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BIFURCATION POINTS AND BRANCHING PATHS IN THE NONLINEAR STABILITY ANALYSIS OF SHELL STRUCTURES

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The very detailed numerical analysis of the shell stability problem comprising coincident bifurcation points and branching paths is presented in the paper. The general routine leading to detection of bifurcation points and to the tracing of post bifurcation paths is described. The spherical shell under uniform normal inward pressure is examined as the example. The principal equilibrium path and all postbifurcation branching paths are determined in the whole range of the loading history. One parameter, conservative loading is taken into account. The linearly elastic material is considered and the whole analysis is performed for big deflections. Finite element method in displacement formulation is used in the program of numerical analysis. The computer program has been run on standard PC.

1. Introduction

From the practical point of view the knowledge of limit points locations on the nonlinear equilibrium paths is maybe more important than the knowledge of bifurcation points, which do not appear in real structures (i.e. structures with natural imperfections). The methods of determination of limit points are far more simple than the techniques of determination of bifurcation points and bifurcation paths. The knowledge of the latter is very important from the point of view of researcher who wants to examine all phenomena associated with a nonlinear stability analysis.

In the literature on the subject there exist many suggestions how to detect bifurcation points, how to switch into postbifurcation paths and how to trace them, respectively. The most up-to-date review of these methods was presented by Choong and Hagai (1993). Many approaches are of theoretical significance only since it is nearly impossible to incorporate them into program. Other are not general enough to apply then to any circumstances. Nearly all of the methods discussed by Choong and Hagai (1993) fail when not a simple but a coincident point of bifurcation is encountered.

In this paper the procedure called by Batoz et al. (1976) the "load perturbation method" will be successfully adopted after appropriate modification to the problem, which exhibits a coincident point of bifurcation. It will be shown that the procedure is general enough to handle with detecting of the bifurcation points of any kind and tracing postbifurcation paths in the whole range of load history.

The main aim of the paper is presentation of the algorithm which enables detection of the bifurcation points and determination of the full postbifurcation paths in the numerical analysis of discrete mechanical systems on the example of shell structures. It is not enough to locate the bifurcation point. One should determine what kind of bifurcation point it is (cf Husein (1975), Thompson and Hunt (1973)). To answer this question one should examine stability of the postbifurcation path in the vicinity of the branching point. To do this, properties of the tangent stiffness matrix must be examined. Main of them are: stability determinant (i.e. the determinant of tangent stiffness matrix in the finite element approach) and the lowest eigenvalues. Calculation of the big matrix determinant or finding its eigenvalues and eigenforms is very laborious and time consuming in the case of big systems (several hundred DOFs or more). It will be shown that there is no need to find these quantities at every step of numerical analysis. It is possible also to find a part of them (e.g. stability determinant) as the by-product of other analysis.

Laterally loaded shell structures are capable of loosing its stability by bifurcation buckling or snapping. Very often the first one precedes the other but the postbifurcation path has to meet the fundamental path and the nonsymmetrical form of deformation must return to the symmetric mode. It is interesting to observe in what manner this process proceeds. Usually (cf Batoz et al. (1976), Srinivasan and Bobby (1976), Leicester (1968)) only the first part of the branching path is determined. In author's opinion it is interesting to trace the path till the moment it meets the fundamental path, i.e. till it returns to the mode of deformation corresponding to the fundamental path. Such complete branching paths will be shown in the example shown in the paper.

2. Fundamental equilibrium path

The equilibrium path is the set of equilibrium configurations in the load displacement space. Those points are obtained as a result of the solution of the nonlinear algebraic set of equations which govern the problem. The detailed procedure and the technique of solution is presented by Marcinowski (1989). Here the attention will be focused on the detection of bifurcation points during the tracing procedure of the fundamental path, switching into the postbifurcation path and tracing the branching paths.

At the first step of analysis the fundamental (or primary) equilibrium path is determined. It is the path passing through the origin. The tracing of this path is accomplished using any displacement as the control parameter (cf Marcinowski (1989)). If the geometry of the examined structure is symmetrical and the load is symmetrical as well, the deformation mode along the fundamental equilibrium path is also symmetrical. This general rule ceases to be true for post bifurcation paths. To answer the question of stability of the current branch of the primary path the tangent stiffness matrix determinant must be calculated. It appears during the solution of the nonlinear algebraic equations of equilibrium set process as the by-product of the solution procedure. After factorization of the matrix into two triangle matrices (Cholesky's procedure of the solution for increments in the Newton's iteration) the determinant can be obtained as the result of multiplication of all diagonal terms in an upper triangle. For big systems the obtained value would be too big. For this reason only the logarithm of this number is calculated. And namely

$$x = \det \mathbf{K} = \prod_{i=1}^{N} T_{ii} \qquad \text{and} \qquad \log x = \sum_{i=1}^{N} \log T_{ii}$$

where

T_{ii} – diagonal elements of the upper triangle

N – number of independent degrees of freedom.

At the same time the number of negative terms among T_{ii} is determined. If all T_{ii} are positive the matrix $\mathbf{K}$ is positive definite (it is only the necessary condition for the positive definiteness (cf Demidovich and Maron (1970)) and one should be very cautious using this condition only) and the current configuration is stable. When at least one T_{ii} is negative the configuration ceases to be stable. The change of the sign takes place at limit points. Those points are easy to distinguish, being local extremes on the path. The change of the determinant sign takes place also at bifurcation points (cf Husein (1975), Thompson and Hunt (1973)). Usually the slope of the fundamental path remains finite at these points and only the change of the determinant sign proves that a bifurcation point has been passed. At such zones also the convergence drops rapidly being an additional indication that the bifurcation point has in the vicinity. After passing this very zone tracing of the fundamental path proceeds in standard way. There is a problem yet. One should answer if the point of bifurcation is the simple critical point or the coincident critical point (cf Husein (1975)). Additional analysis must be performed. At the vicinity of the bifurcation point (on the unstable part of the fundamental path, i.e. a little bit above the bifurcation point) all eigenvalues of the tangent stiffness matrix must be calculated. If only one is zero – the simple critical point is dealt with and if more then one are zero – the coincident critical point is met. This additional analysis is rather laborious but is obligatory to avoid possibility of missing of another paths intersecting the fundamental one at this very point. Fortunately this analysis must be performed only at few points. In the analysis described above the location of a bifurcation point is determined only approximately. There is no need to determine it very accurately. It will be done later when tracing the post bifurcation path. The point of intersection between the post bifurcation and fundamental paths, respectively, will establish the exact location of the bifurcation point.

3. Branching paths

Branching paths are the paths which intersect the fundamental path at the bifurcation point. They exhibit the qualitatively distinct mode of deformation than this of the fundamental path. During the tracing of the primary path by means of any general routine one can easy omit the bifurcation point as well as branching paths. The special procedures must be adopted to determine both of them. In the literature several approaches to the problem of switching into branching path exist (cf Choong and Hangai (1993)), where the most important role plays the eigenvector corresponding to the zero eigenvalue of tangent stiffness matrix which appears exactly at the bifurcation point. But these approaches in the author's opinion seem to be very laborious and not general. Their most important drawback is that they do not work at the coincident bifurcation points. In this paper the load perturbation method similar to the one described by Batoz et al. (1976) will be adopted. A small perturbation load in the form of concentrated force is applied along with a given load distribution. The perturbation load should be applied in such a way to initialize the desired form of deformation, different from the deformation mode on the primary path. The path with the perturbation load (being actually the imperfection path) is traced as long as the level of the bifurcation point is reached and passed and its location was determined approximately during the tracing of the fundamental path. The change of the sign of the tangent stiffness matrix determinant indicated that the bifurcation point had been passed. There was no need to determine its location very precisely. It will be done later during the tracing of the postbifurcation path. The point of intersection between the postbifurcation and the fundamental paths will be the bifurcation point.

So, when the path with perturbation load reaches (or rather slightly passes) the level of bifurcation point estimated during tracing of the primary path, the perturbation load is removed. This removing is accomplished at one or at several steps. It depends on the value of perturbation load taken arbitrarily (but relatively small) at the very beginning. The first equilibrium configuration (the first point) reached at this zone allows to start the tracing of the branching path (the postbifurcation path) in an ordinary way. The tracing is performed in two opposite directions from this very point. Similarly as when tracing the fundamental path the displacements were used as control parameters. They were changed when the convergence of the method dropped. The graph of several equilibrium paths was drawn currently. It was updated after every portion (no more than $30 \div 40$ points) of calculation. With the help of this graph it was easy to decide which displacement parameter was to choose. The step of the control parameter was established by the trial and error method. In the vicinity of bifurcation points the convergence drops and one should shorten the control parameter increment. To answer the question if the post bifurcation path is stable or not it is enough to examine the sign of the stability determinant. In doubtful situations (the positive stability determinant is only the necessary but not the sufficient condition for a stable configuration (cf Demidovich and Maron (1970))) the eigenvalue analysis must be performed. All positive eigenvalues prove that the configuration is stable, otherwise it is unstable.

4. Numerical example

The spherical shell spread over the square base and undergoing the action of the uniform normal inward pressure was chosen as the example of the detailed analysis. This example is of interest since its behavior is highly nonlinear and two branching paths appear below the symmetrical buckling. This problem is suitable for testing the capability and reliability of the technique presented in the paper.

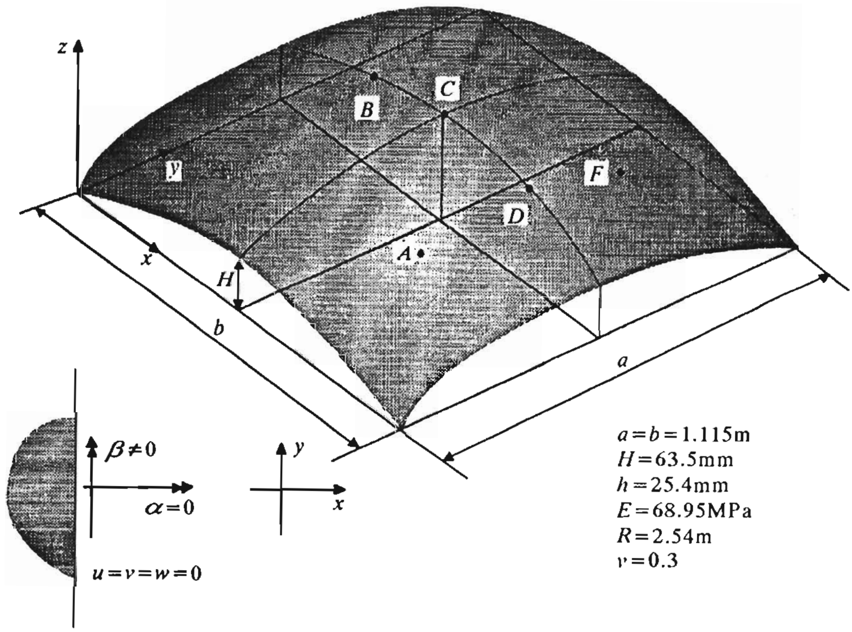

Fig. 1. Spherical shell under uniform inward pressure

The geometry, boundary conditions and material properties are shown in Fig.1. The symmetrical and part of the unsymmetrical paths for this shell were obtained by Leicester (1968). It is difficult yet to compare obtained results with those of Leicester since he had drawn his graphs versus the average displacement. One can compare only the critical pressure values (see Leicester (1968) – Fig.3 and Fig.7). The solution of this problem was presented also by Srinivasan and Bobby (1976). Unfortunately it was also only the fragmentary presentation of results. More detailed analysis of this problem was given by Batoz et al. (1976). But this solution is confined to the part of symmetrical path and initial segments of the postbuckling imperfect paths (see Batoz et al. (1976) – Fig.6) only. It creates however the possibility of comparison of the critical pressure values.

The calculations started from the division of the quarter of shell into four elements. This division leads to 105 DOFs (degrees of freedom) and allows to determine the whole symmetric path labeled by S in figures below. On the first branch in the vicinity of the pressure level p = 5.52 kPa the sta-

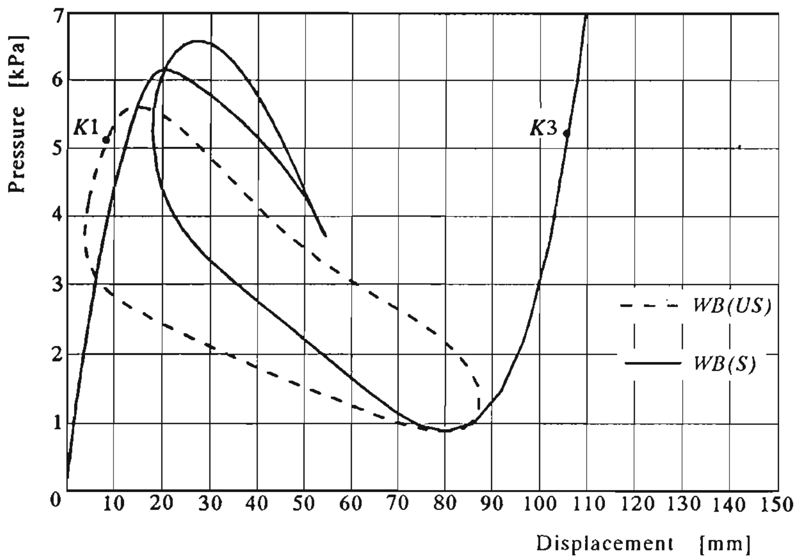

Fig. 2. Fundamental and branching paths (US mode) for the node B

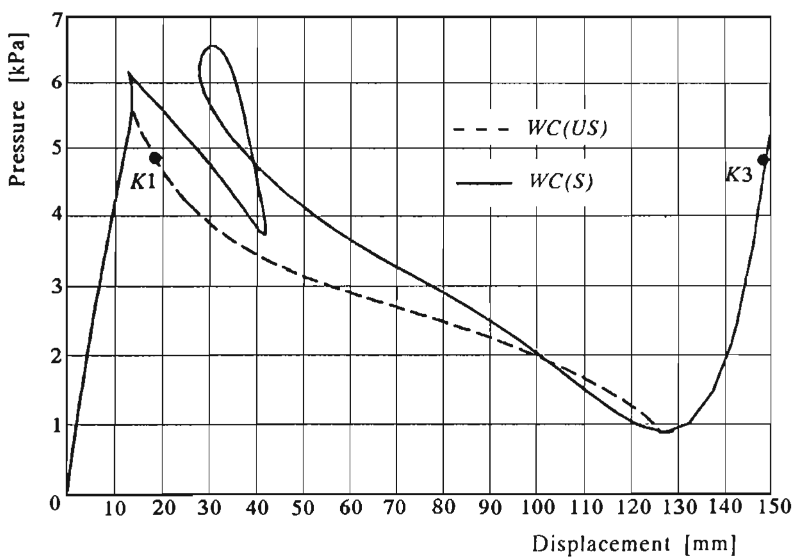

Fig. 3. Fundamental and branching paths (US mode) for the node C

bility determinant changes its sign from positive to negative. The additional eigenvalue analysis at this zone brings about two negative eigenvalues. It was obvious that the bifurcation point was passed. Since this very point till the beginning of the last raising branch on the primary path the stability determinant remains negative. It means that all configurations within this interval are unstable.

To find the branching paths of the US mode (unsymmetrical with respect to one axis linking middle points of the square and symmetrical with respect to the other) a half of the shell was divided into eight elements. This division corresponds to 125 DOFs. The perturbation load was applied at the node D (Fig.1) and the imperfection path was traced till the level $p = 5.52$ kPa was reached. Then the perturbation load was removed and the tracing of the branching paths started. This analysis has been performed till the symmetrical configuration was reached again.

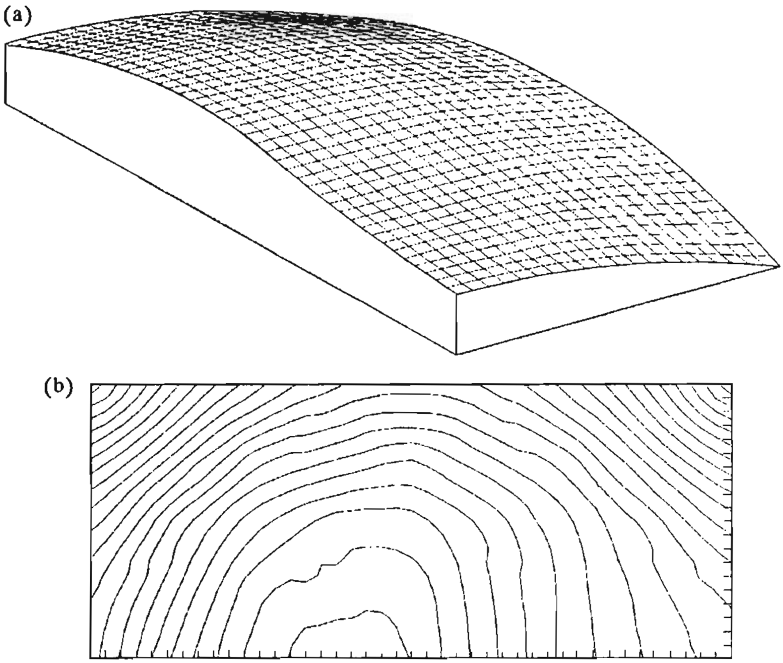

Fig. 4. Deformed configuration $K1$; (a) – view, (b) – contour-map

Figures 2 and 3 show the paths of S (solid line) and US (dashed line) modes for two nodes: C and B (see Fig.1) respectively. In all figures w means the vertical displacement positive downward. The US paths of the node A

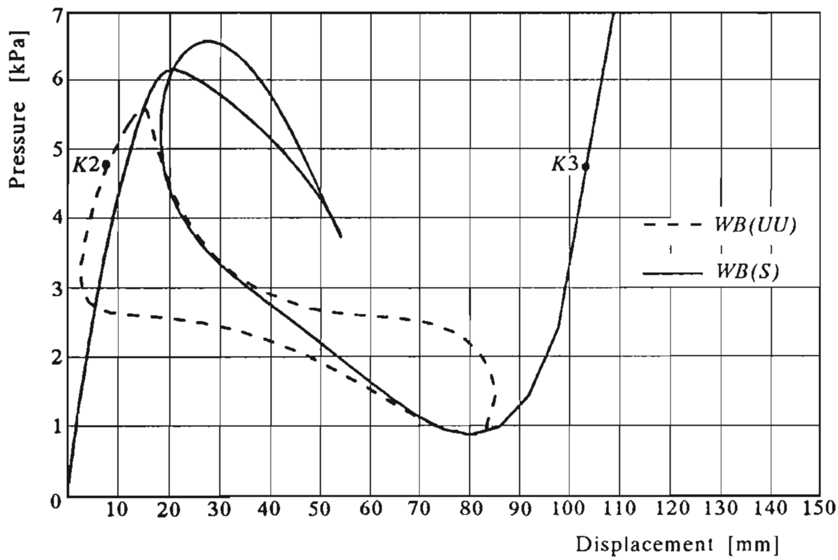

Fig. 5. Fundamental and branching paths (UU mode) for the node B

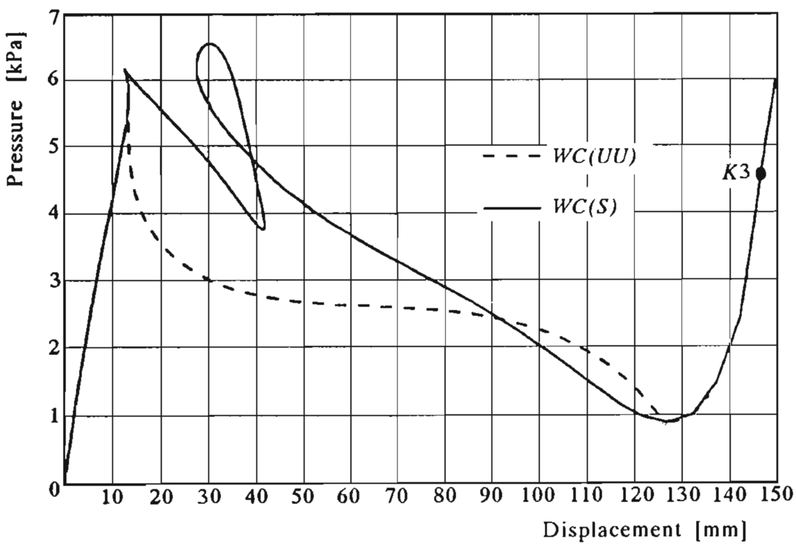

Fig. 6. Fundamental and branching paths (UU mode) for the node C

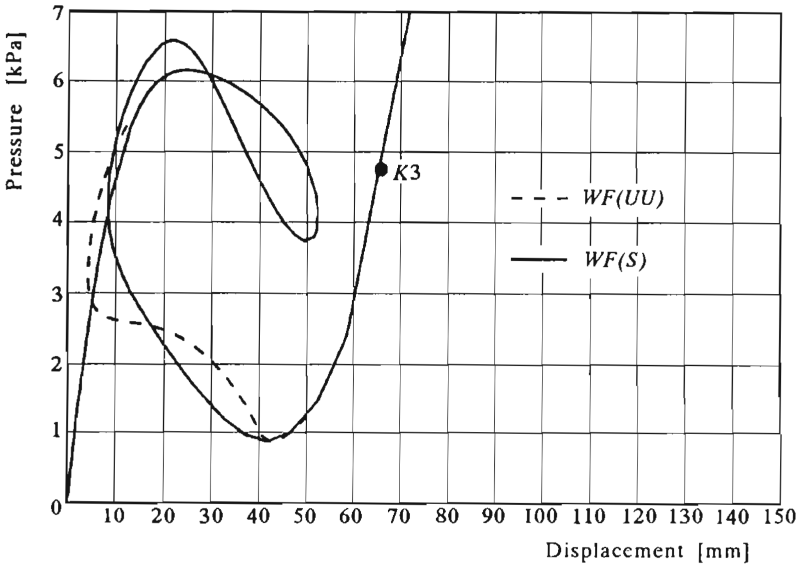

Fig. 7. Fundamental and branching paths (UU mode) for the node F

coincide and it is the reason why on the plot one can see only one not two of them. The deformed configurations labeled $K1$ (Fig.2) were shown in Fig.4 in the forms of the three dimensional plot and the contour-map, respectively.

To determine the branching paths of the UU mode the whole shell was divided into 16 elements (325 DOFs). In this case the perturbation load was applied at the node A. In the manner described earlier two branching paths were traced. Figures 5, 6 and 7 show the symmetrical S (solid line) and the branching UU (dashed line) paths for nodes B, C and F, respectively. The deformed configuration $K2$ (Fig.5) was shown in Fig.8. Fig.9 presents the inverted symmetrical configuration $K3$ on the stable final branch of the primary path. It turned out that the branching paths of US and UU modes, respectively, intersect the fundamental path exactly at the same point $p = 5.592$ kPa. It is seen from Fig.10 and Fig.11 on which paths of both modes (UU and US modes) for nodes B and C were superimposed. It confirms that the bifurcation point met on the fundamental path was the coincident bifurcation point (cf Husein (1975)). Since all the branching paths were unstable it follows that it was the unstable symmetrical point of bifurcation (cf Husein (1975), Thompson and Hunt (1973)). It is note worthy that the phenomenon of buckling will proceed as follows. Immediately after the pressure attains the critical level $p = 5.592$ kPa the shell adopt the nearest stable configuration

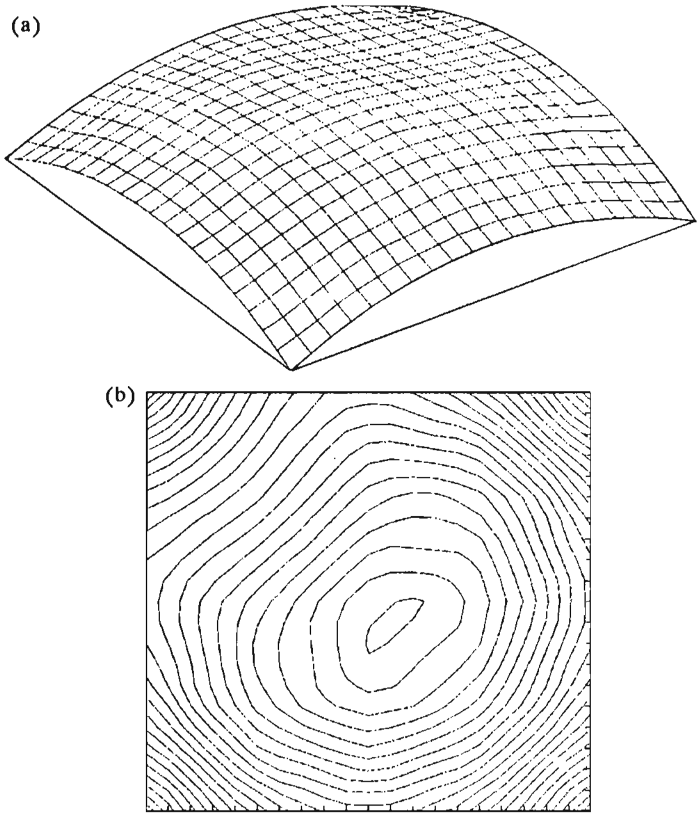

Fig. 8. Deformed configuration $K2$; (a) – view, (b) – contour-map

and it will be the very distant one on the stable raising branch of the fundamental path. This transition will proceed suddenly and dynamically. The inverted configuration reached in this way is stable and it is possible to raise the pressure now till the material failure occurs.

5. Concluding remarks

One may say that the complex analysis of big deformations of the shell was really made. The calculations were performed since the very beginning to the very distant inverted configurations. The critical value of the pressure obtained here $p = 5.592$ kPa corresponds to the value which one obtains

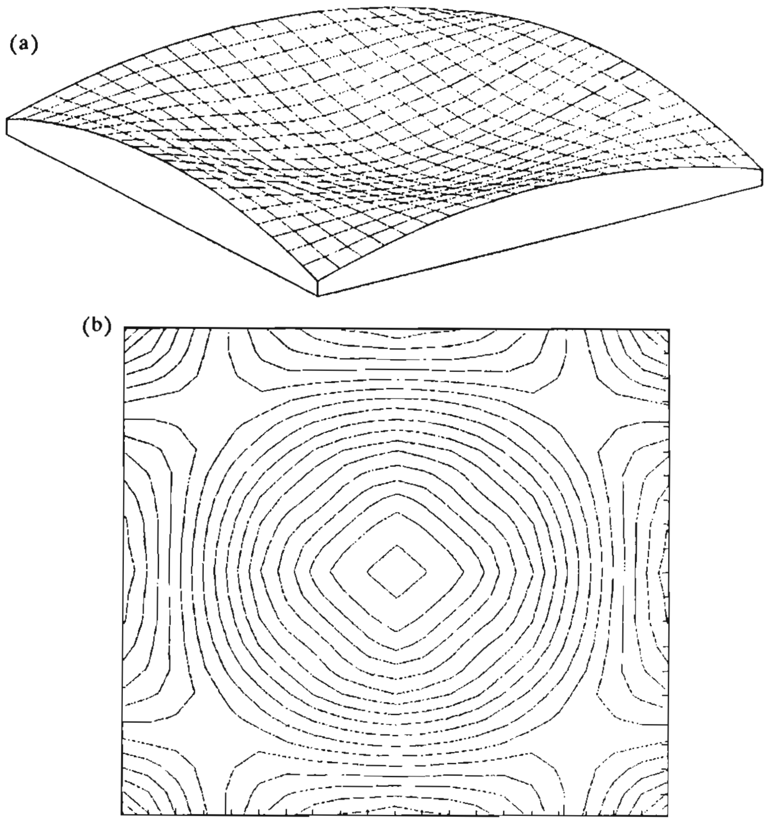

Fig. 9. Deformed configuration $K3$; (a) – view, (b) – contour-map

extrapolating the perturbation paths shown in Fig.6 by Batoz et al. (1976). The most important advantage of the present analysis is the tracing of the whole loading history from the very beginning through the branching paths till the inverted configuration. Only the analysis like this clarifies the all phenomena associated with the big deformations of the analyzed shell. In the author's opinion the presented example confirms that the general strategy adopted in cases like this is fully correct. The procedure dealing with the bifurcation points detection and tracing of the postbifurcation paths really works even in this rather complicated case. The fact that the coincident bifurcation point was detected on the path was no obstacle in this approach. Both postbifurcation paths intersecting the fundamental path at this point were traced without any problems within their whole range. Procedure seems to be stable and convergent at its every step.

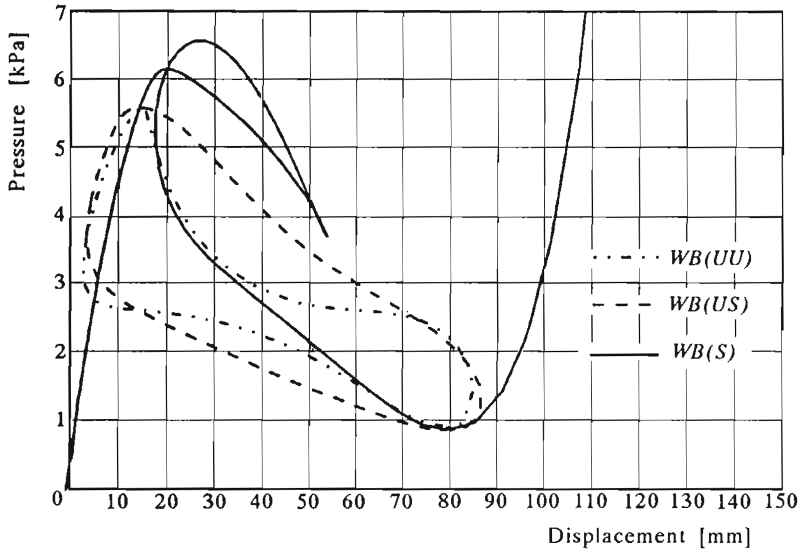

Fig. 10. Fundamental and branching paths (US and UU mode) for the node B

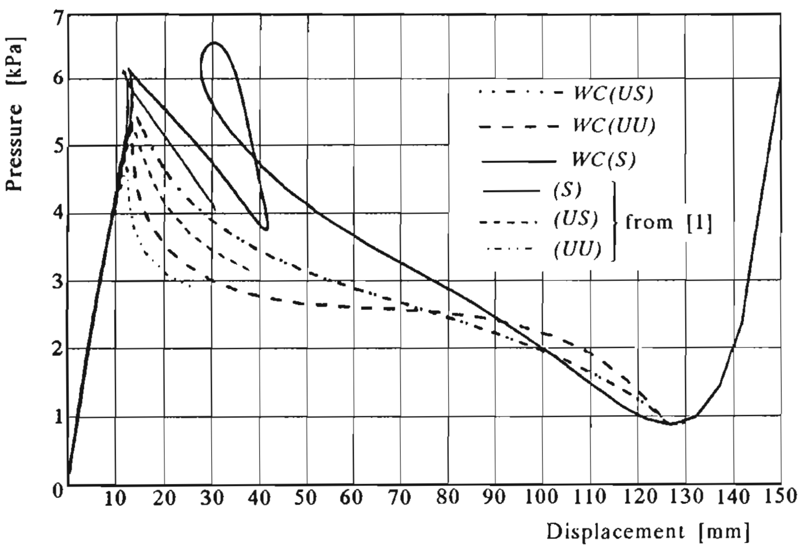

Fig. 11. Fundamental and branching paths (US and UU mode) for the node C

In Fig.11 the present results were compared with the paths obtained by Batoz et al. (1976). The primary paths nearly coincide if one takes into account the poor precision of copying the results from the small figure and units conversion. As far as the postbukcling paths are concerned those presented by Batoz et al. (1976) are just imperfection paths in their initial segments.

The performed analysis was another occasion to test the program presented by Marcinowski (1989). It seems to be capable also of solving problems which deal with bifurcation points and branching paths. Because earlier it was exploited in many benchmarking problems dealing with paths comprising limit points, now one can say that it is the quite versatile tool to solve the problems dealing with all types of elastic stability of shells.

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Punkty bifurkacji i rozgałęzienia ścieżek w nieliniowej analizie stateczności konstrukcji powłokowych

Streszczenie

W pracy przedstawiono szczegółową analizę numeryczną problemu stateczności powłok, w którym wystąpiły wielokrotne punkty bifurkacji i rozgałęzienia ścieżek. Zaprezentowano ogólną procedurę pozwalającą ustalić położenie punktów bifurkacji i śledzić przebieg ścieżek pobifurkacyjnych. Jako przyklad rozważono powlokę sferyczną poddaną dzialaniu równomiernego ciśnienia skierowanego do wewnątrz. Ścieżka podstawowa oraz wszystkie ścieżki pobifurkacyjne odgałęziające się od niej zostaly wyznaczone w pełnym zakresie obciążenia. Rozważano obciążenie jednoparametrowe, zachowawcze. Przyjęto liniowo sprężysty model materialu, a cala analiza była prowadzona dla dużych przemieszczeń. W programie analizy numerycznej wykorzystano metodę elementów skończonych w sformulowaniu przemieszczeniowym. Program byl uruchamiany na standardowym PC.

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