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FUNDAMENTAL SOLUTIONS RELATED TO THE STRESS INTENSITY FACTORS OF MODES I, II AND III. THE ASYMMETRIC PROBLEM

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The Green's functions are defined as a solution to the problem of an elastic, transversely isotropic body with a penny-shaped or an external crack subjected to asymmetric vertical and in-plane lateral concentrated forces distributed along a circumference. Exact solutions are presented in a closed form for the stress intensity factors under each type of ring forces being assumed as fundamental solutions.

1. Introduction

The stress analysis of an axisymmetric cracked body under bending is in general more difficult than in the case of tension and torsion. The fundamental solutions related to the stress intensity factors of a cracked solid under tension and torsion were published recently (Rogowski (1994)). In the present paper, the fundamental solutions (Green's functions) required in the analysis of an axisymmetrical transversely isotropic body with the internal penny-shaped or an external crack under bending are obtained for the stress intensity factors of modes I, II and III.

It is obvious, that the concentrated forces continuously distributed along a circumference as shown in Fig.2 and Fig.3 are sufficient for the bending problems of an axially symmetric body. The inplane lateral force F_x may be decomposed into the radial force $F_r = F_x \cos\theta$ and the tangential force $F_\theta = -F_x \sin\theta$. Then, we assume two ring forces in r and z directions with the intensity $\cos\theta$ and one ring force in the θ direction with the intensity $-\sin\theta$ in cylindrical coordinate system (r, θ, z). The solution to a practical bending problem can be obtain by superposing and integrating these three fundamental solutions. All calculations are straightforward, since the fundamental solutions are presented in terms of elementary functions and need no further transformations. The results presented for general cases are new, but some of those relating to special cases of isotropic or transversely isotropic solids with the crack surface loading are in complete agreement with the already known coresponding ones (cf Murakami (1987); Sneddon and Lowengrub (1969); Kassir and Sih (1975); Rogowski (1986)).

2. Basic equations

In this paper we employ cylindrical coordinates and denote them by (r,θ,z). Let a penny-shaped crack or an external crack be located in the plane $z = 0$ of a homogeneous and transversely isotropic elastic solid. The penny-shaped crack occupies the region $0 \le r \le a$ $(z = 0)$ and the external crack occupies the region $r \ge a$ $(z = 0)$. Both surfaces of the cracks are stress-free. The half-space $z \ge 0$ is subjected to asymmetric (about $\theta = \pi/2$ axis) body forces (2.1)

$$
\begin{aligned}
F_3 &= \frac{1_3}{2\pi r^2}\delta(r-b)\delta(z-h)\cos\theta \\
F_x &= \frac{1_x}{2\pi r}\delta(r-b)\delta(z-h)
\end{aligned}
\tag{2.1}
$$

distributed along a circle $(r = b, z = h)$, where $\delta(\cdot)$ is a Dirac delta function.

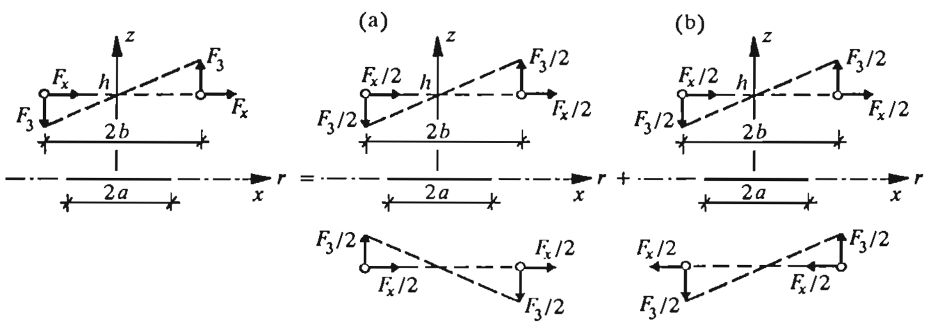

Fig. 1.

Due to the symmetry or antisymmetry of the problem, it can be reduced to the mixed boundary-value problem for a half-space with the following boundary conditions (Fig.1a,b)

— for a penny-shaped crack, $0 \le r \le a$, $0 \le \theta \le 2\pi$

$$\begin{array}{llllll} u_z = 0 & r > a & z = 0 & \sigma_z = 0 & r < a & z = 0 \qquad (2.2a) \\ u_r = 0 = u_\theta & r > a & z = 0 & \sigma_{zr} = 0 = \sigma_{z\theta} & r < a & z = 0 \qquad (2.2b) \end{array}$$

— for an external crack, $r \ge a$, $0 \le \theta \le 2\pi$

$$\begin{array}{llllll} u_z = 0 & r < a & z = 0 & \sigma_z = 0 & r > a & z = 0 \qquad (2.3a) \\ u_r = 0 = u_\theta & r < a & z = 0 & \sigma_{zr} = 0 = \sigma_{z\theta} & r > a & z = 0 \qquad (2.3b) \end{array}$$

for symmetric (a) and antisymmetric (b) loadings, respectively.

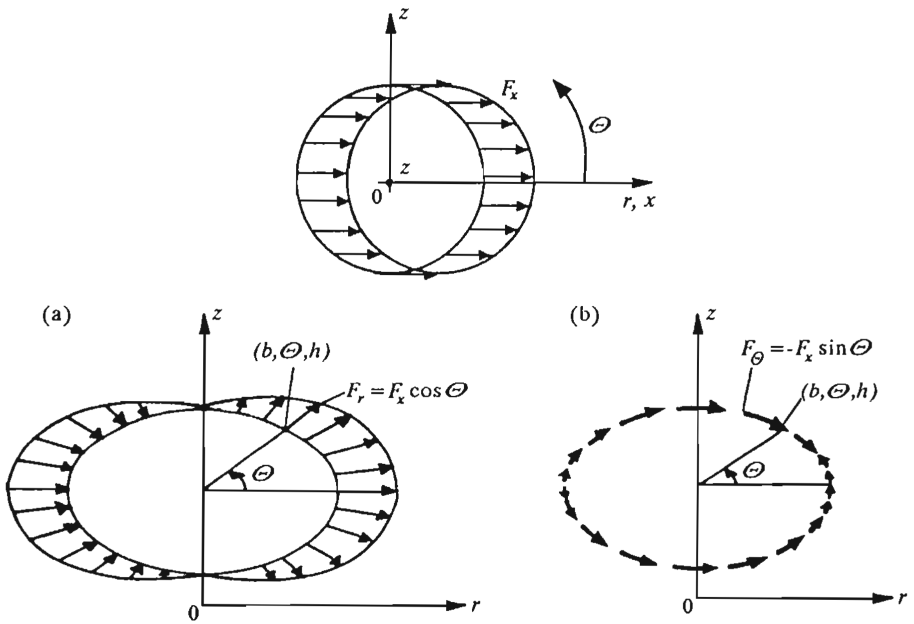

Fig. 2.

The solutions of the cracked elastic body subjected to concentrated loadings acting along a circumference are obtained by using Hankel transforms (cf Sneddon (1972)). Omitting details of calculations it can be shown that the displacement and stress fields associated with the action of the concentrated asymmetric (with respect to $\theta = \pi/2$ axis) ring forces and meeting the mixed boundary conditions (2.2a,b) or (2.3a,b) on the plane where the crack appears are:

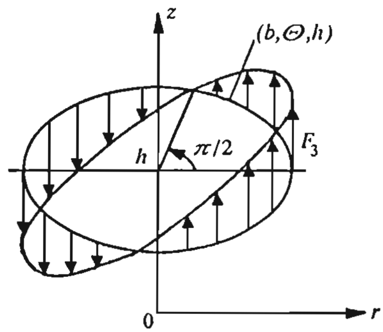

Fig. 3.

(i) For concentrated vertical and in-plane lateral, symmetric ring forces as shown in Fig.2, Fig.3 and Fig.1a

$$u_z(r,\theta,0) = \frac{1}{4\pi G_z C}\cos\theta\Big[\int\limits_0^\infty A(\xi)J_1(\xi r)\,d\xi + \tag{2.4}$$

$$+\frac{1_3}{b}\int\limits_0^\infty H_0(\xi s_i h)J_1(\xi b)J_1(\xi r)\,d\xi + 1_x v_0\int\limits_0^\infty H_1(\xi s_i h)J_0(\xi b)J_1(\xi r)\,d\xi\Big]$$

$$\sigma_z(r,\theta,0) = -\frac{1}{4\pi}\cos\theta\int\limits_0^\infty \xi A(\xi)J_1(\xi r)\,d\xi \tag{2.5}$$

The system reveals symmetry about $\theta = 0$ axis and $z = 0$ plane, and antisymmetry about $\theta = \pi/2$ axis.

(ii) For concentrated in-plane lateral, antisymmetric ring forces as shown in Fig.2 and Fig.1b

$$G_z\Big(\frac{u_r}{\cos\theta} - \frac{u_\theta}{\sin\theta}\Big)\Big|_{z=0} = \frac{1}{4\pi}\int\limits_0^\infty\Big\{\frac{1}{\kappa_1+\kappa_2}\Big[B(\xi) + 1_x H_3(\xi s_i h)J_0(\xi b)\Big] + $$

$$+ \frac{1}{\kappa_1-\kappa_2}\Big[C(\xi) - 1_x e^{-\xi s_3 h}J_0(\xi b)\Big]\Big\}J_0(\xi r)\,d\xi \tag{2.6}$$

$$G_z\Big(\frac{u_r}{\cos\theta} + \frac{u_\theta}{\sin\theta}\Big)\Big|_{z=0} = -\frac{1}{4\pi}\int\limits_0^\infty\Big\{\frac{1}{\kappa_1+\kappa_2}\Big[B(\xi) + 1_x H_3(\xi s_i h)J_0(\xi b)\Big] + $$

$$- \frac{1}{\kappa_1-\kappa_2}\Big[C(\xi) - 1_x e^{-\xi s_3 h}J_0(\xi b)\Big]\Big\}J_2(\xi r)\,d\xi$$

$$\left(\frac{\sigma_{zr}}{\cos\theta} - \frac{\sigma_{z\theta}}{\sin\theta}\right)\Big|_{z=0} = -\frac{1}{4\pi}\int\limits_0^\infty \xi\Big[B(\xi) - C(\xi)\Big]J_0(\xi r)\,d\xi$$
$$\left(\frac{\sigma_{zr}}{\cos\theta} + \frac{\sigma_{z\theta}}{\sin\theta}\right)\Big|_{z=0} = \frac{1}{4\pi}\int\limits_0^\infty \xi\Big[B(\xi) + C(\xi)\Big]J_2(\xi r)\,d\xi \tag{2.7}$$

The system reveals symmetry about $\theta = 0$ axis and antisymmetry about $\theta = \pi/2$ axis and $z = 0$ plane.

(iii) For concentrated vertical, antisymmetric ring forces as shown in Fig.3 and Fig.1b

$$G_z\left(\frac{u_r}{\cos\theta} - \frac{u_\theta}{\sin\theta}\right)\Big|_{z=0} = \frac{1}{4\pi}\int\limits_0^\infty\Big\{\frac{1}{\kappa_1+\kappa_2}\Big[D(\xi) + \frac{1_3 v_1}{b}H_2(\xi s_i h)J_1(\xi b)\Big] + \frac{1}{\kappa_1-\kappa_2}E(\xi)\Big\}J_0(\xi r)\,d\xi$$
$$G_z\left(\frac{u_r}{\cos\theta} + \frac{u_\theta}{\sin\theta}\right)\Big|_{z=0} = -\frac{1}{4\pi}\int\limits_0^\infty\Big\{\frac{1}{\kappa_1+\kappa_2}\Big[D(\xi) + \frac{1_3 v_1}{b}H_2(\xi s_i h)J_1(\xi b)\Big] + - \frac{1}{\kappa_1-\kappa_2}E(\xi)\Big\}J_2(\xi r)\,d\xi \tag{2.8}$$

$$\left(\frac{\sigma_{zr}}{\cos\theta} - \frac{\sigma_{z\theta}}{\sin\theta}\right)\Big|_{z=0} = -\frac{1}{4\pi}\int\limits_0^\infty \xi\Big[D(\xi) - E(\xi)\Big]J_0(\xi r)\,d\xi$$
$$\left(\frac{\sigma_{zr}}{\cos\theta} + \frac{\sigma_{z\theta}}{\sin\theta}\right)\Big|_{z=0} = \frac{1}{4\pi}\int\limits_0^\infty \xi\Big[D(\xi) + E(\xi)\Big]J_2(\xi r)\,d\xi \tag{2.9}$$

Again, the system reveals symmetry about $\theta = 0$ axis and antisymmetry about $\theta = \pi/2$ axis and $z = 0$ plane, respectively.

In Eqs (2.4) ÷ (2.9) J_0, J_1 and J_2 are the Bessel functions of the first kind, the arbitrary functions $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$ and $E(\xi)$ are unknown, the material parameters s_i, C, v_0, v_1, κ_1, κ_2 are defined in Appendix, G_z is the shear modulus along z-axis, and the known functions H_0, H_1, H_2 and H_3 are defined in Appendix.

Within the context of linear elastic fracture mechanics, the stress intensity factors are defined as

$$\left.\begin{array}{l} K_I \\ K_{II} \\ K_{III} \end{array}\right\} = \lim_{r\to a^+} \sqrt{2(r-a)} \left\{\begin{array}{l} \sigma_z(r,\theta,0) \\ \sigma_{zr}(r,\theta,0) \\ \sigma_{z\theta}(r,\theta,0) \end{array}\right. \tag{2.10}$$

for a penny-shaped crack and

$$\left.\begin{array}{l} K_I \\ K_{II} \\ K_{III} \end{array}\right\} = \lim_{r\to a^-} \sqrt{2(a-r)} \left\{\begin{array}{l} \sigma_z(r,\theta,0) \\ \sigma_{zr}(r,\theta,0) \\ \sigma_{z\theta}(r,\theta,0) \end{array}\right. \tag{2.11}$$

for an external crack.

K_I, K_{II} and K_{III} are the modes I, II and III stress intensity factors (Kanninen and Popelar (1985)), respectively. K_I corresponds to the case (i) of loadings, Eqs (2.4), (2.5) and K_{II}, K_{III} correspond to the cases (ii) and (iii) of loadings, Eqs (2.6) ÷ (2.9).

3. Mode I loadings

3.1. A penny-shaped crack

The boundary conditions (2.2a) yield

$$\begin{aligned} \int_0^\infty A(\xi)J_1(\xi r)\,d\xi = -\frac{1_3}{b}\int_0^\infty H_0(\xi s_i h)J_1(\xi b)J_1(\xi r)\,d\xi\, + \\ -1_x v_0 \int_0^\infty H_1(\xi s_i h)J_0(\xi b)J_1(\xi r)\,d\xi \qquad r > a \end{aligned} \tag{3.1}$$

$$\int_0^\infty \xi A(\xi)J_1(\xi r)\,d\xi = 0 \qquad r < a \tag{3.2}$$

The integral representation of $A(\xi)$

$$A(\xi) = \sqrt{\xi}\int_0^a \sqrt{x} f(x) J_{3/2}(x\xi)\,dx - \frac{1_3}{b} H_0(\xi s_i h)J_1(\xi b) - 1_x v_0 H_1(\xi s_i h) J_0(\xi b) \tag{3.3}$$

satisfies identically Eq (3.1), and Eq (3.2) is converted to the Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int_0^r \frac{d\big[xf(x)\big]}{dx}\frac{dx}{\sqrt{r^2-x^2}} = \frac{1_3 r}{b}\int_0^\infty \xi H_0(\xi s_i h)J_1(\xi b)J_1(\xi r)\,d\xi + + 1_x v_0 r\int_0^\infty \xi H_1(\xi s_i h)J_0(\xi b)J_1(\xi r)\,d\xi \tag{3.4}$$

The auxiliary unknown function $f(x)$ is assumed to be continuous in the interval $[0,a]$ and is required to satisfy the condition $\sqrt{x}f(x)\to 0$ as $x\to 0^+$.

The solution to Eq (3.4) is exact and has the form

$$f(x) = \frac{1_3}{b}\sqrt{\frac{2}{\pi}}\int_0^\infty H_0(\xi s_i h)\Big(\frac{\sin(\xi x)}{\xi x}-\cos(\xi x)\Big)J_1(\xi b)\,d\xi + + 1_x v_0\sqrt{\frac{2}{\pi}}\int_0^\infty H_1(\xi s_i h)\Big(\frac{\sin(\xi x)}{\xi x}-\cos(\xi x)\Big)J_0(\xi b)\,d\xi \tag{3.5}$$

Since the improper integrals in Eq (3.5) are calculated analytically (see Appendix, Eq (A3) ÷ (A6)) the final form of an auxiliary function $f(x)$ is given by elementary functions as follows

$$f(x) = \sqrt{\frac{2}{\pi}}\Big\{\frac{1_x v_0}{x(ks_2-s_1)}\Big[ks_2\Big(\frac{\pi}{2}-\tan^{-1}\zeta_1-\frac{\zeta_1}{\zeta_1^2+\eta_1^2}\Big)-s_1\Big(\frac{\pi}{2}-\tan^{-1}\zeta_2+ -\frac{\zeta_2}{\zeta_2^2+\eta_2^2}\Big)\Big]+\frac{1_3}{x^2(k-1)}\Big[\frac{k\eta_2}{(1+\zeta_2^2)(\zeta_2^2+\eta_2^2)}-\frac{\eta_1}{(1+\zeta_1^2)(\zeta_1^2+\eta_1^2)}\Big]\Big\}\cos\theta \tag{3.6}$$

where the oblate spheroidal coordinates ζ_i, η_i are defined in Appendix (Eq (A7)).

From Eqs (2.5) and (3.3) it may be shown that the singular part of axial stress is given by the formula

$$\sigma_z(r,\theta,0) = \sqrt{\frac{2}{\pi}}\frac{f(a)}{4\pi\sqrt{r^2-a^2}}\cos\theta \qquad \text{as} \qquad r\to a^+ \tag{3.7}$$

From Eqs $(2.10)_1$ and (3.7) it follows that, the stress intensity factor at the crack tip is obtained explicitly in terms of the oblate spheroidal coordinates

$\bar{\zeta}_i$, $\bar{\eta}_i$ (the values of ζ_i, η_i for $x = a$, see Appendix) as follows

$$K_I = \frac{1}{2\pi^2\sqrt{a^3}}\Big\{\frac{1_x v_0}{ks_2 - s_1}\Big[ks_2\Big(\frac{\pi}{2} - \tan^{-1}\bar{\zeta}_1 - \frac{\bar{\zeta}_1}{\bar{\zeta}_1^2 + \bar{\eta}_1^2}\Big) - s_1\Big(\frac{\pi}{2} - \tan^{-1}\bar{\zeta}_2 + \\ - \frac{\bar{\zeta}_2}{\bar{\zeta}_2^2 + \bar{\eta}_2^2}\Big)\Big] + \frac{1_3}{a(k-1)}\Big[\frac{k\bar{\eta}_2}{(1+\bar{\zeta}_2^2)(\bar{\zeta}_2^2 + \bar{\eta}_2^2)} - \frac{\bar{\eta}_1}{(1+\bar{\zeta}_1^2)(\bar{\zeta}_1^2 + \bar{\eta}_1^2)}\Big]\Big\}\cos\theta \tag{3.8}$$

In the special cases K_I becomes

— for $b < a$, $h = 0$

$$K_I = \frac{1}{2\pi^2\sqrt{a^3}}\cos\theta\Big(\frac{\pi}{2}1_x v_0 + \frac{1_3}{\sqrt{a^2 - b^2}}\Big) \tag{3.9a}$$

— for $b > a$, $h = 0$

$$K_I = \frac{1}{2\pi^2\sqrt{a^3}}\cos\theta\Big\{1_x v_0\Big[\sin^{-1}\Big(\frac{a}{b}\Big) - \frac{a}{\sqrt{b^2 - a^2}}\Big]\Big\} \tag{3.9b}$$

—for $b = 0$

$$K_I = \frac{1}{2\pi^2\sqrt{a^3}}\cos\theta\Big\{\frac{1_x v_0}{ks_2 - s_1}\Big[ks_2\Big(\tan^{-1}\frac{a}{s_1 h} - \frac{s_1 a h}{a^2 + s_1^2 h^2}\Big) + \\ -s_1\Big(\tan^{-1}\frac{a}{s_2 h} - \frac{s_2 a h}{a^2 + s_2^2 h^2}\Big)\Big] + \frac{1_3 a^3}{k-1}\Big[\frac{k}{(a^2 + s_2^2 h^2)^2} - \frac{1}{(a^2 + s_1^2 h^2)^2}\Big]\Big\} \tag{3.9c}$$

when loading is applied to the faces of a crack, outside of the crack in its plane and as the concentrated force acting on the z-axis at the distance h from the middle of the crack, respectively.

By evaluation of the limits as $s_1 \to 1$, $k \to 1$ and $s_2 \to 1$ in Eqs (3.8) and (3.9) we obtain the value of K_I for an isotropic solid. For isotropic material $v_0 = (1 - 2\nu)/[2(1 - \nu)]$, where ν denotes the Poisson ratio.

3.2. An external crack

For an external crack of inner radius a the dual integral equations (3.1) and (3.2) are defined the ranges $r < a$ and $r > a$, respectively. The representation of $A(\xi)$

$$A(\xi) = \sqrt{\frac{2}{\pi}}\int\limits_0^a h(x)\sin(\xi x)\,dx \tag{3.10}$$

yields the Abel integral equation of the problem in unknown function $h(x)$

$$\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{xh(x)}{\sqrt{r^2-x^2}}\,dx = -\frac{1_3 r}{b}\int\limits_0^\infty H_0(\xi s_i h)J_1(\xi b)J_1(\xi r)\,d\xi + - 1_x v_0 r\int\limits_0^\infty H_1(\xi s_i h)J_0(\xi b)J_1(\xi r)\,d\xi \tag{3.11}$$

The solution to this equation is exact and has the form

$$h(x) = -\sqrt{\frac{2}{\pi}}\Big(\frac{1_3}{b}\int\limits_0^\infty H_0(\xi s_i h)J_1(\xi b)\sin(\xi x)\,d\xi + + 1_x v_0\int\limits_0^\infty H_1(\xi s_i h)J_0(\xi b)\sin(\xi x)\,d\xi\Big) \tag{3.12}$$

Using the integrals (A1) and (A2) defined in Appendix the final formula for an auxiliary function $h(x)$ is given explicitly

$$h(x) = -\sqrt{\frac{2}{\pi}}\Big[\frac{1_x v_0}{x(ks_2-s_1)}\Big(\frac{ks_2\eta_1}{\zeta_1^2+\eta_1^2}-\frac{s_1\eta_2}{\zeta_2^2+\eta_2^2}\Big)+ + \frac{1_3}{x^2(k-1)}\Big(\frac{k\zeta_2}{(1+\zeta_2^2)(\zeta_2^2+\eta_2^2)}-\frac{\zeta_1}{(1+\zeta_1^2)(\zeta_1^2+\eta_1^2)}\Big)\Big] \tag{3.13}$$

The stress intensity factor of mode I is given by

$$K_I = \frac{1}{2\pi^2\sqrt{a^3}}\Big[\frac{1_x v_0}{ks_2-s_1}\Big(\frac{ks_2\bar{\eta}_1}{\bar{\zeta}_1^2+\bar{\eta}_1^2}-\frac{s_1\bar{\eta}_2}{\bar{\zeta}_2^2+\bar{\eta}_2^2}\Big)+ + \frac{1_3}{(k-1)a}\Big(\frac{k\bar{\zeta}_2}{(1+\bar{\zeta}_2^2)(\bar{\zeta}_2^2+\bar{\eta}_2^2)}-\frac{\bar{\zeta}_1}{(1+\bar{\zeta}_1^2)(\bar{\zeta}_1^2+\bar{\eta}_1^2)}\Big)\Big]\cos\theta \tag{3.14}$$

where $\bar{\zeta}_i$, $\bar{\eta}_i$ are the oblate spheroidal coordinates for $x = a$ (see Appendix).

In the special cases K_I takes the values

— for $b < a$, $h = 0$

$$K_I = \frac{1_x v_0}{2\pi^2\sqrt{a(a^2-b^2)}}\cos\theta \tag{3.15a}$$

— for $b > a$, $h = 0$

$$K_I = \frac{1_3 a}{2\pi^2 b^2 \sqrt{a(b^2 - a^2)}} \cos\theta \tag{3.15b}$$

—for $b = 0$

$$K_I = \frac{1}{2\pi^2\sqrt{a^3}} \cos\theta \Big[\frac{1_x v_0}{ks_2 - s_1} \Big(\frac{ks_2 a^2}{a^2 + s_1^2 h^2} - \frac{s_1 a^2}{a^2 + s_2^2 h^2} \Big) +$$

$$+ \frac{1_3 a^2}{k-1} \Big(\frac{ks_2 h}{(a^2 + s_2^2 h^2)^2} - \frac{s_1 h}{(a^2 + s_1^2 h^2)^2} \Big) \Big] \tag{3.15c}$$

4. Modes II and III loadings

4.1. A penny-shaped crack

4.1.1. Concentrated forces distributed along a circumference in the x-direction (Fig.2 and Fig.1b)

Introduce new constants $B_1(\xi)$ and $C_1(\xi)$

$$\begin{aligned} B(\xi) + 1_x \Pi_3(\xi s_i h) J_0(\xi b) &= \frac{1}{2}(\kappa_1 + \kappa_2)[B_1(\xi) - C_1(\xi)] \\ C(\xi) - 1_x e^{-\xi s_3 h} J_0(\xi b) &= \frac{1}{2}(\kappa_1 - \kappa_2)[B_1(\xi) + C_1(\xi)] \end{aligned} \tag{4.1}$$

Thus

$$\begin{aligned} G_z\Big(\frac{u_r}{\cos\theta} - \frac{u_\theta}{\sin\theta}\Big)\Big|_{z=0} &= \frac{1}{4\pi}\int\limits_0^\infty B_1(\xi) J_0(\xi r)\, d\xi \\ G_z\Big(\frac{u_r}{\cos\theta} + \frac{u_\theta}{\sin\theta}\Big)\Big|_{z=0} &= \frac{1}{4\pi}\int\limits_0^\infty C_1(\xi) J_2(\xi r)\, d\xi \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
\Big(\frac{\sigma_{zr}}{\cos\theta} - \frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0} &= \frac{1}{4\pi}\int\limits_0^\infty \xi\Big[\kappa_1 C_1(\xi) - \kappa_2 B_1(\xi)\Big] J_0(\xi r)\, d\xi + \\
&+ \frac{1_x}{4\pi}\int\limits_0^\infty \xi\Big[H_3(\xi s_i h) + \mathrm{e}^{-\xi s_3 h}\Big] J_0(\xi b) J_0(\xi r)\, d\xi \\
\Big(\frac{\sigma_{zr}}{\cos\theta} + \frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0} &= \frac{1}{4\pi}\int\limits_0^\infty \xi\Big[-\kappa_2 C_1(\xi) + \kappa_1 B_1(\xi)\Big] J_2(\xi r)\, d\xi + \\
&- \frac{1_x}{4\pi}\int\limits_0^\infty \xi\Big[H_3(\xi s_i h) - \mathrm{e}^{-\xi s_3 h}\Big] J_0(\xi b) J_2(\xi r)\, d\xi
\end{aligned}
\tag{4.3}
$$

The unknows $B_1(\xi)$ and $D_1(\xi)$ are determined by imposing the mixed boundary conditions in the plane $z = 0$, where the crack appears, Eqs (2.2b), resulting in the following two simultaneous pairs of coupled dual integral equations

— for $r < a$

$$
\begin{aligned}
&\int\limits_0^\infty \Big[\kappa_1 C_1(\xi) - \kappa_2 B_1(\xi)\Big]\xi J_0(\xi r)\, d\xi = \\
&\qquad = -1_x \int\limits_0^\infty \Big[H_3(\xi s_i h) + \mathrm{e}^{-\xi s_3 h}\Big]\xi J_0(\xi b) J_0(\xi r)\, d\xi \\
&\int\limits_0^\infty \Big[-\kappa_2 C_1(\xi) + \kappa_1 B_1(\xi)\Big]\xi J_2(\xi r)\, d\xi = \\
&\qquad = 1_x \int\limits_0^\infty \Big[H_3(\xi s_i h) - \mathrm{e}^{-\xi s_3 h}\Big]\xi J_0(\xi b) J_2(\xi r)\, d\xi
\end{aligned}
\tag{4.4}
$$

— for $r > a$

$$
\begin{aligned}
&\int\limits_0^\infty B_1(\xi) J_0(\xi r)\, d\xi = 0 \\
&\int\limits_0^\infty C_1(\xi) J_2(\xi r)\, d\xi = 0
\end{aligned}
\tag{4.5}
$$

These coupled dual integral equations (4.4) and (4.5) are converted to a pair of partialy uncoupled Abel integral equations by employing the following integral representations for $B_1(\xi)$ and $C_1(\xi)$

$$B_1(\xi) = \sqrt{\xi}\int\limits_0^a \sqrt{x}\Phi_1(x)J_{1/2}(\xi x)\,dx + \frac{\kappa_1}{\kappa_2}\sqrt{\xi}\int\limits_0^a \sqrt{x}\Phi_2(x)J_{5/2}(\xi x)\,dx \tag{4.6}$$

$$C_1(\xi) = \sqrt{\xi}\int\limits_0^a \sqrt{x}\Phi_2(x)J_{5/2}(\xi x)\,dx$$

in which $\Phi_1(x)$ and $\Phi_2(x)$ are unknown functions excited to be continuous in the interval $[0,a]$ and to satisfy the conditions

$$\sqrt{x}\Phi_j(x) \to 0 \qquad \text{as} \qquad x \to 0^+ \qquad (j=1,2)$$

Using the Weber-Schafheitlin integral

$$\int\limits_0^\infty \alpha^{-\lambda}J_\mu(q\alpha)J_\nu(\rho\alpha)\,d\alpha = \begin{cases} 0 & 0<q<\rho \\ \dfrac{\rho^\nu(q^2-\rho^2)^\lambda}{2^\lambda q^\mu \Gamma(1+\lambda)} & 0<\rho<q \end{cases} \tag{4.7}$$

which holds when $\lambda = \mu-\nu-1$, $R(\mu-\nu) > 0$, $R(\nu) > -1$, and take advantage that the integral (4.7) is also equal to zero for $\rho > q$, when $\lambda = -1/2$, $\mu = 5/2$, $\nu = 0$, it may be shown that the representation for $B_1(\xi)$ and $C_1(\xi)$ automatically satisfies Eqs (4.5). Modifying $B_1(\xi)$ by integrating the first integral in Eq $(4.6)_1$ by parts and substituting the resulting expression along with Eq $(4.6)_2$ into Eq $(4.4)_1$ we obtain the following Abel integral equation

$$\kappa_2\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{d\big[\Phi_1(x)\big]}{dx}\frac{dx}{\sqrt{r^2-x^2}}\,dx = 1_x\int\limits_0^\infty \Big[H_3(\xi s_i h) + e^{-\xi s_3 h}\Big]\xi J_0(\xi b)J_0(\xi r)\,d\xi \tag{4.8}$$

Again, modifying both equations (4.6), by integrating by parts and substituting the resulting expressions into Eq $(4.4)_2$ yield the second Abel integral equation

$$\sqrt{\frac{2}{\pi}}\int\limits_0^r \Big[\frac{\kappa_2^2-\kappa_1^2}{\kappa_2}\frac{d\big[x^2\Phi_2(x)\big]}{dx} + \kappa_1 x^3\frac{d}{dx}\Big(\frac{\Phi_1(x)}{x}\Big)\Big]\frac{dx}{\sqrt{r^2-x^2}} =$$
$$= -1_x r^2\int\limits_0^\infty \Big[H_3(\xi s_i h) - e^{-\xi s_3 h}\Big]\xi J_0(\xi b)J_2(\xi r)\,d\xi \tag{4.9}$$

Applying the Abel solution method to Eqs (4.8) and (4.9) with the use of the formula

$$J_n(\xi x) = \sqrt{\frac{2\xi}{\pi}}\frac{1}{x^n}\int\limits_0^x \frac{J_{n-(1/2)}(\xi r)r^{n+(1/2)}}{\sqrt{x^2-r^2}}\,dr \tag{4.10}$$

results in the formulae for $\Phi_1(x)$ and $\Phi_2(x)$

$$\begin{aligned}\Phi_1(x) &= \sqrt{\frac{2}{\pi}}\frac{1_x}{\kappa_2}\int\limits_0^\infty \Big[H_3(\xi s_i h) + \mathrm{e}^{-\xi s_3 h}\Big] J_0(\xi b)\sin(\xi x)\,d\xi \\ \Phi_2(x) &= 1_x\int\limits_0^\infty \Big[-\frac{1}{\kappa_1+\kappa_2}H_3(\xi s_i h) + \frac{1}{\kappa_2-\kappa_1}\mathrm{e}^{-\xi s_3 h}\Big] J_0(\xi b)\sqrt{\xi x}J_{5/2}(\xi x)\,d\xi\end{aligned} \tag{4.11}$$

Integrating the integral from Eq $(4.11)_2$ by parts and using in both solutions (4.11) the integrals (A1), (A5) and (A6), respectively, which are given in Appendix, results in the final formulae for $\Phi_1(x)$ and $\Phi_2(x)$

$$\begin{aligned}\Phi_1(x) &= \sqrt{\frac{2}{\pi}}\frac{1_x}{x\kappa_2}\Big[\frac{1}{k-1}\Big(\frac{k\eta_1}{\zeta_1^2+\eta_1^2} - \frac{\eta_2}{\zeta_2^2+\eta_2^2}\Big) + \frac{\eta_3}{\zeta_3^2+\eta_3^2}\Big] \\ \Phi_2(x) &= \sqrt{\frac{2}{\pi}}\frac{1_x}{x}\Big\{\frac{1}{(\kappa_1+\kappa_2)(k-1)}\Big(\frac{k\eta_1}{\zeta_1^2+\eta_1^2} - \frac{\eta_2}{\zeta_2^2+\eta_2^2}\Big) + \\ &-\frac{\eta_3}{(\kappa_2-\kappa_1)(\zeta_3^2+\eta_3^2)} + \frac{3}{(\kappa_1+\kappa_2)(k-1)}\Big[k\eta_1\Big(\zeta_1\Big(\frac{\pi}{2}-\tan^{-1}\zeta_1\Big)-1\Big) + \\ &-\eta_2\Big(\zeta_2\Big(\frac{\pi}{2}-\tan^{-1}\zeta_2\Big)-1\Big)\Big] - \frac{3}{\kappa_2-\kappa_1}\eta_3\Big(\zeta_3\Big(\frac{\pi}{2}-\tan^{-1}\zeta_3\Big)-1\Big)\Big\}\end{aligned} \tag{4.12}$$

where the oblate spheroidal coordinates ζ_i, η_i $(i = 1,2,3)$ are defined in the Appendix.

The singular parts of the shear stresses are given by

$$\begin{aligned}\Big(\frac{\sigma_{zr}}{\cos\theta} - \frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0} &= \frac{\kappa_2}{4\pi}\sqrt{\frac{2}{\pi}}\frac{\Phi_1(a)}{\sqrt{r^2-a^2}} \\ \Big(\frac{\sigma_{zr}}{\cos\theta} + \frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0} &= \frac{1}{4\pi}\sqrt{\frac{2}{\pi}}\Big[\frac{\kappa_2^2-\kappa_1^2}{\kappa_2}\Phi_2(a) + \kappa_1\Phi_1(a)\Big]\frac{1}{\sqrt{r^2-a^2}}\end{aligned} \tag{4.13}$$

as $r \to a^+$, so the stress intensity factors of modes II and III are

$$K_{II} = \sqrt{\frac{2}{\pi}}\frac{\kappa_2+\kappa_1}{8\pi\sqrt{a}}\cos\theta\Big[\Phi_1(a)+\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Phi_2(a)\Big]$$
$$K_{III} = -\sqrt{\frac{2}{\pi}}\frac{\kappa_2-\kappa_1}{8\pi\sqrt{a}}\sin\theta\Big[\Phi_1(a)-\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\Phi_2(a)\Big] \tag{4.14}$$

Substituting for $\Phi_1(a)$ and $\Phi_2(a)$ the final solutions are

$$\begin{aligned} K_{II} &= \frac{1_x}{2\pi^2\sqrt{a^3}}\cos\theta\Big\{\frac{1}{k-1}\Big(\frac{k\bar\eta_1}{\bar\zeta_1^2+\bar\eta_1^2}-\frac{\bar\eta_2}{\bar\zeta_2^2+\bar\eta_2^2}\Big)+ \\ &+ \frac{3}{2(k-1)}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big[k\bar\eta_1\Big(\bar\zeta_1(\frac{\pi}{2}-\tan^{-1}\bar\zeta_1)-1\Big)+ \\ &- \bar\eta_2\Big(\bar\zeta_2(\frac{\pi}{2}-\tan^{-1}\bar\zeta_2)-1\Big)\Big]-\frac{3}{2}\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\bar\eta_3\Big(\bar\zeta_3(\frac{\pi}{2}-\tan^{-1}\bar\zeta_3)-1\Big)\Big\} \end{aligned} \tag{4.15}$$

$$\begin{aligned} K_{III} &= -\frac{1_x}{2\pi^2\sqrt{a^3}}\sin\theta\Big\{\frac{\bar\eta_3}{\bar\zeta_3^2+\bar\eta_3^2}-\frac{3}{2(k-1)}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\cdot \\ &\cdot \Big[k\bar\eta_1\Big(\bar\zeta_1(\frac{\pi}{2}-\tan^{-1}\bar\zeta_1)-1\Big)-\bar\eta_2\Big(\bar\zeta_2(\frac{\pi}{2}-\tan^{-1}\bar\zeta_2)-1\Big)\Big]+ \\ &+ \frac{3}{2}\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\bar\eta_3\Big(\bar\zeta_3(\frac{\pi}{2}-\tan^{-1}\bar\zeta_3)-1\Big)\Big\} \end{aligned}$$

where the oblate spheroidal coordinates $\bar\zeta_i$, $\bar\eta_i$ are obtained from ζ_i, η_i by substitution $x = a$.

In the special cases K_{II} and K_{III} take the values

— for $b < a$, $h = 0$

$$K_{II} = \frac{1_x}{2\pi^2\sqrt{a^3}}\cos\theta\Big[\frac{a}{\sqrt{a^2-b^2}}+3\frac{\kappa_1}{\kappa_2}\frac{\sqrt{a^2-b^2}}{a}\Big] \tag{4.16a}$$

$$K_{III} = -\frac{1_x}{2\pi^2\sqrt{a^3}}\sin\theta\Big[\frac{a}{\sqrt{a^2-b^2}}-3\frac{\kappa_1}{\kappa_2}\frac{\sqrt{a^2-b^2}}{a}\Big] \tag{4.17a}$$

— for $b > a$, $h = 0$

$$K_{II} = 0 \tag{4.16b}$$

$$K_{III} = 0 \tag{4.17b}$$

— for $b = 0$

$$K_{II} = \frac{1_x}{2\pi^2\sqrt{a^3}} \cos\theta \Big\{ \frac{a^2}{k-1}\Big(\frac{k}{a^2+s_1^2h^2} - \frac{1}{a^2+s_2^2h^2}\Big) + \frac{3\kappa_1}{\kappa_2} +$$
$$+ \frac{3h}{2a(k-1)}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big[ks_1\Big(\frac{\pi}{2} - \tan^{-1}\frac{s_1h}{a}\Big) - s_2\Big(\frac{\pi}{2} - \tan^{-1}\frac{s_2h}{a}\Big)\Big] + \qquad (4.16c)$$
$$- \frac{3h}{2a}\Big(1+\frac{\kappa_1}{\kappa_2}\Big)s_3\Big(\frac{\pi}{2} - \tan^{-1}\frac{s_3h}{a}\Big)\Big\}$$

$$K_{III} = -\frac{1_x}{2\pi^2\sqrt{a^3}} \sin\theta \Big\{ \frac{a^2}{a^2+s_3^2h^2} - \frac{3\kappa_1}{\kappa_2} +$$
$$- \frac{3h}{2a(k-1)}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big[ks_1\Big(\frac{\pi}{2} - \tan^{-1}\frac{s_1h}{a}\Big) - s_2\Big(\frac{\pi}{2} - \tan^{-1}\frac{s_2h}{a}\Big)\Big] + \qquad (4.17c)$$
$$+ \frac{3h}{2a}\Big(1+\frac{\kappa_1}{\kappa_2}\Big)s_3\Big(\frac{\pi}{2} - \tan^{-1}\frac{s_3h}{a}\Big)\Big\}$$

For the case when the stress of half intensity acts on the crack surfaces in the opposite directions parallel to the x-axis we obtain after integration of Eqs (4.16) and (4.17) for $b < a$ and $h = 0$

$$K_{II} = \frac{1}{\pi\sqrt{a}}\Big(1 + \frac{\kappa_1}{\kappa_2}\Big)\cos\theta$$
$$K_{III} = -\frac{1}{\pi\sqrt{a}}\Big(1 - \frac{\kappa_1}{\kappa_2}\Big)\sin\theta \qquad (4.18)$$

This agrees with the author's result (Rogowski (1986)), which was based on separate calculations.

4.1.2. Concentrated forces distributed along a circumference (Fig.3 and Fig.1b)

Eqs (2.8) and (2.9) differ from Eqs (2.6) and (2.7) as follows: $J_0(\xi b)$ and $1_x H_3(\xi s_i h)$ are replaced by $J_1(\xi b)b^{-1}$ and $1_3 v_1 H_2(\xi s_i h)$, respectively, and the terms with $\exp(-\xi s_3 h)$ are omitted in Eqs (2.6). Thus the Abel integral equations are obtained immediately from Eqs (4.8) and (4.9). The solutions are also obtained from Eqs (4.11) in the form

$$\Psi_1(x) = \sqrt{\frac{2}{\pi}}\frac{1_3 v_1}{\kappa_2 b}\int\limits_0^\infty H_2(\xi s_i h)J_1(\xi b)\sin(\xi x)\,d\xi$$
$$\Psi_2(x) = -\frac{1_3 v_1}{(\kappa_1+\kappa_2)b}\int\limits_0^\infty H_2(\xi s_i h)J_1(\xi b)\sqrt{\xi x}J_{5/2}(\xi x)\,d\xi \qquad (4.19)$$

where $\Phi_j(x)$ are replaced by $\Psi_j(x)$ $(j = 1,2)$ since in this problem we have the constants $D(\xi)$ and $E(\xi)$ (in Eqs (2.8), (2.9)) instead the constants $B(\xi)$ and $C(\xi)$ (in Eqs (2.6), (2.7)). Using the analytic expressions for the improper integrals (see Appendix) we have

$$
\begin{aligned}
\Psi_1(x) &= \sqrt{\frac{2}{\pi}} \frac{1_3 v_1}{\kappa_2 x^2 (ks_2 - s_1)} \Big[\frac{ks_2\zeta_2}{(1+\zeta_2^2)(\zeta_2^2+\eta_2^2)} - \frac{s_1\zeta_1}{(1+\zeta_1^2)(\zeta_1^2+\eta_1^2)} \Big] \\
\Psi_2(x) &= \sqrt{\frac{2}{\pi}} \frac{1_3 v_1}{(\kappa_1+\kappa_2) x^2 (ks_2 - s_1)} \Big\{ \frac{ks_2\zeta_2}{(1+\zeta_2^2)(\zeta_2^2+\eta_2^2)} + \\
&- \frac{s_1\zeta_1}{(1+\zeta_1^2)(\zeta_1^2+\eta_1^2)} + \frac{3hx}{b^2} \Big[ks_2^2(1-\eta_2) - s_1^2(1-\eta_1) \Big] + \\
&- \frac{3}{2} \Big[ks_2 \Big(\frac{\pi}{2} - \tan^{-1}\zeta_2 - \frac{\zeta_2}{1+\zeta_2^2} \Big) - s_1 \Big(\frac{\pi}{2} - \tan^{-1}\zeta_1 - \frac{\zeta_1}{1+\zeta_1^2} \Big) \Big] \Big\}
\end{aligned}
\tag{4.20}
$$

The stress intensity factors of modes II and III are defined from equations similar to Eqs (4.14) $(\Phi_j(a) \to \Psi_j(a))$ by substituting Eqs (4.20). The results are

$$
\begin{aligned}
K_{II} &= \frac{1_3 v_1}{2\pi^2\sqrt{a^5}(ks_2 - s_1)} \cos\theta \Big\{ \frac{ks_2\bar{\zeta}_2}{(1+\bar{\zeta}_2^2)(\bar{\zeta}_2^2+\bar{\eta}_2^2)} - \frac{s_1\bar{\zeta}_1}{(1+\bar{\zeta}_1^2)(\bar{\zeta}_1^2+\bar{\eta}_1^2)} + \\
&+ \frac{3}{2}\Big(1 - \frac{\kappa_1}{\kappa_2}\Big) \Big[\frac{ha}{b^2} \Big(ks_2^2(1-\bar{\eta}_2) - s_1^2(1-\bar{\eta}_1) \Big) + \\
&- \frac{1}{2} \Big(ks_2 \Big(\frac{\pi}{2} - \tan^{-1}\bar{\zeta}_2 - \frac{\bar{\zeta}_2}{1+\bar{\zeta}_2^2} \Big) - s_1 \Big(\frac{\pi}{2} - \tan^{-1}\bar{\zeta}_1 - \frac{\bar{\zeta}_1}{1+\bar{\zeta}_1^2} \Big) \Big) \Big] \Big\} \\
K_{III} &= \frac{1_3 3 v_1}{4\pi^2\sqrt{a^5}(ks_2 - s_1)} \sin\theta \Big(1 - \frac{\kappa_1}{\kappa_2}\Big) \Big\{ \frac{ha}{b^2} \Big[ks_2^2(1-\bar{\eta}_2) - s_1^2(1-\bar{\eta}_1) \Big] + \\
&- \frac{1}{2} \Big[ks_2 \Big(\frac{\pi}{2} - \tan^{-1}\bar{\zeta}_2 - \frac{\bar{\zeta}_2}{1+\bar{\zeta}_2^2} \Big) - s_1 \Big(\frac{\pi}{2} - \tan^{-1}\bar{\zeta}_1 - \frac{\bar{\zeta}_1}{1+\bar{\zeta}_1^2} \Big) \Big] \Big\}
\end{aligned}
\tag{4.21}
$$

In the special cases we have

— for $b < a$, $h = 0$

$$
K_{II} = -\frac{1_3 3 v_1}{16\pi\sqrt{a^5}} \Big(1 - \frac{\kappa_1}{\kappa_2}\Big) \cos\theta \tag{4.22a}
$$

$$
K_{III} = -\frac{1_3 3 v_1}{16\pi\sqrt{a^5}} \Big(1 - \frac{\kappa_1}{\kappa_2}\Big) \sin\theta \tag{4.23a}
$$

— for $b > a,\ h = 0$

$$K_{II} = \frac{1_3 v_1}{2\pi^2\sqrt{a^5}}\cos\theta\Big[\frac{a^3}{b^2\sqrt{b^2-a^2}}+$$

$$-\frac{3}{4}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big(\sin^{-1}\frac{a}{b}-\frac{a}{b}\sqrt{1-(\frac{a}{b})^2}\Big)\Big] \tag{4.22b}$$

$$K_{III} = -\frac{1_3 3v_1}{8\pi^2\sqrt{a^5}}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\sin\theta\Big(\sin^{-1}\frac{a}{b}-\frac{a}{b}\sqrt{1-(\frac{a}{b})^2}\Big) \tag{4.23b}$$

— for $b = 0$

$$K_{II} = \frac{1_3 v_1}{2\pi^2\sqrt{a^5}(ks_2-s_1)}\cos\theta\Big\{\frac{ks_2^2ha^3}{(a^2+s_2^2h^2)^2}+$$

$$-\frac{s_1^2ha^3}{(a^2+s_1^2h^2)^2}-\frac{3}{4}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big[ks_2\Big(\frac{\pi}{2}-\tan^{-1}\frac{s_2h}{a}-\frac{s_2ha}{a^2+s_2^2h^2}\Big)+ \tag{4.22c}$$

$$-s_1\Big(\frac{\pi}{2}-\tan^{-1}\frac{s_1h}{a}-\frac{s_1ha}{a^2+s_1^2h^2}\Big)\Big]\Big\}$$

$$K_{III} = -\frac{1_3 3v_1}{8\pi^2\sqrt{a^5}(ks_2-s_1)}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\sin\theta\cdot$$

$$\cdot\Big[ks_2\Big(\frac{\pi}{2}-\tan^{-1}\frac{s_2h}{a}-\frac{s_2ha}{a^2+s_2^2h^2}\Big)-s_1\Big(\frac{\pi}{2}-\tan^{-1}\frac{s_1h}{a}-\frac{s_1ha}{a^2+s_1^2h^2}\Big)\Big] \tag{4.23c}$$

4.2. An external crack

4.2.1. Concentrated forces distributed along a circumference in the x direction (Fig.2 and Fig.1b)

Introduce the new constants

$$D(\xi)-E(\xi)=D_1(\xi) \qquad\qquad D(\xi)+E(\xi)=E_1(\xi)$$

Thus Eqs (2.8) and (2.9) become

$$G_z\Big(\frac{u_r}{\cos\theta}-\frac{u_\theta}{\sin\theta}\Big)\Big|_{z=0}=\frac{1}{4\pi(\kappa_1^2-\kappa_2^2)}\int\limits_0^\infty\Big[-\kappa_2 D_1(\xi)+\kappa_1 E_1(\xi)+$$

$$+1_x\Big((\kappa_1-\kappa_2)H_3(\xi s_i h)-(\kappa_1+\kappa_2)\mathrm{e}^{-\xi s_3 h}\Big)J_0(\xi b)\Big]J_0(\xi r)\;d\xi \tag{4.24}$$

$$G_z\Big(\frac{u_r}{\cos\theta}+\frac{u_\theta}{\sin\theta}\Big)\Big|_{z=0}=-\frac{1}{4\pi(\kappa_1^2-\kappa_2^2)}\int\limits_0^\infty\Big[\kappa_1 D_1(\xi)-\kappa_2 E_1(\xi)+$$

$$+1_x\Big((\kappa_1-\kappa_2)H_3(\xi s_i h)+(\kappa_1+\kappa_2)\mathrm{e}^{-\xi s_3 h}\Big)J_0(\xi b)\Big]J_2(\xi r)\;d\xi$$

$$\Big(\frac{\sigma_{zr}}{\cos\theta}-\frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0}=-\frac{1}{4\pi}\int\limits_0^\infty \xi D_1(\xi)J_0(\xi r)\;d\xi \tag{4.25}$$

$$\Big(\frac{\sigma_{zr}}{\cos\theta}+\frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0}=\frac{1}{4\pi}\int\limits_0^\infty \xi E_1(\xi)J_2(\xi r)\;d\xi$$

If we take the representations

$$D_1(\xi)=\sqrt{\xi}\int\limits_0^a\sqrt{x}\varphi_1(x)J_{-1/2}(\xi x)\;dx+\frac{\kappa_1}{\kappa_2}\sqrt{\xi}\int\limits_0^a\sqrt{x}\varphi_2(x)J_{3/2}(\xi x)\;dx \tag{4.26}$$

$$E_1(\xi)=\sqrt{\xi}\int\limits_0^a\sqrt{x}\varphi_2(x)J_{3/2}(\xi x)\;dx$$

we find that the stresses (4.25) are zero if $r>a$ and that if $r<a$

$$\Big(\frac{\sigma_{zr}}{\cos\theta}-\frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0}=-\frac{1}{4\pi}\sqrt{\frac{2}{\pi}}\Big[\frac{\varphi_1(a)}{\sqrt{a^2-r^2}}-\int\limits_r^a\frac{d\varphi_1(x)}{dx}\frac{dx}{\sqrt{x^2-r^2}}+$$

$$+\frac{\kappa_1}{\kappa_2}\Big(\frac{\varphi_2(a)}{\sqrt{a^2-r^2}}-r^2\int\limits_r^a\frac{d}{dx}\Big(\frac{\varphi_2(x)}{x^2}\Big)\frac{dx}{\sqrt{x^2-r^2}}\Big)\Big] \tag{4.27}$$

$$\Big(\frac{\sigma_{zr}}{\cos\theta}+\frac{\sigma_{z\theta}}{\sin\theta}\Big)\Big|_{z=0}=\frac{1}{4\pi}\sqrt{\frac{2}{\pi}}\Big(\frac{\varphi_2(a)}{\sqrt{a^2-r^2}}-r^2\int\limits_r^a\frac{d}{dx}\Big(\frac{\varphi_2(x)}{x^2}\Big)\frac{dx}{\sqrt{x^2-r^2}}\Big)$$

Substituting the representations for $D_1(\xi)$ and $E_1(\xi)$ into Eqs (4.24) and for the resulting expressions into displacement boundary conditions (2.3b) results in the integral equations in auxiliary functions $\varphi_1(x)$ and $\varphi_2(x)$

$$\kappa_2\sqrt{\frac{2}{\pi}}\int_0^r \frac{\varphi_1(x)}{\sqrt{r^2-x^2}}\,dx = \tag{4.28}$$
$$= 1_x\int_0^\infty\Big[(\kappa_1-\kappa_2)H_3(\xi s_i h)-(\kappa_1+\kappa_2)e^{-\xi s_3 h}\Big]J_0(\xi b)J_2(\xi r)\,d\xi$$

$$\kappa_1\sqrt{\frac{2}{\pi}}\int_0^r \frac{\varphi_1(x)}{\sqrt{r^2-x^2}}\,dx +$$
$$+\frac{1}{r^2}\sqrt{\frac{2}{\pi}}\int_0^r\Big[-2\kappa_1\varphi_1(x)+\frac{\kappa_1^2-\kappa_2^2}{\kappa_2}\varphi_2(x)\Big]\frac{x^2}{\sqrt{r^2-x^2}}\,dx = \tag{4.29}$$
$$= -1_x\int_0^\infty\Big[(\kappa_1-\kappa_2)H_3(\xi s_i h)-(\kappa_1+\kappa_2)e^{-\xi s_3 h}\Big]J_0(\xi b)J_2(\xi r)\,d\xi$$

The solution to Eq (4.28) is

$$\varphi_1(x) = -\sqrt{\frac{2}{\pi}}\frac{1_x}{x}\Big[\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\frac{1}{k-1}\Big(\frac{k\zeta_1}{\zeta_1^2+\eta_1^2}-\frac{\zeta_2}{\zeta_2^2+\eta_2^2}\Big)+\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\frac{\zeta_3}{\zeta_3^2+\eta_3^2}\Big] \tag{4.30}$$

where ζ_i, η_i are the oblate spheroidal coordinates (see Appendix).

Substituting Eq (4.28) into Eq (4.29) and applying the Abel solution method the function $\varphi_2(x)$ is obtained as follows

$$\varphi_2(x) = \sqrt{\frac{2}{\pi}}\frac{1_x}{x}\Big[\frac{1}{k-1}(k\tan^{-1}\zeta_1-\tan^{-1}\zeta_2)-\tan^{-1}\zeta_3 + \tag{4.31}$$
$$+ \frac{1}{k-1}\Big(\frac{k\zeta_1}{\zeta_1^2+\eta_1^2}-\frac{\zeta_2}{\zeta_2^2+\eta_2^2}\Big)-\frac{\zeta_3}{\zeta_3^2+\eta_3^2}\Big]$$

From Eqs (4.27) we conclude that

$$\sigma_{zr}(r,\theta,0) = -\frac{1}{8\pi}\sqrt{\frac{2}{\pi}}\Big[\frac{\varphi_1(a)}{\sqrt{a^2-r^2}}-\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\frac{\varphi_2(a)}{\sqrt{a^2-r^2}}\Big]\cos\theta \tag{4.32}$$
$$\sigma_{z\theta}(r,\theta,0) = \frac{1}{8\pi}\sqrt{\frac{2}{\pi}}\Big[\frac{\varphi_1(a)}{\sqrt{a^2-r^2}}+\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\frac{\varphi_2(a)}{\sqrt{a^2-r^2}}\Big]\sin\theta$$

as $r \to a^-$.

Substituting for $\varphi_1(a)$ and $\varphi_2(a)$ from Eqs (4.30) and (4.31) the final solutions are

$$K_{II} = \frac{1_x}{2\pi^2\sqrt{a^3}}\cos\theta\Big\{\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\frac{1}{k-1}\Big(\frac{k\bar\zeta_1}{\bar\zeta_1^2+\bar\eta_1^2}-\frac{\bar\zeta_2}{\bar\zeta_2^2+\bar\eta_2^2}\Big)+\frac{\kappa_1}{\kappa_2}\frac{\bar\zeta_3}{\bar\zeta_3^2+\bar\eta_3^2}+ \\ +\frac{1}{2}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big[\frac{1}{k-1}(k\tan^{-1}\bar\zeta_1-\tan^{-1}\bar\zeta_2)-\tan^{-1}\bar\zeta_3\Big]\Big\} \tag{4.33}$$

$$K_{III} = \frac{1_x}{2\pi^2\sqrt{a^3}}\sin\theta\Big\{\frac{\kappa_1}{\kappa_2(k-1)}\Big(\frac{k\bar\zeta_1}{\bar\zeta_1^2+\bar\eta_1^2}-\frac{\bar\zeta_2}{\bar\zeta_2^2+\bar\eta_2^2}\Big)-\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\frac{\bar\zeta_3}{\bar\zeta_3^2+\bar\eta_3^2}+ \\ +\frac{1}{2}\Big(1+\frac{\kappa_1}{\kappa_2}\Big)\Big[\frac{1}{k-1}(k\tan^{-1}\bar\zeta_1-\tan^{-1}\bar\zeta_2)-\tan^{-1}\bar\zeta_3\Big]\Big\}$$

where $\bar\zeta_i$, $\bar\eta_i$ are obtained from ζ_i, η_i for $x = a$ (see Appendix).

In the special cases K_{II} and K_{III} take the values

— for $b < a$, $h = 0$

$$K_{II} = 0 \tag{4.34a}$$

$$K_{III} = 0 \tag{4.35a}$$

— for $b > a$, $h = 0$

$$K_{II} = \frac{1_x}{2\pi^2\sqrt{a(b^2-a^2)}}\cos\theta \tag{4.34b}$$

$$K_{III} = -\frac{1_x}{2\pi^2\sqrt{a(b^2-a^2)}}\sin\theta \tag{4.35b}$$

— for $b = 0$

$$K_{II} = \frac{1_x}{2\pi^2\sqrt{a^3}}\cos\theta\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\Big\{\frac{1}{k-1}\Big(\frac{ks_1ha}{a^2+s_1^2h^2}-\frac{s_2ha}{a^2+s_2^2h^2}\Big)+ \\ +\frac{\kappa_1s_3ha}{(\kappa_2-\kappa_1)(a^2+s_3^2h^2)}+\frac{1}{2}\Big[\frac{1}{k-1}\Big(k\tan^{-1}\frac{s_1h}{a}-\tan^{-1}\frac{s_2h}{a}\Big)-\tan^{-1}\frac{s_3h}{a}\Big]\Big\} \tag{4.34c}$$

$$K_{III} = \frac{1_x}{2\pi^2\sqrt{a^3}}\sin\theta\Big\{\frac{\kappa_1}{\kappa_2(k-1)}\Big(\frac{ks_1ha}{a^2+s_1^2h^2}-\frac{s_2ha}{a^2+s_2^2h^2}\Big)+ \\ -\Big(1+\frac{\kappa_1}{2\kappa_2}\Big)\Big[\frac{s_3ha}{a^2+s_3^2h^2}-\frac{1}{2(k-1)}\Big(k\tan^{-1}\frac{s_1h}{a}-\tan^{-1}\frac{s_2h}{a}\Big)+\frac{1}{2}\tan^{-1}\frac{s_3h}{a}\Big]\Big\} \tag{4.35c}$$

4.2.2. Concentrated forces distributed along a circumference (Fig.3 and Fig.1b)

For an external crack the dual integral equations are obtained from Eqs (4.28) and (4.29) by replacying $J_0(\xi b)$ by $J_1(\xi b)/b$ and $H_3(\xi s_i h)$ by $v_1 H_2(\xi s_i h)$, and 1_x by 1_3 and omitting the function $\exp(-\xi s_3 h)$. Thus

$$\kappa_2\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{\Psi_1(x)}{\sqrt{r^2-x^2}}\,dx = \frac{1_3 v_1}{b}(\kappa_1-\kappa_2)\int\limits_0^\infty H_2(\xi s_i h)J_1(\xi b)J_0(\xi r)\,d\xi \tag{4.36}$$

$$\kappa_1\sqrt{\frac{2}{\pi}}\int\limits_0^r \frac{\Psi_1(x)}{\sqrt{r^2-x^2}}dx + \frac{1}{r^2}\sqrt{\frac{2}{\pi}}\int\limits_0^r \Big[-2\kappa_1\Psi_1(x) + \frac{\kappa_1^2-\kappa_2^2}{\kappa_2}\Psi_2(x)\Big]\frac{x^2\,dx}{\sqrt{r^2-x^2}} = \tag{4.37}$$

$$= -\frac{1_3 v_1}{b}(\kappa_1-\kappa_2)\int\limits_0^\infty H_2(\xi s_i h)J_1(\xi b)J_2(\xi r)\,d\xi$$

The solution to Eq (4.36) is

$$\Psi_1(x) = -\sqrt{\frac{2}{\pi}}\Big(1-\frac{\kappa_1}{\kappa_2}\Big)\frac{1_3 v_1}{b^2(ks_2-s_1)}\Big[ks_2\Big(1-\frac{\eta_2(1+\zeta_2^2)}{\zeta_2^2+\eta_2^2}\Big)-s_1\Big(1-\frac{\eta_1(1+\zeta_1^2)}{\zeta_1^2+\eta_1^2}\Big)\Big] \tag{4.38}$$

Substituting Eq (4.36) into Eq (4.37) and applying the Abel solution method the function $\Psi_2(x)$ is obtained as follows

$$\begin{aligned}\Psi_2(x) &= \sqrt{\frac{2}{\pi}}\frac{1_3 v_1}{b^2(ks_2-s_1)}\Big[ks_2(1-\eta_2)-s_1(1-\eta_1)+ \\ &+ ks_2\Big(1-\frac{\eta_2(1+\zeta_2^2)}{\zeta_2^2+\eta_2^2}\Big)-s_1\Big(1-\frac{\eta_1(1+\zeta_1^2)}{\zeta_1^2+\eta_1^2}\Big)\Big]\end{aligned} \tag{4.39}$$

The stresses are given by Eqs (4.32) $(\varphi_i(x)\to\Psi_i(x))$, thus the stress intensity factors of modes II and III are

$$\begin{aligned}K_{II} &= \Big(1-\frac{\kappa_1}{\kappa_2}\Big)\frac{1_3 v_1}{2\pi^2 b^2\sqrt{a}(ks_2-s_1)}\cos\theta\Big[ks_2\Big(1-\frac{\bar\eta_2(1+\bar\zeta_2^2)}{\bar\zeta_2^2+\bar\eta_2^2}\Big)+ \\ &- s_1\Big(1-\frac{\bar\eta_1(1+\bar\zeta_1^2)}{\bar\zeta_1^2+\bar\eta_1^2}\Big)+\frac{1}{2}[ks_2(1-\bar\eta_2)-s_1(1-\bar\eta_1)]\Big] \\ K_{III} &= \frac{1_3 v_1}{2\pi^2 b^2\sqrt{a}(ks_2-s_1)}\sin\theta\Big\{\frac{\kappa_1}{\kappa_2}\Big[ks_2\Big(1-\frac{\bar\eta_2(1+\bar\zeta_2^2)}{\bar\zeta_2^2+\bar\eta_2^2}\Big)+ \\ &- s_1\Big(1-\frac{\bar\eta_1(1+\bar\zeta_1^2)}{\bar\zeta_1^2+\bar\eta_1^2}\Big)\Big]+\frac{1}{2}\Big(1+\frac{\kappa_1}{\kappa_2}\Big)[ks_2(1-\bar\eta_2)-s_1(1-\bar\eta_1)]\Big\}\end{aligned} \tag{4.40}$$

where $\bar{\zeta}_i$, $\bar{\eta}_i$ are obtained from ζ_i, η_i for $x = a$ (see Appendix).

In the special cases K_{II} and K_{III} assume the forms

— for $b < a$, $h = 0$

$$K_{II} = \Big(1 - \frac{\kappa_1}{\kappa_2}\Big)\frac{1_3 v_1}{2\pi^2 b^2 \sqrt{a}}\cos\theta\Big(\frac{3}{2} - \frac{a}{\sqrt{a^2-b^2}} - \frac{\sqrt{a^2-b^2}}{2a}\Big) \tag{4.41a}$$

$$K_{III} = \frac{1_3 v_1}{2\pi^2 b^2\sqrt{a}}\sin\theta\cdot$$

$$\cdot\Big[\frac{\kappa_1}{\kappa_2}\Big(1 - \frac{a}{\sqrt{a^2-b^2}}\Big) + \frac{1}{2}\Big(1 + \frac{\kappa_1}{\kappa_2}\Big)\Big(1 - \frac{\sqrt{a^2-b^2}}{a}\Big)\Big] \tag{4.42a}$$

— for $b > a$, $h = 0$

$$K_{II} = \Big(1 - \frac{\kappa_1}{\kappa_2}\Big)\frac{1_3 3 v_1}{4\pi^2 b^2\sqrt{a}}\cos\theta \tag{4.41b}$$

$$K_{III} = \frac{1_3 3 v_1}{2\pi^2 b^2\sqrt{a}}\sin\theta \tag{4.42b}$$

— for $b = 0$

$$K_{II} = \Big(1 - \frac{\kappa_1}{\kappa_2}\Big)\frac{1_3 v_1}{2\pi^2\sqrt{a^5}}\cos\theta\cdot$$

$$\cdot\Big[-\frac{1}{2(ks_2 - s_1)}\Big(\frac{ks_2 a^2}{a^2 + s_2^2 h^2} - \frac{s_1 a^2}{a^2 + s_1^2 h^2}\Big) + \frac{1}{4}\Big] \tag{4.41c}$$

$$K_{III} = \frac{1_3 v_1}{2\pi^2\sqrt{a^5}}\sin\theta\cdot$$

$$\cdot\Big[-\frac{\kappa_1}{2\kappa_2(ks_2 - s_1)}\Big(\frac{ks_2 a^2}{a^2 + s_2^2 h^2} - \frac{s_1 a^2}{a^2 + s_1^2 h^2}\Big) + \frac{1}{4}\Big(1 + \frac{\kappa_1}{\kappa_2}\Big)\Big] \tag{4.42c}$$

Appendix

The following integrals are used to evaluate the auxiliary functions appearing in this paper

$$\int\limits_0^\infty J_0(\xi b)\sin(\xi x)e^{-\xi s_i h}\, d\xi = \frac{\eta_i}{x(\zeta_i^2 + \eta_i^2)} \tag{A.1}$$

$$\int_0^\infty J_1(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{b\zeta_i}{x^2(1+\zeta_i^2)(\zeta_i^2+\eta_i^2)} \tag{A.2}$$

$$\int_0^\infty J_0(\xi b)\cos(\xi x)e^{-\xi s_i h}\,d\xi = \frac{\zeta_i}{x(\zeta_i^2+\eta_i^2)} \tag{A.3}$$

$$\int_0^\infty J_1(\xi b)\cos(\xi x)e^{-\xi s_i h}\,d\xi = \frac{1}{b}\Big(1-\frac{\eta_i(1+\zeta_i^2)}{\zeta_i^2+\eta_i^2}\Big) \tag{A.4}$$

$$\int_0^\infty \xi^{-1}J_0(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{\pi}{2}-\tan^{-1}\zeta_i \tag{A.5}$$

$$\int_0^\infty \xi^{-1}J_1(\xi b)\sin(\xi x)e^{-\xi s_i h}\,d\xi = \frac{x}{b}(1-\eta_i) \tag{A.6}$$

The oblate spheroidal coordinates ζ_i, η_i are related to b, s_ih, x by the equations

$$b^2 = x^2(1+\zeta_i^2)(1-\eta_i^2) \qquad\qquad s_ih = x\zeta_i\eta_i \tag{A.7}$$

where $-1 \le \eta_i \le 1$ and $\zeta_i \ge 0$. The surfaces $\zeta_i = 0$ and $\eta_i = 0$ are, respectively, the interior and exterior of the circle $b = x$, $h = 0$ we have here, therefore

$$\begin{array}{llll} h=0 & b<x & \zeta_i = 0 & \eta_i = \sqrt{1-(b^2/x^2)} \\ h=0 & b>x & \zeta_i = \sqrt{(b^2/x^2)-1} & \eta_i = 0 \\ b=0 & & \zeta_i = s_ih/x & \eta_i = 1 \end{array} \tag{A.8}$$

For $x = a$, ζ_i and η_i are denoted by $\bar{\zeta}_i$ and $\bar{\eta}_i$.

Three sets of oblate spheroidal coordinates ζ_i, η_i $(i = 1,2,3)$ are associated with three material parameters s_i $(i = 1,2,3)$, which are defined by elastic constants c_{ij} of a transversely isotropic body as follows

$$\begin{gathered} s_1, s_2: \quad c_{33}c_{44}s^4 - [c_{11}c_{33} - c_{13}(c_{13}+2c_{44})]s^2 + c_{11}c_{44} = 0 \\ s_3^2 = \frac{c_{11}-c_{12}}{2c_{44}} = \frac{G_r}{G_z} \end{gathered} \tag{A.9}$$

where G_r and G_z are the shear moduli.

Other material parameters are defined as follows

$$k = \frac{c_{33}s_1^2 - c_{44}}{c_{13}+c_{44}} \qquad\qquad v_0 = \frac{ks_2 - s_1}{(k-1)s_1s_2}$$

$$v_1 = \frac{ks_2 - s_1}{k-1} \qquad C = \frac{(k+1)(s_1-s_2)}{(k-1)s_1s_2} \tag{A.10}$$
$$\kappa_1 = \frac{1}{2}(Cs_1s_2 - s_3) \qquad \kappa_2 = \frac{1}{2}(Cs_1s_2 + s_3)$$

Known functions H_0, H_1, H_2, H_3 are given by the equations

$$\begin{aligned}
H_0(\xi s_i h) &= \frac{1}{k-1}(ke^{-\xi s_2 h} - e^{-\xi s_1 h})\\
H_1(\xi s_i h) &= \frac{1}{ks_2 - s_1}(ks_2 e^{-\xi s_1 h} - s_1 e^{-\xi s_2 h})\\
H_2(\xi s_i h) &= \frac{1}{ks_2 - s_1}(ks_2 e^{-\xi s_2 h} - s_1 e^{-\xi s_1 h})\\
H_3(\xi s_i h) &= \frac{1}{k-1}(ke^{-\xi s_1 h} - e^{-\xi s_2 h})
\end{aligned} \tag{A.11}$$

Each of these functions tends to unity as h tends to zero.

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Rozwiązania podstawowe dla współczynników intensywności naprężenia typów *I*, *II* i *III*. Zagadnienie asymetryczne

Streszczenie

Otrzymano funkcje Greena dla współczynników intensywności naprężenia typów *I*, *II* i *III*. Funkcje Greena zdefiniowano jako rozwiązanie zagadnienia sprężystego poprzecznie izotropowego ciała z kołową lub zewnętrzną szczeliną, gdy na płaszczyźnie równoległej do płaszczyzny szczeliny działają dowolne asymetryczne obciążenia rozłożone na okręgu. Przedstawiono rozwiązania ścisłe, analityczne, w postaci zamkniętej, dla każdego typu obciążeń, jako rozwiązania podstawowe.

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