kg} \end{bmatrix} = \mathbf{T}\begin{bmatrix} u_K \\ v_K \\ w_K \end{bmatrix} \tag{4.18}$$

— the kinematic differential equations of rotational motion about the projectile center of mass (4.8) have the following form (Baranowski *et al.*, 2005; Baranowski, 2006)

$$\begin{aligned}
\frac{d\lambda_0}{dt} &= \frac{1}{2}(-\lambda_1 p - \lambda_2 q - \lambda_3 r) & \qquad \frac{d\lambda_1}{dt} &= \frac{1}{2}(\lambda_0 p - \lambda_3 q + \lambda_2 r)\\
\frac{d\lambda_2}{dt} &= \frac{1}{2}(\lambda_3 p + \lambda_0 q - \lambda_1 r) & \qquad \frac{d\lambda_3}{dt} &= \frac{1}{2}(-\lambda_2 p + \lambda_1 q + \lambda_0 r)
\end{aligned} \tag{4.19}$$

Unlike the description using Euler and Tait-Bryan angles, it is a system of four differential equations in which the solution remains within the $[-1,1]$ range, which facilitates numerical computations.

The main computational problem in the quaternion model is the meeting of combining equation (4.15). The quaternions are "improved" in order to satisfy this equation. The improving algorithm has the following form (Ortyl, 2000)

$$\begin{bmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} - \varepsilon_w \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \tag{4.20}$$

where ε_w – rate of violation of the combining equation (ideally $\varepsilon_w = 0$)

$$\varepsilon_w = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 1 \tag{4.21}$$

The use of quaternions relates to difficulty with physical interpretation of quaternions as they relate to the orientation of the axis of rotation rather than the orientation of the object itself. Therefore, to interpret the computation results correctly, we need to transform these parameters onto aviation angles, which are natural coordinates defining the position of the flying object in space.

Taking advantage of the fact that individual components of the matrix $\mathbf{T}$ are equal one to another, based on equations (4.16) and (4.17), the following relations can be established between the aviation angles and quaternions

$$\begin{aligned}
\sin\Theta &= -T_{31} = 2(\lambda_0\lambda_2 - \lambda_1\lambda_3) & \qquad & -\frac{\pi}{2} \leqslant \Theta \leqslant \frac{\pi}{2}\\
\tan\Psi &= \frac{T_{21}}{T_{11}} = \frac{2(\lambda_1\lambda_2 + \lambda_0\lambda_3)}{\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2} & \qquad & -\pi < \Psi \leqslant \pi\\
\tan\Phi &= \frac{T_{32}}{T_{33}} = \frac{2(\lambda_2\lambda_3 + \lambda_0\lambda_1)}{\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2} & \qquad & 0 < \Phi \leqslant 2\pi
\end{aligned} \tag{4.22}$$

and between quaternions and aviation angles (Gajda, 1990)

$$\begin{aligned}
\lambda_0 &= \cos\frac{\Psi}{2}\cos\frac{\Theta}{2}\cos\frac{\Phi}{2} + \sin\frac{\Psi}{2}\sin\frac{\Theta}{2}\sin\frac{\Phi}{2}\\
\lambda_1 &= \cos\frac{\Psi}{2}\cos\frac{\Theta}{2}\sin\frac{\Phi}{2} - \sin\frac{\Psi}{2}\sin\frac{\Theta}{2}\cos\frac{\Phi}{2}\\
\lambda_2 &= \cos\frac{\Psi}{2}\sin\frac{\Theta}{2}\cos\frac{\Phi}{2} + \sin\frac{\Psi}{2}\cos\frac{\Theta}{2}\sin\frac{\Phi}{2}\\
\lambda_3 &= \sin\frac{\Psi}{2}\cos\frac{\Theta}{2}\cos\frac{\Phi}{2} - \cos\frac{\Psi}{2}\sin\frac{\Theta}{2}\sin\frac{\Phi}{2}
\end{aligned} \tag{4.23}$$

Using quaternions instead of Tait-Bryan angles in kinematic equations of motion for artillery projectiles can provide the following benefits:

- complete elimination of singular points in computing projectile position,

- shorter computation time,
- less numerical errors during simulation thanks to algebraic computations instead of numerical computations of trigonometric functions in the form of expansion in Taylor series with omission of terms of higher orders (which is the case with solving kinematic equations of motion for projectile incorporating Tait-Bryan angles).

5. Summary and conclusions

The paper presents a complete mathematical model of motion of a balanced spin-stabilized projectile considered as a rigid body with 6 degrees of freedom in coordinate systems conforming to ISO 1151. The resulting scalar equations of motion for the projectile, free from singularities, enable simulation of the flight of projectiles fired at the whole range of gun quadrant elevation ($0 < QE < \pi/2$) both in standard and disturbed conditions.

The new method of expressing the aerodynamic force and moment using the total angle of attack α_t and aerodynamic bank angle φ enables correct computation of the whole aerodynamic effect (taking into account the Magnus effect) even for large α_t.

For the modeling of flight of ground artillery projectiles, there is no need for improving quaternions using equation (4.20) because the time of the simulated process is relatively short.

Because of its features, the mathematical model proposed in the paper seems to be particularly suitable for testing stability of flight of projectiles fired at a large quadrant elevation.

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Równania ruchu pocisku stabilizowanego obrotowo na potrzeby badania stabilności lotu

Streszczenie

W pracy przedstawiono model matematyczny ruchu wyważonego pocisku stabilizowanego obrotowo traktowanego jako bryła sztywna o sześciu stopniach swobody. W modelowaniu zastosowano układy odniesienia zgodne z Polską i Międzynarodową Normą ISO 1151. W konstruowaniu kinematycznych równań ruchu dookoła środka masy zaproponowano wykorzystanie układu kątów Taita-Bryana lub parametrów Eulera. Siły i momenty aerodynamiczne wyrażono poprzez kąt nutacji oraz kąt przechylenia aerodynamicznego.

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