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FLEXIBILITY OF CRACKED COMPOSITE BEAMS

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The paper presents a method of creating of the flexibility matrix of a cracked composite beam. The presented method has general character and may be used to determine the flexibility matrix for beams with other types of cracks (double-edge, elliptical, internal, etc.) if their stress intensity factors are known. The presented flexibility matrix can be applied to formation of stiffness matrix of cracked beam finite elements.

1. Introduction

Vibration monitoring can be applied to detection of fatigue cracks in all the cases where typical monitoring methods (ultrasonic, x-ray or visual inspection) are useless i.e. civil engineering structures, rotating shafts and blades, constructional elements made of composite materials (Rytter et al., 1991).

A crack influences stiffness of a structure, and the stiffness, on the other hand, influences dynamic behavior of such a system. It means that measurement of these dynamic characteristics during the lifetime of the structure can be used as a basis for identification of structural damage. The reliability of vibration monitoring depends on determination of these characteristics which are most sensitive on appearance of cracks in structures. Information on the sensitivity of dynamic characteristics can be obtained by constructing theoretical models of changes in the structure stiffness (cf Dimarogonas and Paipetis, 1983; Krawczuk, 1993).

The main aim of this paper is to elaborate the model of local flexibility changes of the beam made of unidirectional composite material due to the transverse, one-edge, nonpropagating, open crack. The method of flexibility matrix formation is based on the Castigliano theorem and laws of fracture mechanics. The elaborated method, described in this paper, may be used to determination the flexibility matrix for beams with other types of cracks (double-edge, elliptical etc.) if their stress intensity factors are known.

2. The method the flexibility matrix of the cracked beam formation

In the case of unidirectional composite materials geometrical form of the crack is a function of the number of loading cycles (Talreja, 1989). At the initial stage of the damage process dominant mechanism is the fiber breakage in the direction perpendicular to the direction of maximal strains. The analysis presented in this paper is restricted to this case.

The terms c_{ij} of flexibility matrix of linear-elastic body can be determined making use of the Castigliano theorem in the form

$$c_{ij} = \frac{\partial^2 U}{\partial P_i \partial P_j} \tag{2.1}$$

where U is the elastic strain energy of elastic body, $P_i - P_j$ denote independent forces acting on the body.

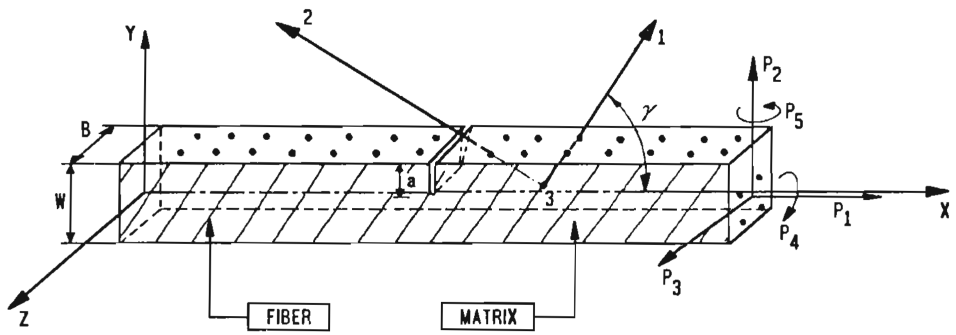


Fig. 1. Composite beam with the fatigue crack

In the case of the composite beam presented in Fig.1 the elastic strain energy U due to the crack is (cf Nikpur and Dimarogonas, 1988)

$$U = \int\limits_A \Big(D_1 \sum_{i=1}^{5} K_{Ii}^2 + D_2 \sum_{i=1}^{5} K_{Ii} \sum_{i=1}^{5} K_{IIi} + D_3 \sum_{i=1}^{5} K_{IIi}^2 + D_4 \sum_{i=1}^{5} K_{IIIi}^2\Big)\, dA \tag{2.2}$$

where A is the area of the crack, K_{ji} denote the stress intensity factors corresponding to three modes of the crack evaluation (j) and with independent forces (i) acting on the beam, $D_1 \div D_4$ are coefficients given by the following relations (cf Sih and Chen, 1981)

$$
\begin{aligned}
D_1 &= -\frac{1}{2}A'_{11}\mathrm{Im}\Big(\frac{s_1+s_2}{s_1s_2}\Big) & D_2 &= \frac{1}{2}A'_{11}\mathrm{Im}(s_1+s_2) \\
D_3 &= A'_{11}\mathrm{Im}(s_1s_2) & d_4 &= \frac{1}{2}\sqrt{A_{44}A_{55}}
\end{aligned}
\tag{2.3}
$$

and s_1, s_2 are the roots of the following characteristic equation (cf Sih and Chen, 1981)

$$
A'_{11}s^4 - 2A'_{16}s^3 + (2A'_{12} + A'_{66})s^2 - 2A'_{26}s + A'_{22} = 0 \tag{2.4}
$$

The method of calculation of the roots s_1 and s_2 is shown in the Appendix. The roots of the characteristic equation (2.4) are either complex or pure imaginary and cannot be real.

Generally, for composite materials, stress intensity factors are not equivalent to those of isotropic bodies of the same geometry and under the same loading conditions (Sih and Chen, 1981). Nevertheless, according to computations performed by Bao et al. (1992) the stress intensity factors K_{ji} for the analyzed composite beam (Fig.1) can be expressed as the stress intensity factors of the isotropic beam multiplied by correction functions Y_i $(i = 1,3)$

$$
\begin{aligned}
K_{I1} &= \frac{P_1}{BW}\sqrt{\pi a}F_1\Big(\frac{a}{W}\Big)Y_1(\zeta) \\
K_{I4} &= \frac{6P_4}{BW^2}\sqrt{\pi a}F_2\Big(\frac{a}{W}\Big)Y_1(\zeta) \\
K_{I5} &= \frac{12P_5z}{B^3W}\sqrt{\pi a}F_2\Big(\frac{a}{W}\Big)Y_1(\zeta) \\
K_{II3} &= \frac{2P_3}{BW\sqrt{\pi a}}F_3\Big(\frac{a}{W}\Big)Y_2(\zeta) \\
K_{III2} &= \frac{2P_2}{BW\sqrt{\pi a}}F_4\Big(\frac{a}{W}\Big)Y_3(\zeta)
\end{aligned}
\tag{2.5}
$$

where $F_1 \div F_4$ are functions which take into account finite dimensions of the beam (cf Dimarogonas and Paipetis, 1983)

$$
F_1 = \sqrt{\frac{\tan\lambda}{\lambda}}\frac{0.752 + 2.02\frac{a}{W} + 0.37(1-\sin\lambda)^3}{\cos\lambda}
$$

$$F_2 = \sqrt{\frac{\tan\lambda}{\lambda}}\frac{0.923 + 0.199(1-\sin\lambda)^4}{\cos\lambda}$$

$$F_3 = \frac{1.30 - 0.65\frac{a}{W} + 0.37\left(\frac{a}{W}\right)^2 + 0.28\left(\frac{a}{W}\right)^3}{\sqrt{1-\frac{a}{W}}}$$

$$F_4 = \sqrt{\frac{\lambda}{\sin\lambda}}$$

and $\lambda = \frac{\pi a}{2W}$.

The correction functions $Y_1 \div Y_3$ which transform stress intensity factors of isotropic body to the stress intensity factors of composite have the following forms (cf Bao et al., 1992)

$$\begin{aligned} Y_1 &= \zeta + 0.1(\zeta - 1) - 0.016(\zeta - 1)^2 + 0.002(\zeta - 1)^3 \\ Y_2 &= Y_3 = 1.0 \end{aligned} \tag{2.6}$$

where

$$\zeta = \frac{\sqrt{E_{11}E_{22}} - 2G_{12}\nu_{12}}{2G_{12}}$$

see Appendix.

Substituting Eqs (2.5), (2.6) and (2.7) into Eq (2.2) and taking Eq (2.1) into account yield the terms of flexibility matrix of the composite beam with the transverse, one-edge, nonpropagating, open crack

$$c_{11} = \frac{2\pi D_1}{B}\int\limits_0^{\bar{a}_k} \bar{a}F_1^2Y_1^2\; d\bar{a} \qquad c_{13} = \frac{2D_2}{B^2}\int\limits_0^{\bar{a}_k} F_1F_3Y_1\; d\bar{a}$$

$$c_{14} = \frac{12\pi D_1}{B^2}\int\limits_0^{\bar{a}_k} \bar{a}F_1^2Y_1^2\; d\bar{a} \qquad c_{15} = \frac{12\pi D_1}{BW}\int\limits_0^{\bar{a}_k} \bar{a}F_1F_2Y_1^2\; d\bar{a}$$

$$c_{22} = \frac{8D_3}{\pi B^2W^2}\int\limits_0^{\bar{a}_k} \frac{F_3^2}{\bar{a}}\; d\bar{a} \qquad c_{33} = \frac{8D_4}{\pi B^2W^2}\int\limits_0^{\bar{a}_k} \frac{F_4^2}{\bar{a}}\; d\bar{a}$$

$$c_{34} = \frac{12D_2}{B^2}\int\limits_0^{\bar{a}_k} F_1F_3Y_1\; d\bar{a} \qquad c_{35} = \frac{12D_2}{B^2W}\int\limits_0^{\bar{a}_k} F_2F_3Y_1\; d\bar{a}$$

$$c_{44} = \frac{96\pi D_1}{B^3}\int\limits_0^{\bar{a}_k} \bar{a}F_1^2Y_1^2\; d\bar{a} \qquad c_{45} = \frac{72\pi D_1}{B^2W}\int\limits_0^{\bar{a}_k} \bar{a}F_1F_2Y_1^2\; d\bar{a}$$

$$c_{55} = \frac{72\pi D_1}{B^2W^2} \int\limits_0^{\bar{a}_k} \bar{a}F_2^2Y_1^2 \, d\bar{a}$$

where $\bar{a} = a/W$.

3. Numerical calculations

The numerical calculations were carried out for the beam made of graphite-fiber reinforced polyamide – Fig.1. The following mechanical properties of the material were assumed (Nikpur and Dimarogonas, 1988)

	E [GPa]	G [GPa]	ν
graphite	276	115	0.2
polyimde	2.76	1.04	0.33

Fig.2 shows the variations of coefficients $D_1 \div D_4$ versus the fiber volume fraction and the fiber angle. The values of the coefficients D_1, D_2 and D_4 decrease as the fiber volume fraction increases. The coefficient D_2 reaches the maximal value for the fiber volume fraction equal to 10%.

Fig.3 presents the changes of the terms of flexibility matrix as a function of the relative depth of the crack. In general, when the depth of the crack increases the value in terms of the flexibility matrix increase, too.

The biggest values are reached by the terms corresponding to the bending moments c_{44}, c_{45}, c_{55}, respectively.

4. Conclusions

The entries of the flexibility matrix of cracked composite prismatic beam are a function of the crack depth. On the other hand the form of the flexibility matrix indicate the presence of various coupling terms which illustrate shearing and bending and also shearing and tension modes deformation due to the crack. In isotropic materials only the coupling of bending and tension deformations is observed.

The coupling effects of deformations due to the crack, complicate static and dynamic behavior of the constructional elements made of unidirectional

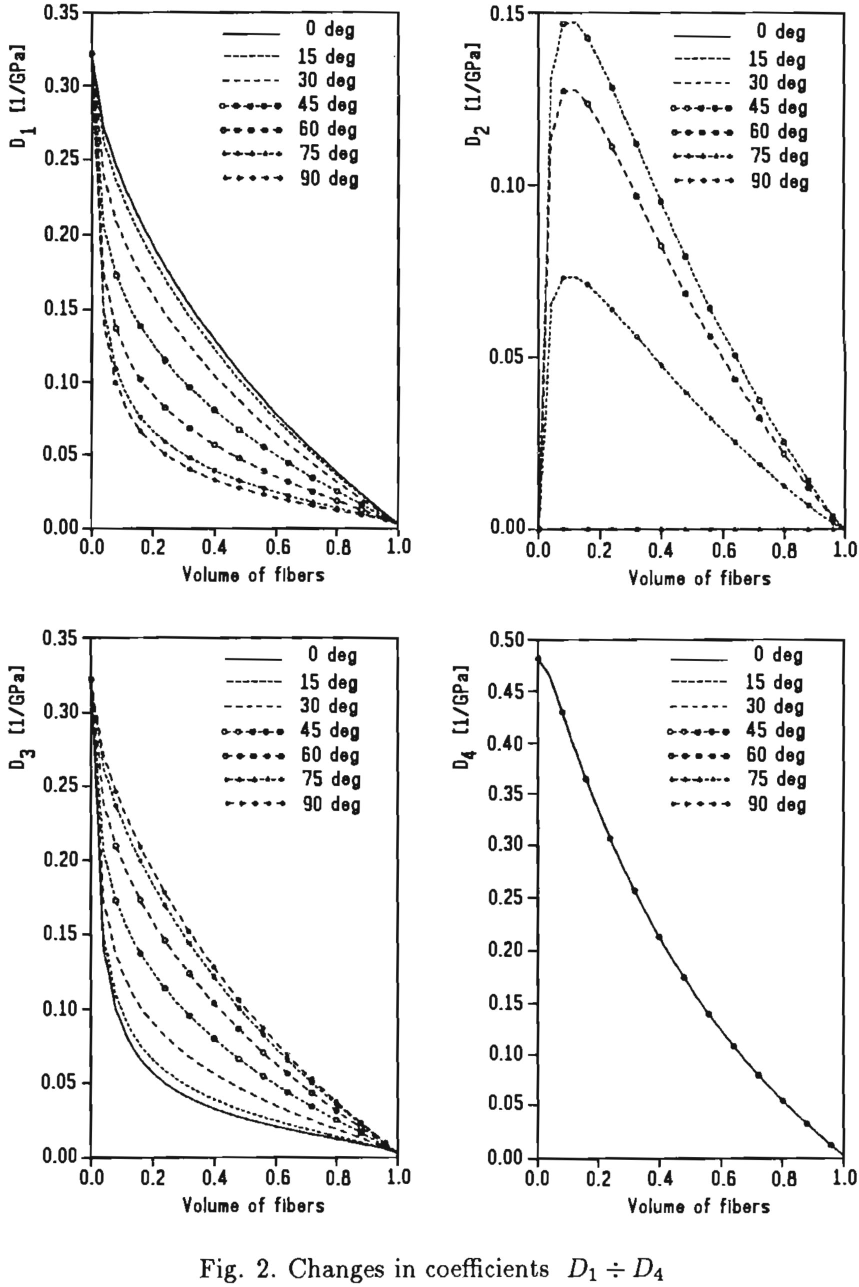
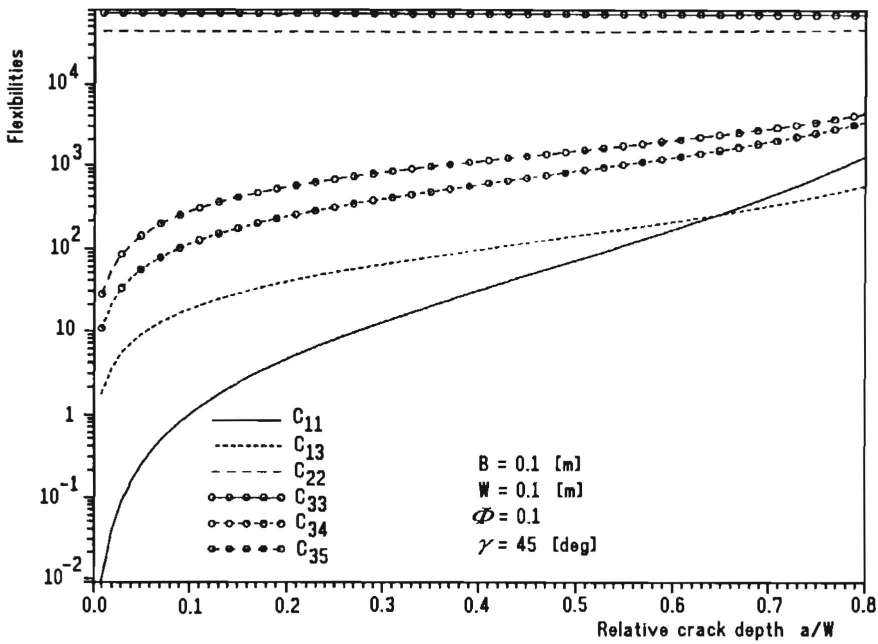


Fig. 2. Changes in coefficients $D_1 \div D_4$



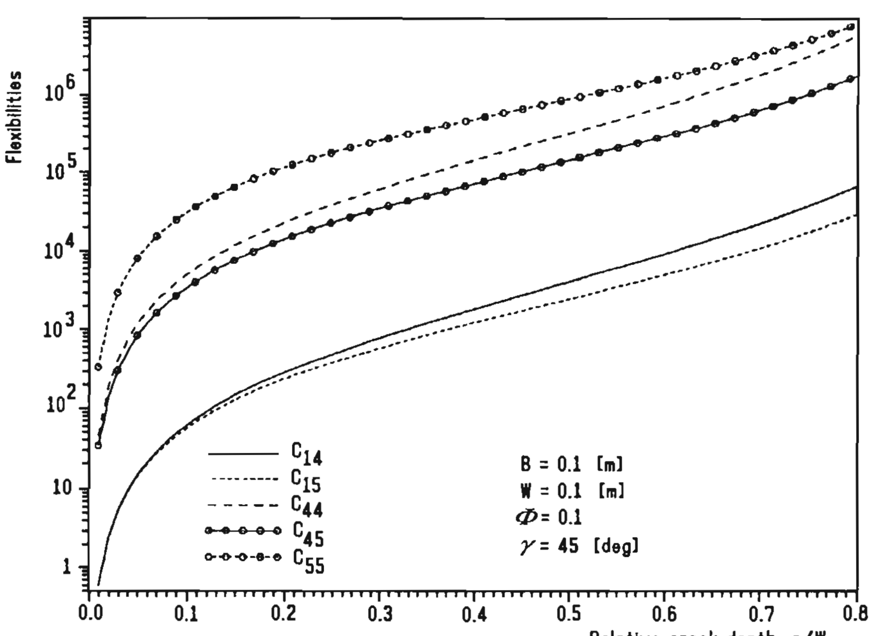


Fig. 3. The changes of the terms of the flexibility matrix

composite materials. This complication, however, provides a more powerful tool to diagnose the existence of cracks in composite bodies.

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Appendix

The mechanical properties of unidirectional composite materials are calculated using the following formulae (cf Sih and Chen, 1981)

$$
\begin{aligned}
E_{11} &= E_f + E_m(1-\Phi) \\
E_{22} = E_{33} &= E_m \frac{(E_f + E_m) + (E_f - E_m)\Phi}{(E_f + E_m) - (E_f - E_m)\Phi} \\
\nu_{12} = \nu_{13} &= \nu_f \Phi + \nu_m(1-\Phi) \\
\nu_{23} = \nu_{32} &= \nu_f \Phi + \nu_m(1-\Phi)\frac{1+\nu_m - \nu_{12}\frac{E_m}{E_{11}}}{1-\nu_m^2 - \nu_m\nu_{12}\frac{E_m}{E_{11}}} \\
G_{12} = G_{13} &= G_m \frac{(G_f + G_m) + (G_f - G_m)\Phi}{(G_f + G_m) - (G_f - G_m)\Phi} \\
G_{23} &= \frac{E_{22}}{2(1+\nu_{23})}
\end{aligned}
\tag{A.1}
$$

where the lower index f denotes fibers and the lower index m denotes a matrix of a composite, E is the Young modulus, G is shear modulus, ν denotes the Poisson ratio, Φ is the fiber volume fraction and lower indexes 1,2,3 denote material principal axes system connected with the fibers of the composite – Fig.1.

In the case when the geometric axes are rotated through γ degrees with respect to the material principal axes, terms of the strain-stress matrix are given as (cf Sih and Chen, 1981)

$$
\begin{aligned}
A'_{11} &= A_{11}m^4 + (2A_{12} + A_{66})m^2n^2 + A_{22}n^4 \\
A'_{22} &= A_{22}m^4 + (2A_{12} + A_{66})m^2n^2 + A_{11}n^4 \\
A'_{12} &= (A_{11} + A_{22} - A_{66})m^2n^2 + A_{12}(m^4 + n^4) \\
A'_{66} &= 2(2A_{11} + 2A_{22} - 4A_{12} - A_{66})m^2n^2 + A_{66}(m^4 + n^4) \\
A'_{16} &= (-2A_{11} + 2A_{12} + A_{66})m^3n + (2A_{22} - 2A_{12} - A_{66})nm^3 \\
A'_{26} &= (-2A_{11} + 2A_{12} + A_{66})mn^3 + (2A_{22} - 2A_{12} - A_{66})m^3n
\end{aligned}
\tag{A.2}
$$

where $m = \cos\gamma$, $n = \sin\gamma$ and A_{ij} are the elements of strain-stress matrix of the composite along the principal axes. Under plane strain conditions, these are related to the mechanical constants of material by (cf Sih and Chen, 1981)

$$
\begin{aligned}
A_{11} &= \frac{1}{E_{11}}\Big(1 - \frac{E_{22}}{E_{11}}\nu_{12}^2\Big) \\
A_{22} &= \frac{1}{E_{22}}(1 - \nu_{23}^2) \\
A_{12} &= -\frac{\nu_{12}}{E_{22}}(1 + \nu_{23}) \\
A_{66} &= \frac{1}{G_{12}} \qquad\qquad A_{44} = \frac{1}{G_{23}} \\
A_{55} &= A_{66} \qquad\qquad A_{45} = 0
\end{aligned}
\tag{A.3}
$$

Podatność belek kompozytowych z pęknięciami

Streszczenie

W pracy przedstawiono metodę wyznaczania macierzy podatności, belki wykonanej z materiału kompozytowego, wywołanej pęknięciem zmęczeniowym. Prezentowana metoda ma ogólny charakter i może być wykorzystana w przypadku analizy różnego rodzaju struktur z pęknięciami o dowolnej postaci geometrycznej, pod warunkiem, że znane są współczynniki intensywności naprężeń analizowanego pęknięcia. Omawiana macierz podatności może być wykorzystana do budowy macierzy sztywności belkowych elementów skończonych z pęknięciami.

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