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FREE VIBRATIONS OF LAYERED BEAM WITH NON-RECTANGULAR CROSS-SECTION COMPOSED OF VISCOELASTIC LAYERS

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The paper deals with two methods of calculating of the eigenfrequencies and the logarithmic decrement for two-layer T-beam composed of viscoelastic stiffness-comparable layers. One of the methods is developed within the linear theory of (visco)elasticity by assuming continuity conditions of forces (instead of stresses) between adjoining layers. Second method is based on Kirchhoff hypothesis and Rayleigh method. A comparison of the methods has been presented and simple, useful relationships for calculation of the logarithmic decrement have been established.

1. Introduction

Layered beams of non-rectangular cross-section composed of stiffness-comparable layers have been applied both in civil and machine engineering and because of this they have been investigated in various aspects. In this short review a few papers are mentioned in order to introduce a reader to the problems considered by researchers. It is noteworth that all the papers discussed here have been devoted to static problems. The author has noticed that dynamic problems of the structures had not in generall been investigated. So, Goodman and Popov (1968) developed a theory enabling prediction of three-layer wood beam displacements caused by a static load, assuming interlayer slip or mechanical connections of layers by means of nails or complete connection between layers by means of glue. Although all considerations of the authors have been restricted to the beam of rectangular cross-section the theory presented there can be extended by taking into account non-rectangular cross-sections. The equation of equilibrium (16) given by the authors is the same as the equation (1a) presented by Itani and Brito (1978) for two-layer T-beam. Problems considered in the latter paper are similar to those found in Goodman and Popov's work. Ansourian and Roederick (1978) have given a theory enabling us calculation of static deflections of two-layer structure composed of concrete plate and steel beam assuming both an interlayer slip and residual stresses within the steel, non-rectangular beam. Troitsky and Zieliński (1989) have analysed a static behaviour of the same structure, however assuming additionally that it is prestressed by means of different tendons. Polensek and Kazic (1991) have introduced a procedure for analyzing I-beams and geometrically similar structures under bending and compressing static loads. Mori et al. (1971) have presented FEM procedure and necessary FORTRAN programs for evaluating stress concentrations of metal-FRP bonded joint where the metal member is of T-cross-section.

In this paper two methods of calculating both the eigenfrequencies and the logarithmic decrement of two-layer T-beam consisting of stiffness-comparable viscoelastic layers have been developed. One of the method has been derived within the linear theory of (visco)elasticity by applying of forces (instead of stresses) continuity conditions between layers. Formulation of the eigenvalue problem within the first method has been developed assuming both isotropic and fibrous layers. The second method is simplified since the Kirchhoff hypothesis of flat cross-sections and the Raylaigh method have been applied. The simplified approach enabled to derive a simple formula for calculating the logarithmic decrement of the structure. Both methods have been compared with respect to their accuracy and the results of comparison have been presented and discussed. According to the author's knowledge the first method is a new one regarding formulation of the problem, however the second one is only an extension of the Baumgarten and Pearce's (1971) approach given for two-layer beam of rectangular cross-section.

2. A new formulation and solution to the eigenvalue problem in the case of isotropic layers

Below we consider an eigenvalue problem of layered simply supported beam of non-rectangular cross-section as for instance shown in Figure 1. The beam is composed of any number of viscoelastic, either isotropic or anisotropic (fibrous), layers and vibrates freely.

In order to formulate the boundary value problem the following kinematic assumptions are applied

$$\begin{aligned} u_{x_j} &= -g_j(z)\frac{dW(x)}{dx}\exp(\mathrm{i}\omega_m t) \qquad\qquad u_{y_j} \neq 0 \\ u_{z_j} &= f_j(z)W(x)\exp(\mathrm{i}\omega_m t) \end{aligned} \tag{2.1}$$

where $i^2 = -1$, $j = 1,2,3,\ldots$ denotes a number of the layer, variable z is the coordinate overlaping the beam deflection, symbol t and ω_m stand for time and

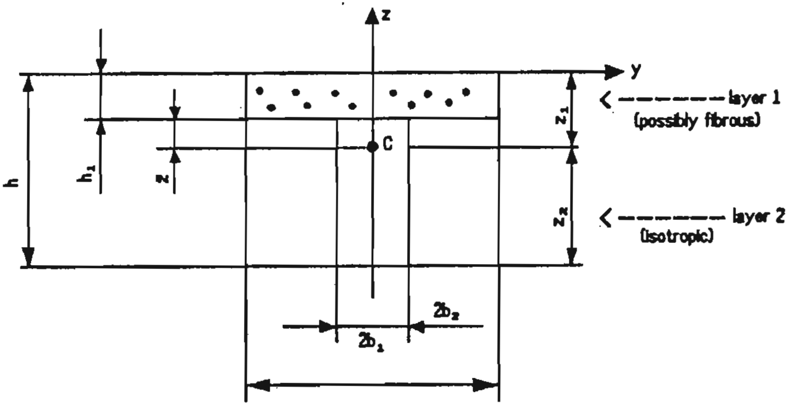

Fig. 1. An exemplary non-rectangular cross-section of the layered beam considered in the paper

for eigenfrequency of mth mode of vibration, respectively. Functions $g_j(z)$, $f_j(z)$ are unknown, however

$$W(x) = W_m \sin\Big(m\pi\frac{x}{L}\Big) \qquad\qquad m = 1,2,3,.... \tag{2.2}$$

where L is the length of beam.

The assumptions (2.1) are not exact in respect of requirements of the set of equations of motion of the theory of elasticity however, it is shown in the paper, that they are exact enough considering correctness of computational results thus practical accuracy of the formulation of problem considered here. Within any layer of the beam occurs plane stress so upon the basis of Hooke's law we obtain

$$(\sigma_{yy})_j \equiv (\sigma_{22})_j = (\sigma_{yz})_j \equiv (\sigma_{23})_j = (\sigma_{xy})_j \equiv (\sigma_{12})_j = 0 \tag{2.3}$$

$$(\varepsilon_{yy})_j \equiv (\varepsilon_{22})_j = -\frac{\nu_j}{1-\nu_j}[(\varepsilon_{xx})_j + (\varepsilon_{zz})_j] \tag{2.4}$$

On the ground of Eq (2.4) one can notice that within the vibrating beam the strains $(\varepsilon_{yy})_j$ are not equal to zero however the assumptions (2.1) satisfy Hooke's law provided that $(\varepsilon_{yy})_j$ does not depend on the space variable y. By taking into account the definition of infinitesimal strain one can write in this case

$$u_{y_j} = C(x,z,t)y + D(x,z,t) \tag{2.5}$$

where $C(x,z,t)$ may be easily evaluated, however $D(x,z,t)$ will be equal to zero when the beam vibrates transversely without twisting and xz plane of the Cartesian coordinate system divides the beam into two parts of equal width.

For an isotropic material the constitutive equation can be written in the well known form

$$(\sigma_{kl})_j = 2\mu_j(\varepsilon_{kl})_j + \delta_{kl}\lambda_j(\varepsilon_{rr})_j \qquad k,l,r = 1,2,3 \tag{2.6}$$

however equations of motion of the jth layer being in plane stress are, within the linear theory of elasticity, as follows

$$\begin{aligned}
&\mu_j\nabla^2 u_{xj} + \mu_j\kappa_j\Big(\frac{\partial^2 u_{xj}}{\partial x^2} + \frac{\partial^2 u_{zj}}{\partial z\partial x}\Big) - \rho_j\frac{\partial^2 u_{xj}}{\partial t^2} = 0\\
&\mu_j\nabla^2 u_{zj} + \mu_j\kappa_j\Big(\frac{\partial^2 u_{zj}}{\partial z^2} + \frac{\partial^2 u_{xj}}{\partial z\partial x}\Big) - \rho_j\frac{\partial^2 u_{zj}}{\partial t^2} = 0\\
&\frac{\partial^2 u}{\partial t^2} = 0
\end{aligned} \tag{2.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$, μ_j, λ_j are the Lame constants and ρ_j is the mass density of jth layer. The factor κ_j is defined as follows

$$\kappa_j = \frac{1+\nu_j}{1-\nu_j} \tag{2.8}$$

where ν_j denotes the Poisson ratio. It is obvious that during sinusoidal vibration of the beam, the third equation of motion is not satisfied. This equation is not taken into account in further considerations because it has been assumed that displacements u_{yj} and accelerations $\partial^2 u_{yj}/\partial t^2$ of the particles within transversely vibrating beam are rather small in comparison with displacements and accelerations in directions x and z, respectively. By using the expressions (2.1) and (2.6) one can transform Eq (2.5) to the following form

$$\begin{aligned}
&-\mu_j\frac{d^2 g_j}{dz^2} + [(\lambda'_j + 2\mu_j)\alpha_m^2 - \rho_j\omega_m^2]g_j + (\lambda'_j + \mu_j)\frac{df_j}{dz} = 0\\
&(\lambda'_j + 2\mu_j)\frac{d^2 f_j}{dz^2} - (\mu_j\alpha_m^2 - \rho_j\omega_m^2)f_j + \alpha_m^2(\lambda'_j + \mu_j)\frac{dg_j}{dz} = 0
\end{aligned} \tag{2.9}$$

$$\alpha_m = m\frac{\pi}{L} \qquad \lambda'_j = 2\mu_j\frac{\nu_j}{1-\nu_j} = \lambda_j\frac{1-2\nu_j}{1-\nu_j} \tag{2.10}$$

After solving the set of equations of motion one obtains functions $f_j(z)$, $g_j(z)$. A form of the functions in the case of elastic layer depends on quantitative relationships between geometrical and material parameters appearing in Eqs (2.9). The problem mentioned has been discussed by Levinson (1985) for elastic, isotropic plate thus it is not discussed here. In the case of viscoelastic layer both the parameters μ_j, λ'_j, ν_j and the functions $f_j(z)$, $g_j(z)$ are complex. Taking into account the correspondence principle one can write the functions for isotropic layer in the following form

$$
\begin{aligned}
f_j(z) &= X_{1_j}\cosh(z\beta_{1j}) + X_{2_j}\sinh(z\beta_{1j}) + X_{3_j}\cosh(z\beta_{2j}) + X_{4_j}\sinh(z\beta_{2j}) \\
g_j(z) &= X'_{1_j}\cosh(z\beta_{1j}) + X'_{2_j}\sinh(z\beta_{1j}) + X'_{3_j}\cosh(z\beta_{2j}) + X'_{4_j}\sinh(z\beta_{2j})
\end{aligned}
\tag{2.11}
$$

where

$$
\beta_{1j}^2 = \alpha_m^2 - \frac{\rho_j\omega_m^2}{\mu_j} \qquad\qquad \beta_{2j}^2 = \alpha_m^2 - \frac{\rho_j\omega_m^2}{\lambda'_j + 2\mu_j}
\tag{2.12}
$$

The vector X'_j is dependent on vector X_j thus there are only five unknown values in Eqs (2.11) i.e., the vector X_j and the natural frequency ω_m. For any layer denoted by subscript $n \neq j$ we have another unknown vector X_n. Thus for mth vibration mode of a beam consisting of p layers one obtains $4p+1$ unknown parameters while one of them is the eigenfrequency ω_m.

After substitution of the functions (2.11) into expressions (2.1) one obtains the displacement field within the jth layer and by using the displacement field functions and constitutive equations one can derive the stress field.

Let us introduce the homogeneous stress boundary conditions on the free surfaces of beam (2.13), the continuity conditions of displacements (2.14) and the continuity conditions of forces (instead of stresses) (2.15), (2.16) between adjoining layers

$$
(\sigma_{zz}(x,0))_1 = (\sigma_{zz}(x,h))_p = 0 \qquad\qquad (\sigma_{zx}(x,0))_1 = (\sigma_{zx}(x,h))_p = 0 \tag{2.13}
$$

$$
(u_x(x,h_j))_j = (u_x(x,h_j))_{j+1} \qquad\qquad (u_z(x,h_j))_j = (u_z(x,h_j))_{j+1} \tag{2.14}
$$

$$
\int\limits_{-b_j}^{+b_j} (\sigma_{zz}(x,h_j))_j\,dz = \int\limits_{-b_{j+1}}^{+b_{j+1}} (\sigma_{zz}(x,h_j))_{j+1}\,dz \tag{2.15}
$$

$$
\int\limits_{-b_j}^{+b_j} (\sigma_{zx}(x,h_j))_j\,dz = \int\limits_{-b_{j+1}}^{+b_{j+1}} (\sigma_{zx}(x,h_j))_{j+1}\,dz \tag{2.16}
$$

where $h = \sum_1^p h_j$, while h_j, b_j denote the thickness and the half of jth layer width, respectively. It is noted that displacement u_y does not appear in the continuity conditions (2.14) and the equations of continuity of forces (instead of stresses) (2.15), (2.16) are not exact in respect of requirements stated in the theory of elasticity. However the simplifications introduced here have enabled the author to solve correctly the problem considered in the present paper.

Taking into account the relationships (2.6) and the Eqs (2.11) ÷ (2.16) one can transform the eigenvalue problem to the form of algebraic, homogeneous, matrix equation

$$\mathbf{AX} = 0 \tag{2.17}$$

The square matrix $\mathbf{A}$ is of size $4p \times 4p$ where p denotes the number of layers of the beam. An ω_m is obtained from the equation

$$\det \mathbf{A} = 0 \tag{2.18}$$

If any layer of the beam is viscoelastic then ω_m consists of real and imaginary parts, respectively

$$\omega_m = \omega_{mR} + \mathrm{i}\omega_{mF} \qquad\qquad \mathrm{i}^2 = -1 \tag{2.19}$$

and the periodic logarithmic decrement is defined as follows

$$\delta_T = 2\pi \frac{\omega_{mF}}{\omega_{mR}} \tag{2.20}$$

After calculating δ_T one can obtain both the loss factor η and the damping capacity Ψ according to the formulas given by Karczmarzyk (1989)

$$\eta = \frac{\delta_T}{\pi} \qquad\qquad \Psi = 1 - \exp(-2\delta_T) \cong 2\delta_T \tag{2.21}$$

The Eq (2.17) can easily be obtained for a beam consisting of any number of stripes using a computer. Computation of eigenfrequencies is however somewhat more difficult. The matrix elements depend on hyperbolic and trigonometric functions of eigenfrequency i.e.

$$A_{kl} = A_{kl}\Big(\sin(\omega_m, ...), \cos(\omega_m, ...), \sinh(\omega_m, ...), \cosh(\omega_m, ...)...\Big) \tag{2.22}$$

and because of this, Eq (2.18) is the transcendental one. Therefore it cannot be transformed to the following form

$$\det\Big(\mathbf{B}_1 - \omega_m \mathbf{B}_2^2\Big) = 0 \tag{2.23}$$

where $\mathbf{B}_1$, $\mathbf{B}_2$ are given matrices.

To obtain the solution to eigenproblem (2.18) the following procedure has been proposed (cf Karczmarzyk, 1989, 1991; Karczmarzyk and Osiński, 1990) – at first a function $F(\omega_m) \equiv \det \mathbf{A}$ is derived, then an eigenfrequency of undamped (elastic) beam is estimated, finally a complex eigenfrequency of the damped (viscoelastic) system is computed from the equation $F(\omega_m) = 0$. All steps of this procedure were followed by the author by means of IBM personal computer. The third step only was taken by using standard subroutine for evaluating roots of an algebraic, nonlinear, complex equation according to the Muller method with deflation. The eigenfrequency of undamped system was useful as an approximative value of the complex solution (eigenfrequency) i.e., as one of input parameters required by the standard subroutine. It was verified that the Fortran code necessary for the calculations has to be prepared in double precision. The formulation of the eigenvalue problem and the way of solution to it given in the present section are acknowledged by the author as a new exact method of vibration analysis of the layered viscoelastic beams. In section 4 an alternative simplified method is presented.

We notice finally that formulation of the problem considered here will be of the same form when the plane strain within layered structure is assumed. In such a case instead of the parameter λ' the Lame constant λ should be placed in Eqs (2.9), (2.12) (Karczmarzyk, Osiński, 1992).

3. Formulation of the problem in the case of fibrous layers

It has been assumed in further considerations that fibres of each layer are parallel to the longitudinal axis of the beam. Such arrangement of the fibres is most desirable considering bending stiffness of beam. Let us assume additionally that material properties of any layer are isotropic within each cross-section of the layer. If the two assumptions are fulfiled we will have so-called hexagonally anisotropic layer.

The constitutive equation of fibrous, hexagonally anisotropic, viscoelastic material can be written in the form

$$\boldsymbol{\sigma}_j = \mathbf{D}_j \boldsymbol{\varepsilon}_j \tag{3.1}$$

where $\boldsymbol{\sigma}_j$ denotes a stress vector, $\boldsymbol{\varepsilon}_j$ is a strain vector and $\mathbf{D}_j$ is a stiffness matrix of jth layer. In the case when the plane stress is considered the matrices are as follows

$$\begin{gathered} \boldsymbol{\sigma}_j \equiv \{\sigma_{xx}, \sigma_{zz}, \sigma_{zx}\}_j \\ \mathbf{D}_j \equiv \begin{bmatrix} b_j & a_j & 0 \\ a_j & q_j & 0 \\ 0 & 0 & 2\mu'_j \end{bmatrix} \\ \boldsymbol{\varepsilon}_j \equiv \{\varepsilon_{xx}, \varepsilon_{zz}, \varepsilon_{zx}\}_j \end{gathered} \tag{3.2}$$

where

$$a_j = E'_j \frac{\nu'_j}{w_j} \qquad b_j = \frac{E'_j}{w_j} \frac{E'_j}{E_j} \qquad q_j = \frac{E'_j}{w_j} \tag{3.3}$$

$$w_j = \frac{E'_j}{E_j} - \nu'_j \nu'_j \tag{3.4}$$

while μ'_j denotes the complex Kirhchoff modulus of jth fibrous viscoelastic layer in $x - z$ plane, E'_j is the Young modulus for direction of fibers, E_j is the Young modulus within $y - z$ plane perpendiculr to direction of fibers (i.e., plane of isotropy), ν'_j denotes the Poisson ratio characterizing an abridgement within the plane of isotropy when direction of force fits in with direction of fibers of viscoelastic hexagonally anisotropic material (cf Ambartyumyan, 1987).

By using the formulas (3.1) ÷ (3.4) and the expressions (2.1), (2.2) one can transform the linear elasticity equations of motion

$$\sigma_{kl,k} = \rho \frac{\partial^2 u_l}{\partial t^2} \tag{3.5}$$

to the following form

$$\begin{aligned} -\mu'_j \frac{d^2 g_j}{dz^2} + (b_j \alpha_m^2 - \rho_j \omega_m^2) g_j + (a_j + \mu'_j) \frac{df_j}{dz} &= 0 \\ q_j \frac{d^2 f_j}{dz^2} - (\mu'_j \alpha_m^2 - \rho_j \omega_m^2) f_j + \alpha_m^2 (a_j + \mu'_j) \frac{dg_j}{dz} &= 0 \end{aligned} \tag{3.6}$$

where $\alpha_m = m\pi/L$ is defined as in the previous section. It can be noticed that the equations of motion (3.6) are of the same type as the equations (2.9) thus formulation of the boundary value problem in this case is of the same form as the previous one described in section 2. Therefore further considerations concerning the formulation of the aforementioned for a beam composed of anisotropic layers have not been conducted here. We notice that formulation of the problem considered here will be of the same form when the plane strain is assumed within layered structure. In such a case, however, the elements a_j, b_j, q_j of the matrix D, occuring in Eqs (3.6), are defined by expressions different from Eq (3.3), (3.4) (cf Karczmarzyk, 1989).

4. An alternative simplified method for analysis of free vibrations of viscoelastic layered beams

The method developed in sections 2,3 is accurate since it is derived within the linear theory of (visco)elasticity. The only simplification is introduced in the stress continuity conditions (2.15), (2.16). (In fact instead of the stress continuity it is considered the continuity of forces.) Thus the method reported in sections

2;3 is accurate and quite versatile i.e., it enables us to investigate the layered beams assuming both stiffness-comparable and incomparable adjoining layers and complete material (viscoelastic) characteristics of all layers. However due to the exactitude of the problem formulation one can not obtain a simple formula for evaluation of an eigenfrequency and a logarithmic decrement for layered structures considered here. In this section we present an alternative simplified method of calculating both the eigenfrequency and the logarithmic decrement of T-beam shown in Figure 1 and composed of stiffness-comparable viscoelastic layers.

Upon a basis of formulas given by Polensek and Kazic (1991) one can derive the following relationship

$$\bar{z} = \frac{h_2}{2} \frac{1 - \xi_t \xi_t \xi_b \xi_E}{1 + \xi_t \xi_b \xi_E} \tag{4.1}$$

where

$$\xi_t = \frac{h_1}{h_2} \qquad \xi_b = \frac{b_1}{b_2} \qquad \xi_E = \frac{E_1}{E_2} \tag{4.2}$$

and E_1, E_2 are the values of static Young moduli of isotropic layers. The parameter $\bar{z}$ denotes coordinate of the cross-section neutral axis (point C) in reference to the interface i.e, the surface of joint of the layeres. When the $\bar{z}$ is greater than zero the point C lies within the layer 2 however in the opposite case the point C lies within the layer 1. The relationship (4.1) is a direct extension of the one given by Baumgarten and Pearce (1971) for a two-layer beam of rectangular cross-section. Few other relationships given by the researchers have just been employed by the author to obtain simple formulas for evaluating both the eigenfrequencies and the logarithmic decrement for the two-layer T-beams.

First of all we notice that the formula (25) derived by Baumgarten and Pearce (1971) can be replaced by the following one

$$\delta_T = \pi \eta_{E1} \frac{(V_{bc})_{max}}{T_{max}} \tag{4.3}$$

where $(V_{bc})_{max}$ is the maximum value of the potential energy of the viscoelastic layer and T_{max} denotes the maximum value of the kinetic energy of the two-layer beam. The symbol η_{E1} is called material loss factor and it is defined as follows

$$\eta_{E1} = \frac{E_{12}}{E_{11}} \qquad E_{11} + \mathrm{i}E_{12} = E_1 \qquad \mathrm{i}^2 = -1 \tag{4.4}$$

while E_1 in this case is the complex Young modulus of viscoelastic material. By replacing the E_{11}, E_{12}, E_1 with ν_{11}, ν_{12}, ν_1, respectively, one obtains the complex Poisson ratio ν_1 for the viscoelastic material. Parameters E_1, ν_1 are dependent on frequency. In the case of non-slender layered beams one has to take into account both the characteristics to calculate accurately damping parameters (cf Karczmarzyk, Osiński, 1992). When the layered beam is slender it is accurate enough to include in a computational algorithm the complex Young modulus, thus the loss factor η_{E1}.

The formula (4.3) is valid for the two-layer beam when the Kirchhoff hypothesis of flat cross-sections is valid for the beam. It is well known that the hypothesis is satisfied for the first mode of vibration of slender beams (cf Huang, 1961). Thus for such beams we can extend the formula (4.3) in order to include different widths and viscoelasticity of adjoining layers. So one can calculate the logarithmic decrement according to the formula

$$\delta_T = \pi \frac{\eta_{E1} V_1 + \eta_{E2} V_2}{T_{12}} \tag{4.5}$$

where

$$V_j = E_{j1} b_j \frac{\alpha_m^4}{6} |z_j^3 - \bar{z}^3| \qquad j = 1,2 \tag{4.6}$$

$$T_{12} = (b_1 h_1 \rho_1 + b_2 h_2 \rho_2) \frac{\omega_m^2}{2} \tag{4.7}$$

and α_m is defined (for a simply supported beam) in section 2, E_{j1} denotes the real part of the complex Young modulus of jth viscoelastic layer. By applying the Rayleigh method one obtains finally

$$\delta_T = \pi \frac{\eta_{E1} V_1 + \eta_{E2} V_2}{V_1 + V_2} \tag{4.8}$$

The expression (4.8) can be further simplified under additional assumptions i.e.

$$\text{for} \qquad V_1 = V_2 \qquad \text{the} \qquad \delta_T = \pi \frac{\eta_{E1} + \eta_{E2}}{2} \tag{4.9}$$

$$\text{for} \qquad \eta_{E1} = \eta_{E2} = \eta_E \qquad \text{the} \qquad \delta_T = \pi \eta_E \tag{4.10}$$

We notice that under assumptions mentioned above the viscoelastic damping of two-layer T-beam vibrations does not depend on geometrical and material parameters of the cross-section appearing in the formulas (4.1), (4.2). By comparing the right-hand side of Eq (4.10) and the left-hand side relationship (2.21) one can say: when both layers are characterized by the same material loss factor (i.e., $\eta_{E1} = \eta_{E2} = \eta_E$) the loss factor of the two-layer T-beam is equal to the material loss factor. The conclusion also referes to the beam of rectangular cross-section.

5. Numerical results and discussion

In order to verify and compare the methods developed for calculating of both the eigenfrequencies and the logarithmic decrement the author has made calculations for three types of beams. Material and geometrical (cross-sectional) parameters of the beams have been given in Table 1 while computational results have been shown in Tables 2-6. We note that the beam 1 is homogeneous i.e., the material parameters of both layers are the same. However to verify an influence of the viscoelasticity of the thin layer (flange) on the logarithmic decrement the calculations have been made assuming for the thick layer (web) at first $\eta_{E2} = 0$ and then $\eta_{E2} = 0.1$. The beams 2,3 consist of two stiffness-comparable layers of different materials. Values of the logarithmic decrement in Table 3 and those in Table 5 with subscripts "12" have been given to prove the right-hand side relationship (4.10) and to show that δ_T is independent of the length of beam when the left-hand side Eq (4.10) is assumed. Parameter ε_2 is defined as follows

$$\varepsilon_2 = \frac{(\delta_{T_A})_2 - (\delta_{T_B})_2}{(\delta_{T_B})_2} 100 \tag{5.1}$$

Table 1. Material and geometrical (cross-sectional) parameters of the beams investigated in the paper. Symbols b_j, h_j have been shown in Fig.1, while symbols E_{j1}, ν_j, ρ_j denote the Young modulus, the Poisson ratio and the mass density, respectively – for $j = 1, 2$.

	$2b_1$	$2b_2$	h_1	h_2	E_{11}	E_{21}	ν_1	ν_2	ρ_1	ρ_2
	[mm]				[Pa]				[kg/m^3]	
beam 1	315	35	35	105	$0.3 \cdot 10^{11}$	$0.3 \cdot 10^{11}$	0.05	0.05	600	600
beam 2	105	35	35	105	$207 \cdot 10^9$	$1.6 \cdot 10^{10}$	0.25	0.30	7860	1750
beam 3	105	35	10	140	$207 \cdot 10^9$	$1.6 \cdot 10^{10}$	0.25	0.30	7860	1750

Table 2. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of beam 1. Subscript A denotes values calculated according to the method developed in section 2, while subscript B denotes values obtained according to the method developed in section 4. Values with subscript 1 are obtained for $\eta_{E1} = 0.1$, $\eta_{E2} = 0$ while those with subscripts 12 are calculated for $\eta_{E1} = \eta_{E2} = 0.1$.

L [mm]	1000	1500	2000	2500	3000	3650
ω_A [rad/s]	2349.3	1066.1	604.4	388.2	270.2	182.9
ω_B [rad/s]	2442.6	1085.6	610.7	390.8	271.4	183.3
$(\delta_{T_A})_1$	0.0757	0.0769	0.0774	0.0777	0.0778	0.0779
$(\delta_{T_B})_1$	0.0785	0.0785	0.0785	0.0785	0.0785	0.0785
$(\delta_{T_A})_{12}$	0.31288	0.31288	0.31288	0.31288	0.31288	0.31288
$(\delta_{T_B})_{12}$	0.31414	0.31414	0.31414	0.31414	0.31414	0.31414

Table 3. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of beam 2. Subscript A denotes values calculated according to the method developed in section 2, while subscript B denotes values obtained according to the method developed in section 4. Values with subscript 12 are calculated for $\eta_{E1} = \eta_{E2} = 0.1$.

L [mm]	1000	1500	2000	2500	3000	3650
ω_A [rad/s]	1000.83	456.25	259.10	166.53	115.93	78.46
ω_B [rad/s]	1049.22	466.31	262.30	167.87	116.58	78.76
$(\delta_{T_A})_{12}$	0.31338	0.31338	0.31338	0.31338	0.31338	0.31338
$(\delta_{T_B})_{12}$	0.31416	0.31416	0.31416	0.31416	0.31416	0.31416

Table 4. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of beam 3. Subscript A denotes values calculated according to the method developed in section 2, while subscript B denotes values obtained according to the method developed in section 4. Decrement values with subscript 2 are calculated for $\eta_{E2} = 0.1$, $\eta_{E1} = 0$.

L [mm]	1000	1500	2000	2500	3000	3650
ω_A [rad/s]	1492.1	693.0	396.3	255.7	178.3	120.8
ω_B [rad/s]	1621.0	720.5	405.3	259.4	180.1	121.7
$(\delta_{T_A})_2$	0.26212	0.25730	0.25534	0.25438	0.25384	0.25344
$(\delta_{T_B})_2$	0.25345	0.25345	0.25345	0.25345	0.25345	0.25345
ε_2	3.42	1.52	0.75	0.37	0.15	0.00

Table 5. Eigenfrequencies and logarithmic decrements for the 3rd mode of vibration of beam 3. Subscript A denotes values calculated according to the method developed in section 2, while subscript B denotes values obtained according to the method developed in section 4. Values with subscript 2 are obtained for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$ while those with subscripts 12 are calculated for $\eta_{E1} = \eta_{E2} = 0.1$.

L [mm]	1000	1500	2000	2500	3000	3650
ω_A [rad/s]	9155.7	4974.2	3083.2	2081.1	1492.1	1033.6
ω_B [rad/s]	14589.3	6484.1	3647.3	2334.3	1621.0	1095.0
$(\delta_{T_A})_2$	0.28813	0.27746	0.27014	0.26530	0.26212	0.25943
$(\delta_{T_B})_2$	0.25345	0.25345	0.25345	0.25345	0.25345	0.25345
$(\delta_{T_A})_{12}$	0.31338	0.31338	0.31338	0.31338	0.31338	0.31338
$(\delta_{T_B})_{12}$	0.31416	0.31416	0.31416	0.31416	0.31416	0.31416
ε_2	13.68	9.47	6.59	4.68	3.42	2.36

Table 6. Eigenfrequencies and logarithmic decrements for the 5th mode of vibration of beam 3. Subscript A denotes values calculated according to the method developed in section 2, while subscript B denotes values obtained according to the method developed in section 4. Values with subscript 2 are calculated for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$.

L [mm]	1000	1500	2000	2500	3000	3650
ω_A [rad/s]	17656.8	10590.9	7026.9	4974.2	3688.3	2633.2
ω_B [rad/s]	40525.9	18011.5	10131.5	6484.1	4502.9	3042.0
$(\delta_{T_A})_2$	0.29584	0.29052	0.28349	0.27746	0.27268	0.26804
$(\delta_{T_B})_2$	0.25345	0.25345	0.25345	0.25345	0.25345	0.25345
ε_2	16.73	14.63	11.85	9.47	7.59	5.76

We notice that the higher is mode of vibration the higher value of the parameter ε_2 occurs. This parameter is strongly dependent on the length of beam. Generally when the web of the two-layer beam is viscoelastic the values of logarithmic decrement calculated according to the method developed in section 2 are higher than those obtained according to the method given in section 4. It is because of the shear deformations omitting in the method described in section 4 while the deformations have been included in the method shown in sections 2 and 3. The difference between the methods is much more perceptible when the eigenfrequencies are compared. For each mode of the beams vibration the eigenfrequencies ω_B are different and higher than the eigenfrequencies ω_A. The difference is dependent on the mode of vibration and the slenderness of beams. Of course the values ω_B are not exact for the 1st mode of vibration and false for the higher vibration modes. The latter conclusion, based upon the results given in Tables 2 $\div$ 6, fits in well with that given by Huang (1961).

The method given in section 4 can be useful inspite of its general inaccuracy since it is far less complicated than that given in section 2 and the formulas (4.1) $\div$ (4.10) are simple and quite accurate when the 1st mode of vibration is investigated.

6. Final conclusion

Two methods for calculating of both the eigenfrequencies and the logarithmic decrement for two-layer T-beam composed of isotropic or fibrous stiffness-comparable layers have been developed in the paper. The method given in section 2 has been obtained within the linear theory of (visco)elasticity, however that one from section 4 is derived by applying the Kirchhoff hypothesis of flat cross-sections and the Rayleigh method. By comparing results obtained according to these methods one can conclude: 1) damping of the 1st mode of vibration of the two-layer

T-beam resulting from the viscoelasticity of layers is accurately predicted by both the methods, 2) both the logarithmic decrement and (especially) the eigenfrequencies of higher modes of vibrations are predicted accurately only by the method derived within the linear theory of (visco)elasticity, 3) validity of the right-hand side expression (4.10), for any mode of vibration, has been confirmed by applying both the methods.

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Drgania swobodne belek warstwowych o przekroju nieprostokątnym złożonym z warstw lepkosprężystych

Streszczenie

Praca dotyczy dwóch metod obliczania częstości własnych i logarytmicznego dekrementu tłumienia dla dwuwarstwowej belki T-owej złożonej z lepkosprężystych warstw o porównywalnej sztywności. Jedna z tych metod została opracowana w ramach liniowej teorii (lepko)sprężystości przy założeniu warunków ciągłości sił (zamiast naprężeń) między przylegającymi warstwami. Druga metoda jest oparta na założeniu Kirchhoffa i metodzie Rayleigh'a. Przedstawiono porównanie obu metod i ustalono proste i użyteczne zależności do obliczania dekrementu tłumienia.

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