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FREE VIBRATION OF A SYSTEM COMPOSED OF TWO BEAMS SEPARATED BY AN ELASTIC LAYER

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The problem of transverse free vibration of a system of two axially loaded beams which are separated by an elastic layer is considered. The solution of the problem for different types of boundary conditions is performed. The frequency equation is obtained in the case when the first beam of the system is compressed under the force P_1 while the second one under goes action of tensile force $P_2 = -P_1$. Influence of the stiffness modulus of the elastic layer on vibration frequencies of the compound system is investigated.

1. Introduction

Vibration problems in the compound systems are of great importance since their solutions very often have direct reference to real systems. In the case of vibration of beam systems for many practical applications the classic Bernoulli-Euler theory of beams is aften applied. The systems composed of beams separated by an elastic layer have been considered by Khatua and Cheung (1973), Oniszczuk (1974) Yankelevsky (1991), Roy and Ganesan (1992).

Khatua and Cheung (1973) applied the finite element method to the free vibration problem of the sandwich type structure which is composed of multilayer beams or plates. Oniszczuk (1974) studied transverse vibration of the system composed of two prismatic pinned-pinned and free-free beams coupled by an elastic element. In that paper a complete solution of the problem of the free and forced vibration of beams, neglecting the effect of axial forces, has been given. Yankelevsky (1991) presented analysis of the system consisting of N beams and N elastic layers. Deflection lines, bending moments and shear forces have been determined for all the beams of the system. Investigations concerning the effect of the damping layer on the vibration of the two-beam system has been presented by Roy and Ganesan (1992). The analysis of some problems of free vibration damping in homogeneous and layered beams and bands has been carried out by Karczmarzyk and Osiński (1992). Kukla and Skalmierski (1993) presented a solution of the problem of single beam vibration under an axial force varying along the length. In the case of forced vibration the flux of energy, which is emitted by a vibrating beam, has been determined.

In the present paper a solution of the free vibration problem of the system of two beams under axial forces and separated by an elastic layer is demonstrated. Stiffness modulus of the elastic layer has been considered as being constant and in the second case as described by the stepped function. Numerical examples have proved the influence of an elastic layer on the free vibration frequencies of the system.

2. Formulation of the problem

Consider a system of two beams of the length L which are separated by an elastic layer with the stiffness modulus k. Let us assume that the beams are loaded by longitudinal forces p_1 and p_2, respectively. The differential equations of motion at a small amplitude of vibration of the beams are given by

$$E_1 I_1 \frac{\partial^4 y_1}{\partial x^4} + p_1 \frac{\partial^2 y_1}{\partial x^2} + k(y_2 - y_1) + \rho_1 A_1 \frac{\partial^2 y_1}{\partial t^2} = 0 \tag{2.1}$$

$$E_2 I_2 \frac{\partial^4 y_2}{\partial x^4} + p_2 \frac{\partial^2 y_2}{\partial x^2} + k(y_1 - y_2) + \rho_2 A_2 \frac{\partial^2 y_2}{\partial t^2} = 0 \tag{2.2}$$

Here

$E_i I_i$	–	modulus of flexural rigidity of the ith beam $(i = 1,2)$
A_i	–	cross-sectional area
ρ_i	–	mass density
y_i	–	lateral deflection of the ith beam
x	–	distance along the beam length
t	–	time.

The boundary conditions for beams of the system may be written as follows

$$\frac{\partial^m y_i}{\partial x^m}(0,t) = \frac{\partial^n y_i}{\partial x^n}(0,t) = 0 \qquad i = 1,2 \tag{2.3}$$

$$\frac{\partial^{m'} y_i}{\partial x^{m'}}(L,t) = \frac{\partial^{n'} y_i}{\partial x^{n'}}(L,t) = 0 \qquad i = 1,2 \tag{2.4}$$

where $m, m' = 0, 1$ and $n, n' = 1, 2, 3$ $(m < n,\ m' < n')$. The boundary conditions (2.3) and (2.4) represent three basic ways of the beam ends supporting: clamped $(m = 0, n = 1;\ m' = 0, n' = 1)$, pinned $(m = 0, n = 2;\ m' = 0, n' = 2)$ and sliding $(m = 1, n = 3;\ m' = 1, n' = 3)$. If the beam is free at the end $x = 0$ or $x = L$ then the boundary conditions have the form

$$\frac{\partial^2 y_i}{\partial x^2} = \frac{\partial^3 y_i}{\partial x^3} + \frac{p_i}{E_i I_i}\frac{\partial y_i}{\partial x} = 0 \qquad \text{for} \quad i = 1,2 \quad \text{and} \quad x = 0 \quad \text{or} \quad x = L \tag{2.5}$$

All possible combinations of the boundary conditions (2.3), (2.4) and (2.5) consist of ten cases of different attachments of the beam ends.

For the free vibration of the system we can assume

$$\begin{aligned} y_1(x,t) &= Y_1(x)\cos\omega t \\ y_2(x,t) &= Y_2(x)\cos\omega t \end{aligned} \tag{2.6}$$

Substituting Eq (2.6) into Eqs (2.1) and (2.2) and introducing dimensionless coordinates we get

$$\frac{d^4 Z_1}{d\xi^4} + P_1\frac{d^2 Z_1}{d\xi^2} + K_1\Big(Z_1(\xi) - Z_2(\xi)\Big) - \Omega_1^4 Z_1(\xi) = 0 \tag{2.7}$$

$$\frac{d^4 Z_2}{d\xi^4} + P_2\frac{d^2 Z_2}{d\xi^2} + K_2\Big(Z_2(\xi) - Z_1(\xi)\Big) - \Omega_2^4 Z_2(\xi) = 0 \tag{2.8}$$

where

$$\xi = \frac{x}{L} \qquad Z_i = \frac{Y_i}{L} \qquad K_i = \frac{kL^4}{E_i I_i}$$

$$P_i = \frac{p_i L^2}{E_i I_i} \qquad \Omega_i^4 = \frac{\rho_i A_i L^4}{E_i I_i}\omega^2$$

3. Analysis

3.1. Free vibration of beams separated by an elastic layer with constant stiffness modulus

Assuming from Eqs (2.7) and (2.8) the functions $Z_1(\xi)$ and $Z_2(\xi)$ as follows

$$Z_1(\xi) = W_1 e^{\lambda\xi} \qquad Z_2(\xi) = W_2 e^{\lambda\xi} \tag{3.1}$$

one obtains the system of equations

$$(\lambda^4 + P_1\lambda^2 + K_1 - \Omega_1^4)W_1 - K_1W_2 = 0 \tag{3.2}$$

$$K_2W_1 - (\lambda^4 + P_2\lambda^2 + K_2 - \Omega_2^4)W_2 = 0 \tag{3.3}$$

The non-zero solutions of the system of equations (3.2) and (3.3) exist if λ satisfies the characteristic equation. This equation may be written as follows

$$\begin{aligned}\lambda^8 + (P_1 + P_2)\lambda^6 + (K_1 + K_2 + P_1P_2 - \Omega_1^4 - \Omega_2^4)\lambda^4 + [P_1(K_2 - \Omega_2^4) + \\ +P_2(K_1 - \Omega_1^4)]\lambda^2 + (K_1 - \Omega_1^4)(K_2 - \Omega_2^4) - K_1K_2 = 0\end{aligned} \tag{3.4}$$

Below we will discuss the case of beam systems in which the beams have identical flexural rigidity and the same mass per unit length, i.e. we assume that the conditions: $E_2I_2 = E_1I_1$, $\rho_2A_2 = \rho_1A_1$ are satisfied. Moreover, we assume that for the longitudinal forces the relationship: $P_1 = -P_2$ is valid. In the case under consideration Eq (3.4) has the form

$$\lambda^8 + [2(K - \Omega^4) - P^2]\lambda^4 + (\Omega^4 - 2K)\Omega^4 = 0 \tag{3.5}$$

where $K = K_1 = K_2$, $P = P_1 = -P_2$ and $\Omega = \Omega_1 = \Omega_2$. The roots of Eq (3.5) are as follows

— for $\Omega^4 \geq 2K$

$$\lambda_{1,2} = \pm\alpha \qquad \lambda_{3,4} = \pm i\alpha$$

$$\lambda_{5,6} = \pm\beta \qquad \lambda_{7,8} = \pm i\beta$$

— for $\Omega^4 < 2K$

$$\lambda_{5,6} = \pm(1 + i)\gamma \qquad \lambda_{7,8} = \pm(1 - i)\gamma$$

where

$$\alpha = \sqrt[4]{\frac{1}{2}P^2 - K + \Omega^4 + \sqrt{\Delta}} \qquad \beta = \sqrt[4]{\frac{1}{2}P^2 - K + \Omega^4 - \sqrt{\Delta}}$$

$$\gamma = \sqrt[4]{\frac{1}{4}\Big(K - \frac{1}{2}P^2 - \Omega^4 + \sqrt{\Delta}\Big)} \qquad \Delta = \Big(K - \frac{1}{2}P^2\Big)^2 + P^2\Omega^4$$

The solution of the system of differential equations (2.7) and (2.8) will be derived for two cases.

- Case 1, for $\Omega^4 > 2K$

General solution of the system of equations (2.7) and (2.8) may be presented in the form

$$\begin{aligned} Z_1(\xi) &= C_1\Phi_0(\alpha\xi) + C_2\Phi_1(\alpha\xi) + C_3\Phi_2(\alpha\xi) + C_4\Phi_3(\alpha\xi) + C_5\Phi_0(\beta\xi) + \\ &+ C_6\Phi_1(\beta\xi) + C_7\Phi_2(\beta\xi) + C_8\Phi_3(\beta\xi) \end{aligned} \tag{3.6}$$

$$\begin{aligned} Z_2(\xi) &= C_1[a_1\Phi_0(\alpha\xi) + b_1\Phi_2(\alpha\xi)] + C_2[a_1\Phi_1(\alpha\xi) + b_1\Phi_3(\alpha\xi)] + \\ &+ C_3[a_1\Phi_2(\alpha\xi) + b_1\Phi_0(\alpha\xi)] + C_4[a_1\Phi_3(\alpha\xi) + b_1\Phi_1(\alpha\xi)] + \\ &+ C_5[a_2\Phi_0(\beta\xi) + b_2\Phi_2(\beta\xi)] + C_6[a_2\Phi_1(\beta\xi) + b_2\Phi_3(\beta\xi)] + \\ &+ C_7[a_2\Phi_2(\beta\xi) + b_2\Phi_0(\beta\xi)] + C_8[a_2\Phi_3(\beta\xi) + b_2\Phi_1(\beta\xi)] \end{aligned} \tag{3.7}$$

where C_i $(i = 1,...,8)$ are arbitrary constants and the functions $\Phi_j(u)$ $(j = 1,...,4)$ are defined as follows (cf Milne (1991))

$$\Phi_0(u) = \sinh u - \sin u \qquad \Phi_1(u) = \cosh u - \cos u$$
$$\Phi_2(u) = \sinh u + \sin u \qquad \Phi_3(u) = \cosh u + \cos u$$

and

$$a_1 = \frac{1}{K}\Big(\frac{1}{2}P^2 + \sqrt{\Delta}\Big) \qquad b_1 = \frac{P\alpha^2}{K}$$
$$a_2 = \frac{1}{K}\Big(\frac{1}{2}P^2 - \sqrt{\Delta}\Big) \qquad b_2 = \frac{P\beta^2}{K}$$

- Case 2, for $\Omega^4 < 2K$

General solution of the system of equations (2.7) and (2.8) has the form

$$\begin{aligned} Z_1(\xi) &= C_1\Phi_0(\alpha\xi) + C_2\Phi_1(\alpha\xi) + C_3\Phi_2(\alpha\xi) + C_4\Phi_3(\alpha\xi) + \\ &+ C_5\Psi_0(\gamma\xi) + C_6\Psi_1(\gamma\xi) + C_7\Psi_2(\gamma\xi) + C_8\Psi_3(\gamma\xi) \end{aligned} \tag{3.8}$$

$$\begin{aligned} Z_2(\xi) &= C_1[a_1\Phi_0(\alpha\xi) + b_1\Phi_2(\alpha\xi)] + C_2[a_1\Phi_1(\alpha\xi) + b_1\Phi_3(\alpha\xi)] + \\ &+ C_3[a_1\Phi_2(\alpha\xi) + b_1\Phi_0(\alpha\xi)] + C_4[a_1\Phi_3(\alpha\xi) + b_1\Phi_1(\alpha\xi)] + \\ &+ C_5[a_2\Psi_0(\gamma\xi) + b_3\Psi_2(\gamma\xi)] + C_6[a_2\Psi_1(\gamma\xi) + b_3\Psi_3(\gamma\xi)] + \\ &+ C_7[a_2\Psi_2(\gamma\xi) - b_3\Psi_0(\gamma\xi)] + C_8[a_2\Psi_3(\gamma\xi) - b_3\Psi_1(\gamma\xi)] \end{aligned} \tag{3.9}$$

where

$$\Psi_0(u) = \sin u \sinh u \qquad \Psi_1(u) = \cos u \sinh u + \sin u \cosh u$$
$$\Psi_2(u) = \cos u \cosh u \qquad \Psi_3(u) = \cos u \sinh u - \sin u \cosh u$$

and $b_3 = 2P\gamma^2/K$.

Substituting for the functions $Z_1(\xi)$ and $Z_2(\xi)$, from formulae (3.6) and (3.7) or (3.8) and (3.9), into boundary conditions one obtains a linear system of eight homogeneous equations in unknown constants $C_1, C_2, ..., C_8$. For a non-trivial solution the determinant of the coefficient matrix is set equal to zero, yielding the frequency equation. This equation may be written in the form

$$|A_{ij}(\Omega)| = 0 \tag{3.10}$$

Exemplary non-zero coefficients A_{ij} corresponding to the beam system consisting of the clamped-clamped beam and the free-free one (the case, for $\Omega^4 > 2K$), are the following

$$A_{12} = \alpha^2 \qquad A_{16} = \beta^2$$

$$A_{21} = \alpha^3 \qquad A_{23} = \alpha P \qquad A_{25} = \beta^3 \qquad A_{27} = \beta P$$

$$A_{32} = b_1 \qquad A_{34} = a_1 \qquad A_{36} = b_2 \qquad A_{38} = a_2$$

$$A_{42} = \alpha^2 a_1 \qquad A_{44} = \alpha^2 b_1 \qquad A_{46} = \beta^2 a_2 \qquad A_{48} = \beta^2 b_2$$

$$A_{51} = \Phi_0(\alpha) \qquad A_{52} = \Phi_1(\alpha) \qquad A_{53} = \Phi_2(\alpha) \qquad A_{54} = \Phi_3(\alpha)$$
$$A_{55} = \Phi_0(\beta) \qquad A_{56} = \Phi_1(\beta) \qquad A_{57} = \Phi_2(\beta) \qquad A_{58} = \Phi_3(\beta)$$

$$A_{61} = \alpha^3\Phi_3(\alpha) + \alpha P\Phi_1(\alpha) \qquad A_{62} = \alpha^3\Phi_0(\alpha) + \alpha P\Phi_2(\alpha)$$
$$A_{63} = \alpha^3\Phi_1(\alpha) + \alpha P\Phi_3(\alpha) \qquad A_{64} = \alpha^3\Phi_2(\alpha) + \alpha P\Phi_0(\alpha)$$
$$A_{65} = \beta^3\Phi_3(\beta) + \beta P\Phi_1(\beta) \qquad A_{66} = \beta^3\Phi_0(\beta) + \beta P\Phi_2(\beta)$$
$$A_{67} = \beta^3\Phi_1(\beta) + \beta P\Phi_3(\beta) \qquad A_{68} = \beta^3\Phi_2(\beta) + \beta P\Phi_0(\beta)$$

$$A_{71} = a_1\Phi_0(\alpha) + b_1\Phi_2(\alpha) \qquad A_{72} = a_1\Phi_1(\alpha) + b_1\Phi_3(\alpha)$$
$$A_{73} = a_1\Phi_2(\alpha) + b_1\Phi_0(\alpha) \qquad A_{74} = a_1\Phi_3(\alpha) + b_1\Phi_1(\alpha)$$
$$A_{75} = a_2\Phi_0(\beta) + b_2\Phi_2(\beta) \qquad A_{76} = a_2\Phi_1(\beta) + b_2\Phi_3(\beta)$$
$$A_{77} = a_2\Phi_2(\beta) + b_2\Phi_0(\beta) \qquad A_{78} = a_2\Phi_3(\beta) + b_2\Phi_1(\beta)$$

$$A_{81} = \alpha^2(a_1\Phi_2(\alpha) + b_1\Phi_0(\alpha)) \qquad A_{82} = \alpha^2(a_1\Phi_3(\alpha) + b_1\Phi_1(\alpha))$$
$$A_{83} = \alpha^2(a_1\Phi_0(\alpha) + b_1\Phi_2(\alpha)) \qquad A_{84} = \alpha^2(a_1\Phi_1(\alpha) + b_1\Phi_3(\alpha))$$
$$A_{85} = \beta^2(a_2\Phi_2(\beta) + b_2\Phi_0(\beta)) \qquad A_{86} = \beta^2(a_2\Phi_3(\beta) + b_2\Phi_1(\beta))$$
$$A_{87} = \beta^2(a_2\Phi_0(\beta) + b_2\Phi_2(\beta)) \qquad A_{88} = \beta^2(a_2\Phi_1(\beta) + b_2\Phi_3(\beta))$$

Eq (3.10) is solved numerically.

3.2. Free vibration of two beams separated by an elastic layer with a step-wise varying stiffness modulus

Assuming now that the stiffness modulus of elastic layer, k appearing in Eqs (2.1) and (2.2) is a piecewise constant function. It means that the dimensionless stiffness modulus, K can be written as follows

$$K = \sum_{j=1}^{m} \kappa_j \Big[H(\xi - \xi_{j-1}) - H(\xi - \xi_j) \Big]$$

where $0 = \xi_0 < \xi_1 < ... < \xi_m = 1$, κ_j is the constant stiffness modulus of the elastic layer in interval (ξ_{j-1}, ξ_j) for $j = 1, 2, ..., m$ and $H(\cdot)$ denotes the Haeviside function. In particular intervals of constant stiffness modulus, the functions $Z_i(\xi)$ are given by Eqs (3.6) and (3.7) or (3.8) and (3.9). At points $\xi = \xi_j$, $(j = 1, 2, ..., m - 1)$ the continuity conditions are satisfied

$$Z_{ij}^{(k)}(\xi_j) = Z_{ij+1}^{(k)}(\xi_j) \qquad i = 1,2 \quad j = 1, ..., m-1 \quad k = 0, 1, 2, 3$$

where $Z_{ij}^{(k)}(\xi)$ denotes the kth derivative of the $Z_i(\xi)$ function for $\xi_{j-1} < \xi < \xi_j$. For $\xi = 0$ and $\xi = 1$ the boundary conditions in the form (2.3) $\div$ (2.5) are satisfied.

Employing the boundary and continuity conditions we get the linear system of 4m homogeneous equations. The free vibration frequencies Ω are found from the condition that the determinant of this system of equations vanishes.

4. Numerical examples

The above theoretical considerations have been applied to a numerical analysis of beam systems for different types of beam-ends supporting.

In example 1 the systems consisting of beams separated by an elastic layer with constant stiffness modulus are considered. Influence of the stiffness modulus of the layer which separates the beams, on free vibration frequencies of the system is studied. Various boundary conditions for beams have been investigated. The beams of the system are loaded by axial forces: the first beam is compressed ($P_1 = 1$) whereas the second beam of the system under goes the action of tensile force ($P_2 = -1$). In Fig.1 the curves of dimensionless vibration frequencies of the system in relation to the value of dimensionless stiffness modulus of the elastic layer, have been shown. It can be observed

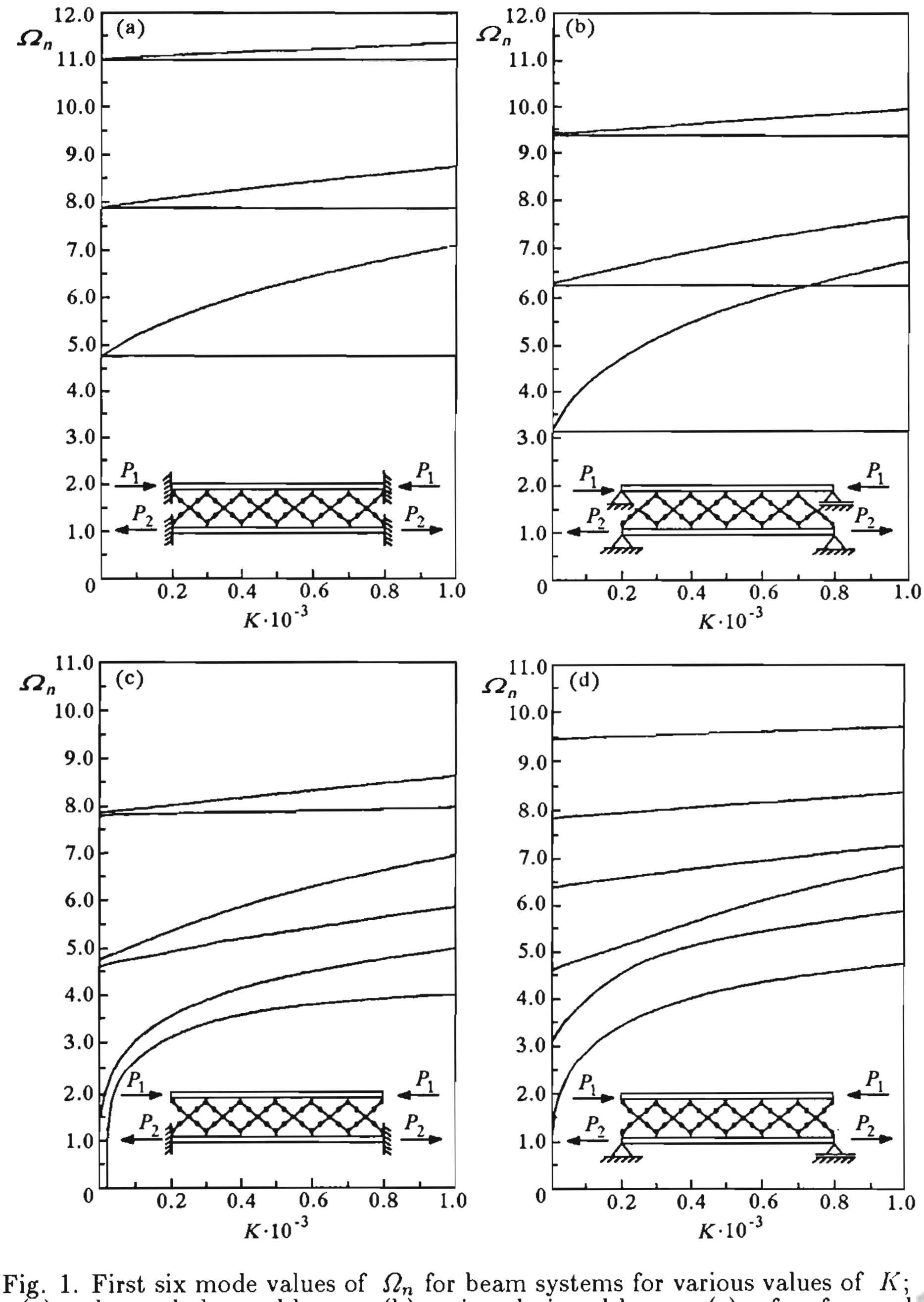

Fig. 1. First six mode values of Ω_n for beam systems for various values of K; (a) – clamped-clamped beams, (b) – pinned-pinned beams, (c) – free-free and clamped-clamped beams, (d) – free-free and pinned-pinned beams

that each frequency of the single beam corresponds to two frequencies of the compound system.

In example 2 the effect of an elastic layer which separates the beams upon the system vibration has been investigated in the case when the elastic layer appears only in the interval $<\xi_1, 1-\xi_1>$. In Fig.2 the curves of dimensionless free vibration frequencies of the system in relation to ξ_1 and for modulus $K = 1000$, have been shown.

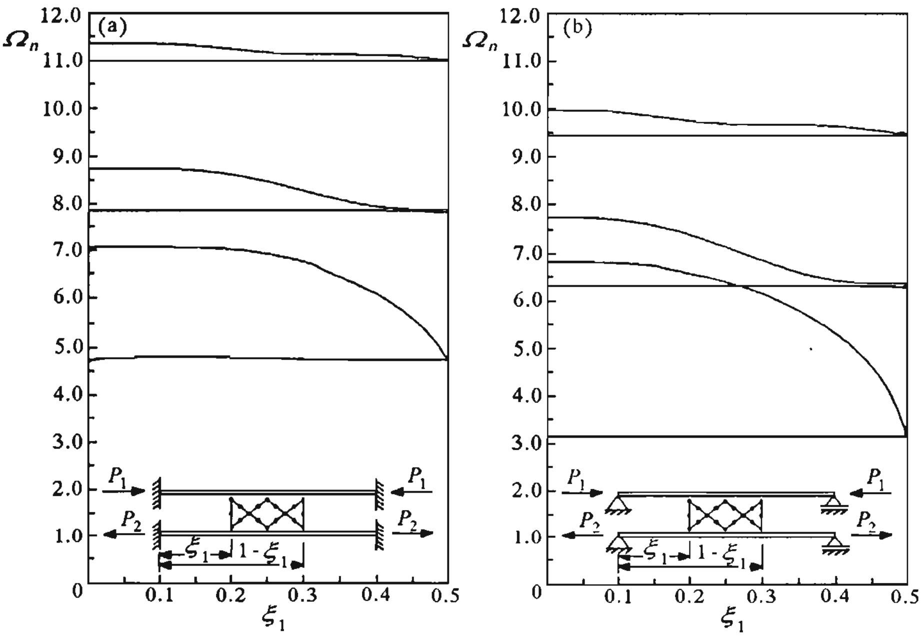

Fig. 2. First six mode values of Ω_n for beam systems with elastic layer at interval $< \xi_1, 1-\xi_1 >$ for various values ξ_1, $K = 1000.0$; (a) – clamped-clamped beams, (b) – pinned-pinned beams

References

1. KARCZMARZYK S., OSIŃSKI Z., 1992, On free vibration damping analysis of layered beams and bands composed of isotropic viscoelastic layers, *Mech. Teoret. i Stos.*, **30**, 721-735

2. KHATUA T.P., CHEUNG Y.K., 1973, Bending and vibration of multilayer sandwich beams, *International Journal for Numerical Methods in Engineering*, **6**, 11-24

3. KUKLA S., SKALMIERSKI B., 1993, The effect of axial loads on transverse vibrations of an Euler-Bernoulli beam, *Mech.Teoret.i Stos.*, **31**, 413-430

4. MILNE H.K., 1991, On the receptance functions of a uniform beam, *Journal of Sound and Vibration*, **146**, 176-180

5. ONISZCZUK Z., 1974, Transversal vibration of the system of two beams connected by means of an elastic element, (in Polish), *Mech.Teoret.i Stos.*, **12**, 72-83

6. ROY P.K., GANESAN N., 1992, Influence of constrained stepped layer on the response of a beam, *Journal of Sound and Vibration*, **154**, 187-190

7. YANKELEVSKY D.Z., 1991, Analysis of a composite layered elastic foundation, *International Journal of Mechanical Sciences*, **33**, 169-177

Drgania swobodne układu dwóch belek oddzielonych warstwą sprężystą

Streszczenie

W pracy przedstawiono problem drgań swobodnych ukladu skladającego się z dwóch obciążonych osiowo belek, które są oddzielone warstwą sprężystą. Rozwiązanie problemu obejmuje różne przypadki warunków brzegowych. Równanie częstości drgań wlasnych otrzymano dla przypadku ukladu, w którym jedna belka jest ściskana silą P_1, a druga jest rozciągana silą $P_2 = -P_1$. Zbadano wplyw współczynnika sprężystości warstwy oddzielającej belki na częstości drgań wlasnych ukladu zlożonego.

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