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GOVERNING EQUATIONS AND BOUNDARY CONDITIONS OF A GENERALIZED MODEL OF ELASTIC FOUNDATION[1]

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Differential equations and admissible boundary conditions describing a new model of a generalized elastic foundation consisting of n SBS-layers (Fig.1) are proposed. Each SBS-layer (Shear layer – Bending layer – Spring layer) consists at outmost of three layers: a shear-sensitive layer, a bending layer and a layer formed by springs (Fig.2). The algorithm of finding the stress and displacements state in such a foundation is presented.
The model of foundation being proposed has a straightforward mechanical interpretation and may by utilized as a representation of many real structures e.g.: embankments, roads, runways, railway subgrades respectively, etc. Assuming true values for the constans describing properties of corresponding SBS-layers one can arrive at a model representing any elastic foundation.

1. Introduction

An elastic foundation can be modelled by a number of well-known methods (cf Jemielita and Szcześniak (1993); Selvadurai (1979); Henry (1986)). The equation of unidirectional multiparameter model may be written in the following, generalized form

$$\mathcal{L}_n\Big(p(x^\alpha)\Big) = \mathcal{R}_n\Big(w(x^\alpha)\Big) \tag{1.1}$$

where

$w(x^\alpha)$ – deflection of the foundation

$p(x^\alpha)$ – load acting upon the plane bounding.

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The differential operators $\mathcal{L}_n$ and $\mathcal{R}_n$ can be expressed as follows

$$\mathcal{L}_n = \sum_{i=0}^{n} A_i \nabla^{2i} \qquad\qquad \mathcal{R}_n = \sum_{i=0}^{n} B_i \nabla^{2i} \tag{1.2}$$

Bharatha and Levinson 1980, have proved that the relations $B_{2i} > 0$, $B_{2i+1} < 0$ are satisfied. It can be proven that the same relations $A_{2i} > 0$, $A_{2i+1} < 0$ are satisfied. The physical interpretation of coefficients A_i and B_i depends on the foundation model being assumed. There are several attainable models of the elastic foundation described by Eq (1.1), one of them is the generalized model consisting of n SBS-layers set forth by Jemielita (1992). Each SBS-layer (Shear layer – Bending layer – Spring layer) consists at outmost of three layers: a shear-sensitive layer, a bending layer and a layer formed by springs (Fig.2).

The model of foundation being proposed has a straightforward mechanical interpretation and may by utilized as a representation of many real structures e.g.: embankments, roads, runways, tracks, respectively, etc. Assuming true values (0 or ∞) for the constans describing properties of corresponding SBS-layers one can arrive at a model representing any elastic foundation[2].

In Jemielita (1992) Eqs (1.1) have been derived employing a direct approach, which is particularly useful when the equation representing the (1.1) model has to be obtained, there are however obstacles in determination of admissible boundary conditions on the cylindrical lateral edge surface of the foundation. When the foundation has a form of infinite layer with the discontinuous loading acting upon it might be difficult to establish properly the continuity conditions which should be imposed on the loading discontinuity. If the plate lies on such a foundation, difficulties might arise in proper establishing of natural boundary conditions on cylindrical lateral surface of the foundation.

All the aforementioned questions can be satisfactorily answered only if the variational approach to derivation of the governing equations of both the plate and the foundation is applied.

The aim of the present contribution is to derive governing equations of the foundation shown in Fig.1, along with the boundary conditions, using the virtual work principle. These equations represent e.g. behaviour of the embankment lying on the native foundation. Some exemplary applications of the equations being obtained will be presented in one the forthcoming papers.

The domain occupied by the foundation is parameterized by a right-handed normal coordinate system $\{x^\alpha, x^3\}$. This system is defined by one family of

[2]It will be shown in details in Section 5, where the equations representing models shown in Fig.4 ÷ Fig.8 will be derived.

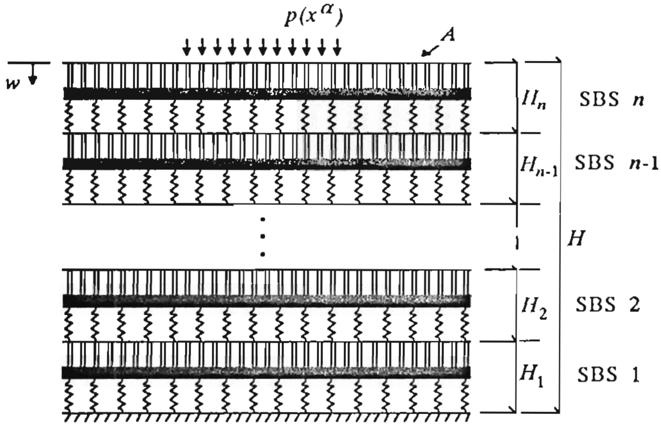


Fig. 1.

planes parallel to a reference plane and two families of cylindrical surfaces orthogonal to these planes. The coordinate x^3 is denoted by $(\bar{z}, z)$ or the capital letter Z. Partial differentation with respect to x^α variables will be denoted briefly by a commma.

A stands for a reference plane of the foundation in the initial configuration. The area occupied by the foundation in the initial configuration is denoted by $\Omega = A \times (0, H)$, where H is the overall thickness of the foundation.

The whole area $\partial\Omega$ of the foundation is

$$\partial\Omega = A \cup A_0 \cup A_s \qquad\qquad A_s = S \times (0, H)$$

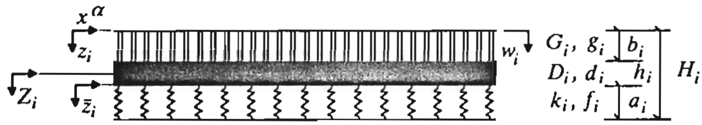
where A and A_0 stand for the planes bounding the foundation and A_s being its lateral cylindrical surface. S denotes the boundary line of the reference plane A. This boundary line is determined by the intersection of the A_s surface and the reference plane.

We assume a pure normal loading $p_3 = p(x^\alpha)$ acting upon the reference plane A. The summation convention over Greek indices running over $1, 2$ will be adopted.

2. Equations of equilibrium and boundary conditions of a unidirectional multiparameter foundation

Consider a unidirectional foundation consisting of n layers shown in Fig.1.

The ith SBS-layer is formed of three layers: the first, elastic one of thickness a_i bears only σ_{33} stresses, the second layer of thickness h_i is a bending one, while the last one of thickness b_i undergoes shearing (Fig.2)[3].



SBS layer (i)

Fig. 2.

We seek for equations expressed in terms of displacements representing the foundation together with the boundary conditions which have to be satisfied on cylindrical lateral side surface of the foundation.

We write the virtual work principle in terms of the linear elastic theory

$$\begin{aligned}&\int\limits_{\Omega}\Big[S^{\beta\alpha}\delta u_{\alpha,\beta}+S^{3\alpha}\delta(u_{\alpha,3}+u_{3,\alpha})+S^{33}\delta u_{3,3}\Big]\,dV=\\&=\int\limits_{\Omega}X\delta u_3\,dV+\int\limits_{\partial\Omega}\Big(p^{\alpha}\delta u_{\alpha}+p^3\delta u_3\Big)\,dA\end{aligned}\tag{2.1}$$

where

$S^{\alpha\beta}, S^{3\alpha}, S^{33}$	–	components of the stress tensor
u_{α}, u_3	–	components of the strain vector
X	–	third component of the body forces vector
p_{α}, p_3	–	components of the body surface loading vector.

Body forces acting in the ith SBS-layer (Fig.2) are

$$X=\begin{cases} g_i & z_i\in(0,b_i)\\ d_i & Z_i\in\Big(-\frac{h_i}{2},\frac{h_i}{2}\Big)\\ f_i & \bar{z}_i\in(0,a_i)\end{cases}$$

We assume the body forces g_i, d_i and f_i to be constant through the thickness in each layer.

[3]The replacement of layers undergoing bending and shearing, respectively does not affect the final results. Hence we can use the term "SBS-layer" or "BSS-layers system" (Bending layer – Shear layer – Spring layer).

Formulae for the displacement vector components in the ith SBS-layer can be written in the form

$$u_3(x^\beta, z) = \begin{cases} w_i(x^\beta) & z_i \in (0, b_i) \\ w_i(x^\beta) & Z_i \in \left(-\frac{h_i}{2}, \frac{h_i}{2}\right) \\ (1-\xi_i)w_i + \xi_i w_{i-1} & \bar{z}_i \in (0, a_i) \end{cases} \tag{2.2}$$

$$u_\alpha(x^\beta, z) = \begin{cases} 0 & z_i \in (0, b_i) \\ -\frac{h_i}{2}\zeta w_i(x^\beta)_{,\alpha} & Z_i \in \left(-\frac{h_i}{2}, \frac{h_i}{2}\right) \\ 0 & \bar{z}_i \in (0, a_i) \end{cases} \tag{2.3}$$

$$i = 1, 2, 3, ..., n$$

where

$$\begin{aligned} \xi_i &= \frac{\bar{Z}_i}{a_i} & \xi \in (0,1) \\ \zeta_i &= \frac{2Z_i}{h_i} & \zeta \in (-1,1) \end{aligned} \tag{2.4}$$

It can be easily seen from the above equations that the strains u_3 are represented by continuous functions, while in formulae for u_α discontinuos functions appear. $w_0(x^\alpha)$ stands for a known function representing displacements of the native foundation.

Substituting Eqs (2.2) and (2.3) into the virtual work principle (2.1) yields

$$\begin{aligned} &\sum_{i=1}^{n}\Big\{\int_A \Big(-M_i^{\beta\alpha}{}_{,\beta\alpha} + N_{i+1} - N_i - Q_i^\alpha{}_{,\alpha} - q_i\Big)\delta w_i dA + \\ &+ \int_S \Big[(M_{i_n} - \widehat{M}_{i_n})\delta\frac{\partial w_i}{\partial n} - (V_{i_n} - \widehat{V}_{i_n})\delta w_i\Big] dS\Big\} + \sum_{i=1}^{n}\sum_{k=1}^{m}(R_{ik} - \widehat{R}_{ik})\delta w_{ik} = 0 \end{aligned} \tag{2.5}$$

with the following notation

$$M_{i_{\alpha\beta}}(x^\gamma) = \int_{-\frac{h_i}{2}}^{\frac{h_i}{2}} Z_i S_{\alpha\beta}\, dZ_i \qquad N_i(x^\alpha) = \frac{1}{a_i}\int_0^{a_i} S_{33}\, d\bar{z}_i$$

$$Q_{i_\alpha}(x^\beta) = \int_0^{b_i} S_{\alpha 3}\, dz_i \qquad R_{ik} = M_{i_{ns}}(s_k + 0) - M_{i_{ns}}(s_k - 0)$$

$$M_{i_n} = M_i^{\beta\alpha} n_\beta n_\alpha \qquad\qquad Q_{i_n} = Q_i^\alpha n_\alpha \tag{2.6}$$

$$V_{i_n} = Q_{i_n} + M_i^{\beta\alpha}{}_{,\beta} n_\alpha + \frac{\partial M_{i_{ns}}}{\partial s}$$

$$\begin{aligned} q_i(x^\alpha) &= \int\limits_0^{a_{i+1}} \xi_{i+1} f_{i+1}\, d\bar{z}_{i+1} + \int\limits_0^{a_i} (1-\xi_i) f_i\, d\bar{z}_i + \int\limits_{-\frac{h_i}{2}}^{\frac{h_i}{2}} d_i\, dZ_i + \\ &+ \int\limits_0^{b_i} g_i\, dz_i \qquad\qquad i = 1,2,...,n-1 \\ q_n(x^\alpha) &= \int\limits_0^{a_n} (1-\xi_n) f_n\, d\bar{z}_n + \int\limits_{-\frac{h_n}{2}}^{\frac{h_n}{2}} d_n\, dZ_n + \int\limits_0^{b_n} g_n\, dz_n \end{aligned} \tag{2.7}$$

$$\begin{aligned} M_{i_{ns}} &= M_i^{\beta\alpha} n_\beta s_\alpha = \epsilon^\gamma{}_\alpha M_i^{\beta\alpha} n_\beta n_\gamma & N_{n+1} &= -p \\ \widehat{V}_{i_n} &= \widehat{Q}_{i_n} + \frac{\partial \widehat{M}_{i_s}}{\partial s} & \widehat{M}_{i_n} &= \widehat{M}_i^\alpha n_\alpha \\ \widehat{M}_{i_\alpha} &= \int\limits_{-\frac{h_i}{2}}^{\frac{h_i}{2}} Z_i \widehat{p}_\alpha\, dZ_i & \widehat{M}_{i_s} &= \widehat{M}_i^\alpha s_\alpha \\ \widehat{Q}_{i_n} &= \int\limits_0^{b_i} \widehat{p}_3\, dz_i + \int\limits_{-\frac{h_i}{2}}^{\frac{h_i}{2}} \widehat{p}_3\, dZ_i \end{aligned} \tag{2.8}$$

where

n_α	–	components of the unit vector $\boldsymbol{n} = [\cos\varphi, \sin\varphi]$ normal to the curve S (Fig.3)
$\epsilon^\beta{}_\alpha$	–	Ricci permutation symbol
s_α	–	components of the unit vector tangent to the curve S (see Fig.3), $s_\alpha = \epsilon^\beta{}_\alpha n_\beta$
$\widehat{R}_{ik}$	–	value of the $\widehat{M}_{i_s}$ difference at the discontinuity s_k on the plate edge of the ith foundation layer
m	–	number of discontinuities
$\widehat{p}_3, \widehat{p}_\alpha$	–	loading acting on the foundation lateral surface.

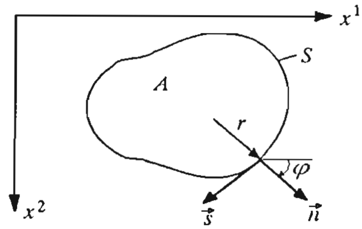


Fig. 3.

The equations of foundation equilibrium follow from Eq (2.4)

$$
\begin{aligned}
&-M_i^{\beta\alpha}{}_{,\beta\alpha} + N_{i+1} - N_i - Q_i^{\alpha}{}_{,\alpha} = q_i \qquad && i = 1,2,...,n-1 \\
&-M_n^{\beta\alpha}{}_{,\beta\alpha} - N_n - Q_{n,\alpha}^{\alpha} = q_n + p &&
\end{aligned}
\tag{2.9}
$$

together with 2^{2n} admissible combinations of $2n$ boundary conditions stipulated on the cylindrical lateral surface of the foundation.

Most important of them are the natural boundary conditions

$$
M_{i_n}\Big|_S = \widehat{M}_{i_n} \qquad V_{i_n}\Big|_S = \widehat{V}_{i_n} \qquad i = 1,2,3,...,n \tag{2.10}
$$

and the rigid boundary conditions

$$
w_i\Big|_S = \widehat{w}_i \qquad \frac{\partial w_i}{\partial n}\Big|_S = \widehat{\varphi}_i \qquad i = 1,2,3,...,n \tag{2.11}
$$

The terms supplied with circumflexes appearing in Eqs (2.10) $\div$ (2.12) stand for known functions.

From the formula (2.5) it also follows that on the foundation boundary, at the twisting moment discontinuities s_k in the ith bending layer, the following condition has to be satisfied

$$
R_{ik} = \widehat{R}_{ik} \qquad \text{for} \quad i = 1,2,3,..,n \qquad k = 1,2,...,m \tag{2.12}
$$

From Eq $(2.6)_4$ it can be seen that the quantities R_{ik}, called corner forces, might appear on the plate layer edge at the points where the twisting moment jumps.

3. Internal forces and stresses

The stresses appearing in Eqs (2.6) can be determined by applying the following governing equations

$$S_{\alpha\beta}(x^\gamma,z)=\begin{cases} 0 & z_i\in(0,b_i)\\ -\dfrac{Z_iE_i}{1-\nu_i^2}\Big[(1-\nu_i)w_{i,\alpha\beta}+\nu_i\nabla^2 w_i\delta_{\alpha\beta}\Big] & Z_i\in\Big(-\frac{h_i}{2},\frac{h_i}{2}\Big)\\ 0 & \bar{z}_i\in(0,a_i)\end{cases} \tag{3.1}$$

$$S_{\alpha 3}(x^\gamma,z)=G_iw_{i,\alpha} \qquad \text{for} \qquad z_i\in(0,b_i) \tag{3.2}$$

$$S_{33}(x^\gamma,z)=-k_iv_i \qquad \text{for} \qquad \bar{z}_i\in(0,a_i) \tag{3.3}$$

where v_i stands for the spring layer deflection in the ith SBS-layer, which can be obtained from

$$v_i(x^\alpha)=w_i(x^\alpha)-w_{i-1}(x^\alpha) \tag{3.4}$$

$$k_i=\frac{\widetilde{E}_i}{a_i} \tag{3.5}$$

where

$\widetilde{E}_i$ – Young modulus value for the spring layer
E_i – Young modulus value for the bending layer
ν_i – bending layer Poisson ratio
G_i – Kirchhoff modulus value for the layer sensitive to shear
$\delta_{\alpha\beta}$ – Kronecker delta.

The values of $S_{\alpha\beta}$ stress in the bending layer (second one) have been calculated from the governing equations of the isotropic Hookean body under the plane stress-state assumption.

The values of S_{33} stress in the spring layer have been calculated on the assumption that the stress-state is uniaxial.

Substituting for the stresses from Eqs (3.1) ÷ (3.3) into formulae for the internal forces (2.6) yields

$$M_{i_{\alpha\beta}}(x^\gamma)=\int\limits_{-\frac{h_i}{2}}^{\frac{h_i}{2}} Z_iS_{\alpha\beta}\,dZ_i=-D_i\Big[(1-\nu_i)w_{i,\alpha\beta}+\nu_i\nabla^2w_i\delta_{\alpha\beta}\Big] \tag{3.6}$$

$$Q_{i_\alpha}(x^\beta)=\int\limits_0^{b_i} S_{\alpha 3}\,dz_i=\widetilde{G}_iw_{i,\alpha} \qquad \widetilde{G}_i=G_ib_i \tag{3.7}$$

$$N_i(x^\alpha) = \frac{1}{a_i}\int\limits_0^{a_i} S_{33}\,d\bar{z}_i = -k_i v_i \tag{3.8}$$

where D_i represents the flexural rigidity of an isotropic plate given in the following form

$$D_i = \frac{E_i h_i^3}{12(1-\nu_i^2)} \tag{3.9}$$

The stresses S_{33} in the ith SBS-layer, together with the stresses $S_{\alpha 3}$ in the bending layer can be determined by imposing the local equilibrium conditions on a 3D-body[4]

$$S^{\beta\alpha}{}_{,\beta} + S^{3\alpha}{}_{,3} = 0 \tag{3.10}$$

$$S^{\beta 3}{}_{,\beta} + S^{33}{}_{,3} + X = 0 \tag{3.11}$$

Substituting Eqs (3.1) into Eq (3.10) and imposing the compatibility conditions of tangent stress on the shearing and bending layers interface, after integration yields

$$S_{\alpha 3}(x^\gamma, z) = \begin{cases} G_i w_{i,\alpha} & z_i \in (0, b_i) \\ (1-\zeta_i)\Big[\dfrac{G_i w_{i,\alpha}}{2} - \dfrac{E_i h_i^2}{8(1-\nu_i^2)}(1+\zeta_i)\nabla^2 w_{i,\alpha}\Big] & \zeta_i \in (-1,1) \\ 0 & \bar{z}_i \in (0, a_i) \end{cases} \tag{3.12}$$

$$i = 1, 2, ..., n$$

It should be noticed that in the present model the shear stress $S_{\alpha 3}$ reveals a discontinuity on the shearing and spring layers interface.

Substituting Eq (3.12) into the equilibrium equation (3.11) we have

$$S_{33}(x^\gamma, z) = \begin{cases} S_{33}(x^\gamma, a_{i+1}) - (g_i + G_i\nabla^2 w_i) z_i & z_i \in (0, b_i) \\ S_{33}(x^\gamma, b_i) + \frac{1}{4}(1+\zeta_i)\Big[D(2+\zeta_i-\zeta_i^2)\nabla^4 w_i + & \\ \quad -\frac{h_i}{2}(3-\zeta)G_i\nabla^2 w_i - \frac{1}{8}h_i d_i\Big] & Z_i \in \Big(-\frac{h_i}{2}, \frac{h_i}{2}\Big) \\ S_{33}\Big(x^\gamma, \frac{h_i}{2}\Big) - \bar{z}_i f_i & \bar{z}_i \in (0, a_i) \end{cases} \tag{3.13}$$

where

$$i = n, n-1, ..., 1 \qquad\qquad S_{33}(x^\alpha, a_{n+1}) = -p(x^\alpha) \tag{3.14}$$

[4]This way of S_{33} determination demands some comment. It is impossible to determine them from the governing equations of bending and shearing layers, respectively, due to the obvious reasons. In the spring layer theses values can be calculated from the governing equation (3.3), more accurate values, however should be solved from the equilibrium condition (3.11).

4. Equations in terms of displacements

The following differential equations system for the multi-layered foundation can be arrived at upon substituting the forces from Eqs (3.6) ÷ (3.8) into the equilibrium equations (2.9)

$$\mathsf{R}_n \mathsf{W} = \mathsf{Q} \tag{4.1}$$

where

$$\mathsf{R}_n = \begin{bmatrix} \mathcal{K}_1 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & \mathcal{K}_2 & -k_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_3 & \mathcal{K}_3 & -k_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-1} & \mathcal{K}_{n-1} & -k_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_n & \mathcal{H}_n \end{bmatrix} \tag{4.2}$$

$\dim \mathsf{R}_n = (n \times n)$

$$\begin{aligned} &\mathsf{W}^\top = \Big[w_1, w_2, w_3, ..., w_n\Big] && \dim \mathsf{W} = (n \times 1) \\ &\mathsf{Q}^\top \Big[q_1, q_2, q_3, ..., \widetilde{q}_n\Big] && \dim \mathsf{Q} = (n \times 1) \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\mathcal{K}_i = \mathcal{H}_i + k_{i+1} && \mathcal{H}_i = k_i - \widetilde{G}_i \nabla^2 + D_i \nabla^4 \\ &\widetilde{q}_n = q_n + p \end{aligned} \tag{4.4}$$

In the author's opinion the differential equation of the foundation deflection $w(x^\alpha) = w_n(x^\alpha)$ is most important. Employing formally the Cramer's rule yields

$$\det \mathsf{R}_n(w) = \det \mathsf{C}_n \tag{4.5}$$

where

$$\mathsf{C}_n = \begin{bmatrix} \mathcal{K}_1 & -k_2 & 0 & 0 & \cdots & 0 & 0 & q_1 \\ -k_2 & \mathcal{K}_2 & -k_3 & 0 & \cdots & 0 & 0 & q_2 \\ 0 & -k_3 & \mathcal{K}_3 & -k_4 & \cdots & 0 & 0 & q_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-1} & \mathcal{K}_{n-1} & q_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_n & \widetilde{q}_n \end{bmatrix} \tag{4.6}$$

$\dim \mathsf{C}_n = (n \times n)$.

Taking into account the foundation model shown in Fig.1 it can be seen that the above equation describes the displacements of the reference plane.

Applying Eqs (4.1) we can write the formula for each w_i in terms of one function $w_n = w$ only. Neglecting the body forces in Eq (4.1) we arrive at Eq (2.16) in Jemielita (1992)[5].

5. Example

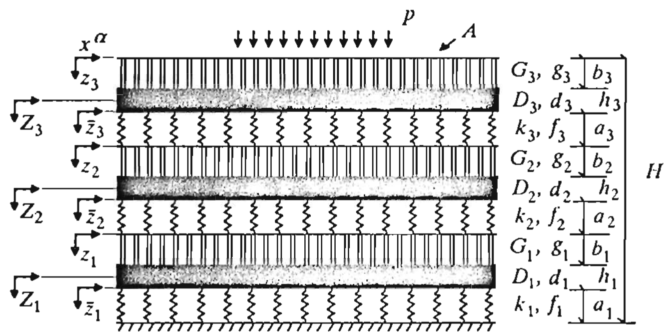


Fig. 4.

Consider the foundation consisting of three SBS-layers (Fig.4). For this type of foundation Eq (2.24) can be rewritten as

$$\det \mathsf{R}_3(w) = \det \mathsf{C}_3 \tag{5.1}$$

where

$$\det \mathsf{R}_3 w = \det \begin{bmatrix} \mathcal{K}_1 & -k_2 & 0 \\ -k_2 & \mathcal{K}_2 & -k_3 \\ 0 & -k_3 & \mathcal{H}_3 \end{bmatrix} w =$$

$$= \Big\{ k_1 k_2 k_3 - \Big[k_2 k_3 \widetilde{G}_1 + k_3(k_1 + k_2)\widetilde{G}_2 + (k_1 k_3 + k_1 k_2 + k_2 k_3)\widetilde{G}_3 \Big] \nabla^2 +$$

$$+ \Big[(k_2 + k_3)\widetilde{G}_1 \widetilde{G}_3 + (k_1 + k_2)\widetilde{G}_2 \widetilde{G}_3 + k_3 \widetilde{G}_1 \widetilde{G}_2 + k_2 k_3 D_1 +$$

[5]It can be easily shown that for $q_i = 0$, $i = 1, 2, ..., n-1$ and $\widetilde{q}_n = p$ the operator $\mathcal{L}_n = \det \mathsf{C}_n$. In Jemielita (1992) the printer's error appeared in Eq (2.2). In the formula for B_{in} matrix only the operator $\mathcal{K}$ appearing in the ith row should be supplied with the mark $\widetilde{}$ over it.

$$+k_3(k_1+k_2)D_2+(k_1k_2+k_1k_3+k_2k_3)D_3\Big]\nabla^4+ \tag{5.2}$$
$$-\Big[\widetilde{G}_1\widetilde{G}_2\widetilde{G}_3+\Big(k_3D_2+(k_2+k_3)D_3\Big)\widetilde{G}_1+\Big(k_3D_1+(k_1+k_2)D_3\Big)\widetilde{G}_2+$$
$$+\Big((k_2+k_3)D_1+(k_1+k_2)D_2\Big)\widetilde{G}_3\Big]\nabla^6+\Big[\widetilde{G}_2\widetilde{G}_3D_1+\widetilde{G}_1\widetilde{G}_3D_2+$$
$$+\widetilde{G}_1\widetilde{G}_2D_3+k_3D_1D_2+(k_1+k_2)D_2D_3+(k_2+k_3)D_1D_3\Big]\nabla^8+$$
$$-(\widetilde{G}_1D_2D_3+\widetilde{G}_2D_1D_3+\widetilde{G}_3D_1D_2)\nabla^{10}+D_1D_2D_3\nabla^{12}\Big\}\,w$$

$$det\mathsf{C}_3=\det\begin{bmatrix}\mathcal{K}_1 & -k_2 & q_1\\ -k_2 & \mathcal{K}_2 & q_2\\ 0 & -k_3 & \widetilde{q}_3\end{bmatrix}=k_2k_3q_1+$$
$$+k_3(k_1+k_2-\widetilde{G}_1\nabla^2+D_1\nabla^4)q_2+\Big\{k_1k_2+k_1k_3+k_2k_3+$$
$$-\Big[(k_2+k_3)\widetilde{G}_1+(k_1+k_2)\widetilde{G}_2\Big]\nabla^2+\Big[\widetilde{G}_1\widetilde{G}_2+(k_2+k_3)D_1+ \tag{5.3}$$
$$+(k_1+k_2)D_2\Big]\nabla^4-(\widetilde{G}_1D_2+\widetilde{G}_2D_1)\nabla^6+D_1D_2\nabla^8\Big\}\widetilde{q}_3$$

From Eqs (4.1) it follows

$$w_1=\frac{1}{k_2k_3}\Big[(\mathcal{K}_2\mathcal{H}_3-k_3^2)w-\mathcal{K}_2\widetilde{q}_3-k_3q_2\Big]=\frac{1}{k_2k_3}\Big\{\Big[k_2k_3+$$
$$-\Big((k_2+k_3)\widetilde{G}_3+k_3\widetilde{G}_2\Big)\nabla^2+\Big((k_2+k_3)D_3+\widetilde{G}_2\widetilde{G}_3+k_3D_2\Big)\nabla^4+ \tag{5.4}$$
$$-\Big(\widetilde{G}_2D_3+\widetilde{G}_3D_2\Big)\nabla^6+D_2D_3\nabla^8\Big]w-(k_2+k_3-\widetilde{G}_2\nabla^2+D_2\nabla^4)\widetilde{q}_3-k_3q_2\Big\}$$

$$w_2=\frac{1}{k_3}\Big[\mathcal{H}_3w-q_3\Big]=\frac{1}{k_3}\Big[(k_3-\widetilde{G}_3\nabla^2+D_3\nabla^4)w-\widetilde{q}_3\Big] \tag{5.5}$$

Taking the above relations into account we can write the formulae for displacements, stresses and internal forces, respectively, in terms of the deflection $w(x^\alpha)$.

Upon neglecting body forces ($q_1=q_2=0$, $q_3=p$), we arrive at the equation given by Jemielita (1992) (Eqs (4.2) and (4.3))[6].

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[6]In Eq (4.3) given by Jemielita (1992) the printer's error appears. The coefficient of ∇^6 operator is $\Big[(k_2+k_3)D_1+k_1D_2+k_2D_3\Big]\widetilde{G}_3$, while it should be $\Big[(k_2+k_3)D_1+(k_1+k_2)D_2\Big]\widetilde{G}_3$.

Applying the above equations we can obtain the equations for model shown in Fig.5 ÷ Fig.8. Substituting into Eqs (5.2) ÷ (5.5) for

$$\begin{aligned} D_1 = D_2 = D_3 = 0 \qquad & \widetilde{G}_3 = 0 \\ d_1 = d_2 = d_3 = 0 \qquad & g_3 = 0 \end{aligned}$$

we obtain the equation

$$\begin{aligned} \Big\{k_1 - \Big[\widetilde{G}_1 + \Big(1+\frac{k_1}{k_2}\Big)\widetilde{G}_2\Big]\nabla^2 + \frac{1}{k_2}\widetilde{G}_1\widetilde{G}_2\nabla^4\Big\}w = q_1 + \Big(1+\frac{k_1}{k_2} - \frac{\widetilde{G}_1}{k_2}\nabla^2\Big)q_2 + \\ +\Big\{1+\frac{k_1}{k_2}+\frac{k_1}{k_3} - \Big[\Big(\frac{1}{k_2}+\frac{1}{k_3}\Big)\widetilde{G}_1 + \frac{1}{k_3}\Big(1+\frac{k_1}{k_2}\Big)\widetilde{G}_2\Big]\nabla^2 + \frac{1}{k_2k_3}\widetilde{G}_1\widetilde{G}_2\nabla^4\Big\}\widetilde{q}_3 \end{aligned} \tag{5.6}$$

representing the foundation shown in Fig.5, together with the relations

$$w_1 = \Big(1 - \frac{\widetilde{G}_2}{k_2}\nabla^2\Big)w - \frac{q_2}{k_2} - \Big(\frac{1}{k_2} + \frac{1}{k_3} - \frac{\widetilde{G}_2}{k_2k_3}\nabla^2\Big)\widetilde{q}_3 \tag{5.7}$$

$$w_2 = w - \frac{\widetilde{q}_3}{k_3} \tag{5.8}$$

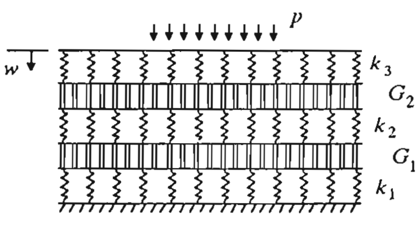

Fig. 5.

If $\widetilde{G}_1 = \widetilde{G}$, $D_1 = D$, $D_2 = D_3 = 0$, $d_2 = d_3 = 0$, $\widetilde{G}_2 = \widetilde{G}_3 = 0$, $g_2 = g_3 = 0$, $f_3 = 0$, $k_3 = \infty$ the equation of the foundation shown in Fig.6 appears

$$(k_1 - \widetilde{G}\nabla^2 + D\nabla^4)w = q_1 + \Big(1 + \frac{k_1}{k_2} - \frac{\widetilde{G}}{k_2}\nabla^2 + \frac{D}{k_2}\nabla^4\Big)(q_2 + p) \tag{5.9}$$

and

$$w_1 = w - \frac{1}{k_2}(q_2 + p) \tag{5.10}$$

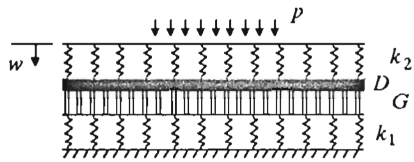

Fig. 6.

While for $D_1 = D_2 = D_3 = 0$, $d_1 = d_2 = d_3 = 0$, $G_3 = 0$, $g_3 = 0$, $f_3 = 0$, $k_3 = \infty$, the equation of the model shown in Fig.7 can be obtained

$$\Big\{k_1 - \Big[\tilde{G}_1 + \Big(1 + \frac{k_1}{k_2}\Big)\tilde{G}_2\Big]\nabla^2 + \frac{1}{k_2}\tilde{G}_1\tilde{G}_2\nabla^4\Big\}w = q_1 + \Big(1 + \frac{k_1}{k_2} - \frac{\tilde{G}_1}{k_2}\nabla^2\Big)(q_2 + p) \tag{5.11}$$

and

$$w_1 = \Big(1 - \frac{\tilde{G}_2}{k_2}\nabla^2\Big)w - \frac{1}{k_2}(q_2 + p) \tag{5.12}$$

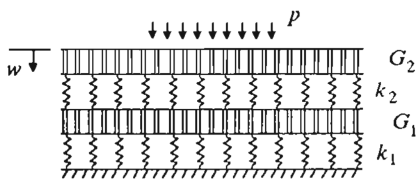

Fig. 7.

Upon putting $D_1 = D_2 = D_3 = 0$, $d_1 = d_2 = d_3 = 0$, $G_1 = G$, $\tilde{G}_2 = \tilde{G}_3 = 0$, $g_2 = g_3 = 0$, $f_3 = 0$, $k_3 = \infty$ one can obtain the equation representing the model presented in Fig.8

$$(k_1 - \tilde{G}\nabla^2)w = q_1 + \Big(1 + \frac{k_1}{k_2} - \frac{\tilde{G}}{k_2}\nabla^2\Big)(q_2 + p) \tag{5.13}$$

and

$$w_1 = w - \frac{1}{k_2}(q_2 + p) \tag{5.14}$$

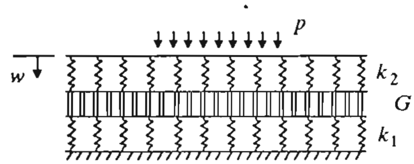

Fig. 8.

The equations given by Kerr (1984) can be arrived at upon neglecting the body forces in Eqs (5.6), (5.9), (5.11) and (5.13), respectively.

References

1. BHARATHA S., LEVINSON M., 1980, A Theory of Elastic Foundations, *Arch.Rat.Mech.Anal.*, **73**, 3, 249-266
2. HENRY F.D.C., 1986, *The Desingn and Construction of Engineering Foundations*, London, New York: Chapman and Hall
3. JEMIELITA G., 1992, Generalization of the Kerr Foundation Model, *Journal of Theoretical and Applied Mechanics*, 30, 4, 843-853
4. JEMIELITA G., SZCZEŚNIAK W., 1993, Sposoby modelowania podłoża, *Prace Naukowe Politechniki Warszawskiej, Budownictwo*, 120, 5-49
5. KERR A.D., 1984, On the Formal Development of Elastic Foundation Models, *Ing.-Archiv.*, **54**, 6, 455-464
6. SELVADURAI A.P.S., 1979, Elastic Analysis of Soil-Foundation Interaction, *Developments in Geotechnical Engineering*, **17**, Amsterdam, Oxford, New York: Elsevier

Równania i warunki brzegowe uogólnionego podłoża sprężystego

Streszczenie

W pracy wyprowadzono równania różniczkowe i dopuszczalne warunki brzegowe modelu uogólnionego podłoża sprężystego składającego się z układu n warstw SBS (rys.1). Każda warstwa SBS (Shear leyer – Bending layer – Spring layer) składa się co najwyżej z trzech warstw: warstwy czułej na ścinanie, warstwy przenoszącej zginanie oraz warstwy sprężyn (rys.2). Wyznaczono przemieszczenia i naprężenia w takim podłożu. Proponowany model podłoża ma prostą interpretację mechaniczną i jest możliwy do odwzorowania w naturze w postaci np. wielowarstwowych nasypów, dróg, pasów startowych, itp. Przyjmując odpowiednie wartości stałych charakteryzujących poszczególne warstwy SBS możemy uzyskać praktycznie dowolny model podłoża.

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