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ON THE CONTROL SYNTHESIS FOR MECHANICAL SYSTEMS IN SPECIFIED MOTION

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The paper deals with the dynamic analysis and the synthesis of control of mechanical systems in specified motion. The program requirements are treated as so-called program constraints and the control reactions ensuring the realization of the constraints are determined. In general, available control forces of the system may not project explicitly into directions orthogonal to the manifolds of particular constraints. Thus, the problem at hand is more general then the classical problem of the dynamic analysis of constrained systems and the determination of constraint reactions. It is proved (and illustrated by examples) that the program constraints can be realized by control forces which have any directions respective the constraint manifolds, and in extreme, by tangent control forces. Criteria for solvability of the problem (controllability of a system in a given program motion) are formulated, and a systematic formulation of the solution of the problem is proposed. Two illustrative examples are included.

1. Introduction

The problem at hand is as follows. Given a controlable mechanical system and a prespecified motion of the system, find how to control the system in order to ensure the realization of the programmed motion. Do Sahn (1984) and Galyullin (1989) solved the problem by direct application of the inverse dynamics approach and the theory of constrained systems. The program requirements concerning the system (expressed analytically) are considered as specific constraints inposed on the system, so-called *program constraints*, and the required *control reactions* ensuring the realization of program of motion are supposed to correspond to the reactions of program constraints treated as *material* ones. The well known procedures for determination of constraint reaction forces (Lagrange multipliers) as functions of the system position, velocity, and applied forces (cf Wittenburg, 1977; Schiehlen, 1986;) are then applied to determine the required control reactions (control parameters).

The aforementioned approach may be lacking in practical applications. By using it, the determined control reactions are, in principle, orthogonal to the manifolds of particular constraints, or more general, the solution of the problem exists only if the control forces of the system are represented explicitly in the directions orthogonal to the constraint manifolds. For a given mechanical system, however, the available control forces may not satisfy the above requirement, i.e. the control forces may not be represented in the directions orthogonal to some or even all constraint manifolds. In this case, the control forces cannot regulate directly the balance of forces orthogonal to those particular constraints. Nevertheless, when the applied forces on the system depend on the system position and velocity, it may be possible to match the control by *tangent forces* so that to accommodate appropriately the changes in the system state of motion and assure the balance of the orthogonal forces without adding any control forces in these directions. To this end, a specific approach to the problem has to be undertaken and some necessary conditions for the existence of tangent realization of program constraints have to be fulfilled. In any way, the statement of the fact that the (program) constraints can be realized by tangent (control) reactions may be considered as an extension of the classical theory of constrained systems, which postulates that the constraint reactions are orthogonal to the corresponding constraint manifolds (cf Wittenburg, 1977; Schiehlen, 1986).

Parczewski and Blajer (1989) present some general remarks on how to solve the dynamics/control problem of systems in specified motion, and classify different possible ways of realization of program constraints (including tangent realization). Blajer (1990), (1991) and (1992c), Blajer and Parczewski (1989), (1990) and (1991) demonstrate then some applications of that formulation with the emphasis on tangent realization of program constraints. In this paper, the mathematical formulation for the dynamic analysis and synthesis of control of mechanical systems in specified motion is developed and modified. Due to the applied linear algebra formalism, a geometrical insight into the problems solved is gained and some simplifications and generalizations are achieved. Two simple and illustrative examples are reported to clarify the mathematical formulation introduced in this paper.

2. Problem formulation

Consider an n-degree-of-freedom controlable mechanical system subject to m ($m \leq n$) independent program constraints. Assume that there are m independet control inputs in the system, and let us limit ourselves to the linear dependence on control parameters. The governing equations of the considered problem can be written as follows

$$\mathbf{M}(\mathbf{x},t)\dot{\mathbf{v}} = \mathbf{h}^*(\mathbf{v},\mathbf{x},t) + \mathbf{B}^\top(\mathbf{v},\mathbf{x},t)\boldsymbol{\lambda} \tag{2.1}$$

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x},t)\mathbf{v} \tag{2.2}$$

$$\mathbf{C}(\mathbf{v},\mathbf{x},t)\dot{\mathbf{v}} + \mathbf{c}_0^*(\mathbf{v},\mathbf{x},t) = \mathbf{0} \tag{2.3}$$

In the dynamic equations (2.1), $\mathbf{M}$ is an $n \times n$ symmetric positive-definite inertia matrix (metric tensor matrix of the base $\mathbf{e}_v$, $\mathbf{M} = \mathbf{e}_v\mathbf{e}_v^\top$); $\mathbf{v} = [v_1, ..., v_n]^\top$ and $\mathbf{x} = [x_1, ..., x_n]^\top$ are the (contravariant) representations of quasi-velocity vector $\boldsymbol{v}$ and position vector $\boldsymbol{x}$ in the bases $\mathbf{e}_v = [e_{v1}, ..., e_{vn}]^\top$ and $\mathbf{e}_x = [e_{x1}, ..., e_{xn}]^\top$, $\boldsymbol{v} = \mathbf{v}^\top\mathbf{e}_v$ and $\boldsymbol{x} = \mathbf{x}^\top\mathbf{e}_x$, respectively; $\mathbf{h}^*$ is the (covariant) representation of forces applied on the system and gyroscopic terms, $\boldsymbol{h} = \mathbf{h}^{*\top}\mathbf{e}_v^*$; and $\mathbf{r}^* = \sum\limits_{i=1}^{m} \mathbf{r}_i^* = \sum\limits_{i=1}^{m} \mathbf{b}_i^*\lambda_i = \mathbf{B}^\top\boldsymbol{\lambda}$ is the (covariant) representation of the total of control forces, $\boldsymbol{r} = \mathbf{r}^{*\top}\mathbf{e}_v^*$, where $\mathbf{B}$ is an $m \times n$ full-rank matrix, $\mathbf{b}_i^*$ is the ith column of $\mathbf{B}^\top$, and $\boldsymbol{\lambda} = [\lambda_1, ..., \lambda_m]^\top$ contains the control parameters. The aspects of contravariant and covariant representations of vectors in a multi-dimensional space, the motivation to distinguish between them, and the meaning of superscript * are explained in more detail by Blajer (1992a,b), and are condensed in Appendix.

Due to the generality of Eq (2.2), $\mathbf{v}$ may be any combination of quasi-velocities and/or generalized velocities, and the analysis can be carried out without paying any attention to distinguishing between the cases, refer also to Blajer (1992a,b); $\mathbf{A}$ is an $n \times n$ invertible transformation matrix, $\mathbf{e}_v = \mathbf{A}^\top\mathbf{e}_x$. Finally, Eqs (2.3) denote the program constraint equations in the second-order kinematic form, where $\mathbf{C}$ is an $m \times n$ full-rank constraint matrix, and $\mathbf{c}_0^*$ is an $m \times 1$ matrix. If the program requirements concerning the system are in the form of geometric and/or first-order kinematic constraints, $\boldsymbol{f}(\mathbf{x},t) = \mathbf{0}$ and $\boldsymbol{\varphi}(\mathbf{v},\mathbf{x},t) = \mathbf{0}$, respectively, they have to be transformed to the form (2.3) by differentiating with respect to time twice or once, respectively. Then $\mathbf{C} = \mathbf{f}_x\mathbf{A}$ and $\mathbf{c}_0^* = (\mathbf{f}_x\mathbf{A})^\cdot + (\mathbf{f}_t)^\cdot$ for $\boldsymbol{f}(\mathbf{x},t) = \mathbf{0}$, and $\mathbf{C} = \boldsymbol{\varphi}_v$ and $\mathbf{c}_0^* = \boldsymbol{\varphi}_x\mathbf{A}\mathbf{v} + \boldsymbol{\varphi}_t$ for $\boldsymbol{\varphi}(\mathbf{v},\mathbf{x},t) = \mathbf{0}$; where the subscripts denote partial differentiation. Obviously, the lower-order constraint conditions must be satisfied by the initial values $\mathbf{v}_0$ and $\mathbf{x}_0$; $\boldsymbol{f}(\mathbf{x}_0,t_0) = \mathbf{0}$, $\dot{\boldsymbol{f}}(\mathbf{v}_0,\mathbf{x}_0,t_0) = \mathbf{0}$, and $\boldsymbol{\varphi}(\mathbf{v}_0,\mathbf{x}_0,t_0) = \mathbf{0}$.

Following the method of classical mechanics for elimating the reaction forces (cf Wittenburg, 1977; Schiehlen, 1986), the required values of control parameters may be determined after substituting Eq (2.1) for $\dot{\mathbf{v}}$ in Eq (2.3), i.e.

$$\boldsymbol{\lambda} = -(\mathbf{C}\mathbf{M}^{-1}\mathbf{B}^\top)^{-1}(\mathbf{C}\mathbf{M}^{-1}\mathbf{h} + \mathbf{c}_0^*) = \boldsymbol{\lambda}(\mathbf{v},\mathbf{x},t) \tag{2.4}$$

Using this, Eqs (2.1) and (2.2) can be solved for $\mathbf{v}(t)$ and $\mathbf{x}(t)$, and the control of the system in the prespecified motion, $\boldsymbol{\lambda}(t)$, can be determined from Eq (2.4). However, the above formulation is valid only if the matrix $\mathbf{C}\mathbf{M}^{-1}\mathbf{B}^\top$ is invertible, which suits the case when the control forces are explicitly represented in the directions orthogonal to all constraint manifolds. In a general case, $\mathbf{CM}^{-1}\mathbf{B}^{\mathsf{T}}$ may not be of maximal rank, and a specific approach to the problem solution has to be undertaken.

3. Method of solution

The columns of $\mathbf{B}^{\mathsf{T}}$ (full-rank control matrix) can be interpreted as (covariant) representations of independent vectors $\boldsymbol{b}_i$ $(i = 1, ..., m)$ which define the directions of control forces in the system configuration space; $\boldsymbol{b}_i = \mathbf{b}_i^{*\mathsf{T}} \mathbf{e}_v^* = \mathbf{b}_i^{*\mathsf{T}} \mathbf{M}^{-1} \mathbf{e}_v$, where $\mathbf{b}_i^*$ is the ith column of $\mathbf{B}^{\mathsf{T}}$, and $\mathbf{M}$ (the inertia matrix defined in Eq (2.1)) is the metric tensor matrix of the base $\mathbf{e}_v$. Thus, the vectors $\mathbf{e}_b = [\boldsymbol{b}_1, ..., \boldsymbol{b}_m]^{\mathsf{T}}$ span an m-dimensional subspace of the n-space, and let us call the subspace a *controlled subspace*. Then, a k-dimensional $(k = n - m)$ subspace which is complementary to the controlled subspace can be defined as spanned by independent vectors $\boldsymbol{d}_j$ $(j = 1, ..., k)$. Assuming that these vectors are represented by (contravariant) components gathered as columns in $\mathbf{D}^{\mathsf{T}}$ ($\mathbf{D}$ is a $k \times n$ full-rank matrix; $\boldsymbol{d}_j = \mathbf{d}_j^{\mathsf{T}} \mathbf{e}_v$; and $\mathbf{d}_j$ is the jth column of $\mathbf{D}^{\mathsf{T}}$), the complementary condition of the two subspaces can be written as

$$\mathbf{DB}^{\mathsf{T}} = \mathbf{0} \tag{3.1}$$

i.e. $\mathbf{D}$ is an orthogonal complement matrix to $\mathbf{B}$ in the n-space. In other words, each vector $\boldsymbol{d}_j$ is orthogonal to any vector $\boldsymbol{b}_i$, $\boldsymbol{d}_j \circ \boldsymbol{b}_i = 0$ $(j = 1, ..., k; \ i = 1, ..., m)$. The vectors $\mathbf{e}_d = [\boldsymbol{d}_1, ..., \boldsymbol{d}_k]^{\mathsf{T}}$ form the base of the *uncontrolled subspace.*

As the vectors $\mathbf{e}_{bd} = [\mathbf{e}_b^{\mathsf{T}} \ \mathbf{e}_d^{\mathsf{T}}]^{\mathsf{T}} = [\boldsymbol{b}_1, ..., \boldsymbol{b}_m, \boldsymbol{d}_1, ..., \boldsymbol{d}_k]^{\mathsf{T}}$ are linearly independent, they form a new base in the n-space. The trasformation formula between the (covariant) bases $\mathbf{e}_{bd}$ and $\mathbf{e}_v$ is (refer also to Blajer (1992a,b))

$$\mathbf{e}_{bd} = \begin{bmatrix} \mathbf{e}_b \\ \mathbf{e}_d \end{bmatrix} = \begin{bmatrix} \mathbf{BM}^{-1} \\ \mathbf{D} \end{bmatrix} \mathbf{e}_v = \mathbf{He}_v \tag{3.2}$$

Since the dynamic equations (2.1) are represented in the (contravariant) base $\mathbf{e}_v^*$, their covariant representation in the base $\mathbf{e}_{bd}^*$ is equivalent to the left-sided multiplication of these equations by $\mathbf{H}$, see Appendix. This leads to the following decomposition of the dynamic equations

$$\mathbf{B}\dot{\mathbf{v}} = \mathbf{BM}^{-1}\mathbf{h}^* + \mathbf{BM}^{-1}\mathbf{B}^{\mathsf{T}}\boldsymbol{\lambda} \tag{3.3}$$

$$\mathbf{DM}\dot{\mathbf{v}} = \mathbf{Dh}^* \tag{3.4}$$

which corresponds to the projection of the dynamic equations into the controlled and uncontrolled subspaces, respectively.

The above partition of the dynamic equations enables one to separate the problem of synthesis of program control from the problem of dynamic analysis of the program motion. Namely, since the matrix $\mathbf{BM}^{-1}\mathbf{B}^{\mathsf{T}}$ is invertible in principle, Eq (3.3) can be solved for $\boldsymbol{\lambda}$, whereas, Eqs (2.3), (3.4) and (2.2) can be assembled to form the following reduced-dimension (control-reaction free) governing equations

$$\mathbf{R}\dot{\mathbf{v}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{DM} \end{bmatrix} \dot{\mathbf{v}} = \mathbf{h}^*_{cd} \begin{bmatrix} -\mathbf{c}^*_0 \\ \mathbf{Dh}^* \end{bmatrix}$$
$$\dot{\mathbf{x}} = \mathbf{Av} \tag{3.5}$$

Now, an important characteristic is the rank of matrix $\mathbf{R}$. If rank($\mathbf{R}$) is maximal (rank($\mathbf{R}$) $= n$) Eqs (3.5) are ordinary differential equations (ODEs) and can be solved for $\mathbf{v}(t)$ and $\mathbf{x}(t)$ using a range of ODE methods. Then, the required program control $\boldsymbol{\lambda}(t)$ can be synthesized from Eq (3.3) which, after substituting $\dot{\mathbf{v}} = \mathbf{R}^{-1}\mathbf{h}^*_{bd}$, can be manipulated to

$$\boldsymbol{\lambda} = (\mathbf{BM}^{-1}\mathbf{B}^{\mathsf{T}})^{-1}\mathbf{B}(\mathbf{R}^{-1}\mathbf{h}^*_{bd} - \mathbf{M}^{-1}\mathbf{h}^*) = \boldsymbol{\lambda}(\mathbf{v},\mathbf{x},t) \tag{3.6}$$

However, if rank($\mathbf{R}$) $< n$, Eqs (3.5) are differential-algebraic equations (DAEs), and the formulation (3.6) fails ($\mathbf{R}$ is noninvertable). Nevertheless, if a solution of these DAEs exists, the program control can be found from

$$\boldsymbol{\lambda} = (\mathbf{BM}^{-1}\mathbf{B}^{\mathsf{T}})^{-1}\mathbf{B}(\dot{\mathbf{v}} - \mathbf{M}^{-1}\mathbf{h}^*) = \boldsymbol{\lambda}(\dot{\mathbf{v}},\mathbf{v},\mathbf{x},t) \tag{3.7}$$

and $\dot{\mathbf{v}}$ may be determined either analytically by transforming the DAEs (3.5) into an equivalent set of ODEs (by differentiating the algebraic equations) and then solving the ODEs for $\dot{\mathbf{v}}$, or numerically from the solution $\mathbf{v}(t)$ to the DAEs (3.5). Prior to discussing these problems, let us interprete the loss in rank of matrix $\mathbf{R}$, which will be of importance in the following.

The rank of matrix $\mathbf{R}$ can be conveniently evaluated by inspecting how the *constraint vectors* $\mathbf{c}_i = \mathbf{c}_i^{*\mathsf{T}}\mathbf{e}^*_v$ ($i = 1,...,m$; $\mathbf{c}^*_i$ is the ith column of $\mathbf{C}^{\mathsf{T}}$) project into the controlled and uncontrolled subspaces. This is equivalent to the following factorization of $\mathbf{C}^{\mathsf{T}}$ (see Appendix for the transformation formula of covariant vector components)

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = \mathbf{HC}^{\mathsf{T}} = \begin{bmatrix} \mathbf{BM}^{-1}\mathbf{C}^{\mathsf{T}} \\ \mathbf{DC}^{\mathsf{T}} \end{bmatrix} \tag{3.8}$$

The $m \times m$ matrix $\mathbf{P} = \mathbf{BM}^{-1}\mathbf{C}^{\mathsf{T}}$ expresses the representation of the constraint vectors in the controlled subspace, and the $k \times m$ matrix $\mathbf{Q} = \mathbf{DC}^{\mathsf{T}}$ is the representation of these vectors in the uncontrolled subspace.

The metric tensor matrix of the base $\mathbf{e}_{bd}$ is

$$\mathbf{M}_{bd} = \mathbf{HMH}^{\mathsf{T}} = \begin{bmatrix} \mathbf{M}_b & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{M}_d \end{bmatrix} \tag{3.9}$$

where $\mathbf{M}_b = \mathbf{B}\mathbf{M}^{-1}\mathbf{B}^{\mathsf{T}}$ and $\mathbf{M}_d = \mathbf{D}\mathbf{M}\mathbf{D}^{\mathsf{T}}$ are the metric tensor matrices of the bases $\mathbf{e}_b$ and $\mathbf{e}_d$, respectively; and $\mathbf{0}$ denotes the $k \times m$ null matrix. After considering that $(\mathbf{H}^{-1})^{\mathsf{T}} = \mathbf{M}_{bd}^{-1}\mathbf{H}\mathbf{M}$, it comes from Eq (3.8) that

$$\mathbf{C} = \mathbf{P}^{\mathsf{T}}\mathbf{M}_b^{-1}\mathbf{B} + \mathbf{Q}^{\mathsf{T}}\mathbf{M}_d^{-1}\mathbf{D}\mathbf{M} \tag{3.10}$$

Using this, the matrix $\mathbf{R}$ defined in Eq $(3.5)_1$ can be stated as follows

$$\mathbf{R} = [\mathbf{P}^{\mathsf{T}}\mathbf{M}_b^{-1} \quad \alpha\mathbf{I}]\mathbf{H}^{\mathsf{T}}\mathbf{M} + [\mathbf{Q}^{\mathsf{T}}\mathbf{M}_d^{-1} \quad \beta\mathbf{I}]\mathbf{D}\mathbf{M} \tag{3.11}$$

where $\mathbf{I}$ is the $k \times k$ identity matrix; and α and β are real constants satisfying the condition $\alpha + \beta = 1$. After some inspection, one can deduce from Eq (3.11) that $\operatorname{rank}(\mathbf{R}) = p + k$, where $p = \operatorname{rank}(\mathbf{P})$. Thus, only the maximal rank of $\mathbf{P}$, $p = m$, assures that Eqs (3.5) are ODEs and Eq (3.6) is valid; for $p < m$, Eqs (3.5) are DAEs.

The case when $\operatorname{rank}(\mathbf{P}) = m$ has been classified by Parczewski and Blajer (1989) as an *orthogonal realization* of program constraints. This means that the constraint vectors $\mathbf{c}_i$ $(i = 1, ..., m)$ give nonzero projections in all directions of $\mathbf{e}_b$, or inversly, the control forces are represented explicitly in all the directions orthogonal to the program constraints (are represented explicitly in the *constrained subspace* spanned by $\mathbf{e}_c = [\mathbf{c}_1, ..., \mathbf{c}_m]^{\mathsf{T}}$). Irrespectively of the control forces give also projections in the *unconstrained subspace*, which occurs when $\operatorname{rank}(\mathbf{Q}) > 0$ and is referred to a *nonideal orthogonal realization* of program constraints, the matrix $\mathbf{R}$ in Eq $(3.5)_1$ is of maximal rank and the program control can be synthesized from Eq (3.6). As mentioned in Section 2, for the case when $\operatorname{rank}(\mathbf{P}) = m$ the program control can be synthesized from Eq (2.4) as well, and the dynamics of program motion can be analysed using Eqs (2.1) and (2.2) after substituting Eq (2.4) for $\boldsymbol{\lambda}$ in Eq (2.1). Nevertheless, the formulations Eqs (2.4) and (3.6) are identical only if $\mathbf{B} = \mathbf{C}$, i.e. when the control forces replace exactly the reactions of program constraints treated as material ones.

Rank$(\mathbf{P}) = p < m$ indicates that $l = m - p$ constraint vectors do not project into the controlled subspace but are represented in the uncontrolled subspace, or in other words, the control forces are not represented in the directions orthogonal to l corresponding program constraints. Thus, the realization of those particular constraints must be *tangent*, if the realization is possible at all. When $\operatorname{rank}(\mathbf{P}) = 0$, all the program constraints have to be realized by tangent control reactions. Since $\operatorname{rank}(\mathbf{P}) + \operatorname{rank}(\mathbf{Q}) = p + q \geq m$ and $\operatorname{rank}(\mathbf{Q}) = q \leq k = n - m$, the necessary condition of existence of mixed (orthogonal-tangent) realization of program constraints is $2m \leq n + p$, and for the case of pure tangent realization $(p = 0)$, the condition $2m \leq n$ must be fulfilled. Obviously, these are not the sufficient conditions of existence of orthogonal-tangent and tangent realizations of program constraints. As the program control can be synthesized from Eq (3.7) only if $\dot{\mathbf{v}}(t)$, $\mathbf{v}(t)$ and $\mathbf{x}(t)$ are available from the dynamic analysis of the program motion, the clue of the subsequent formulation for the case when $\text{rank}(\mathbf{P}) < m$ lies in the treatment of the DAEs (3.5) and in the inspection whether the DAEs are solvable.

As previously stated, $\mathbf{P} = \mathbf{B}\mathbf{M}^{-1}\mathbf{C}^{\top}$ containes in columns the (covariant) components of constraint vectors c_i $(i = 1, ..., m)$ in the base $\mathbf{e}_b^*$ of controlled subspace. On the other hand, the columns of $\mathbf{P}^{\top} = \mathbf{C}\mathbf{M}^{-1}\mathbf{B}^{\top}$ express the (covariant) components of control vectors b_j $(j = 1, ..., m)$ in the base $\mathbf{e}_c^*$ of constrained subspace (spanned by vectors c_i; $i = 1, ..., m$). If $\text{rank}(\mathbf{P}) = p < m$, only p from m control vectors project explicitly into the constrained subspace. Thus, by introducing a full-rank $l \times m$ matrix $\mathbf{U}$ which satisfies the condition

$$\mathbf{U}\mathbf{P}^{\top} = \mathbf{0} \tag{3.12}$$

$l = m - p$ independent vectors u_j $(j = 1, ..., l)$ can be defined in the constrained subspace, which are orthogonal to the control forces represented in this subspace; $u_j = \mathbf{u}_j^{\top}\mathbf{e}_c$, where $\mathbf{u}_j$ is the jth column of $\mathbf{U}^{\top}$. Then, by formulating a matrix $\mathbf{W}$ ($p \times m$ full-rank matrix), which is an orthogonal complement matrix to the matrix $\mathbf{U}$ in the m-subspace, i.e.

$$\mathbf{W}\mathbf{U}^{\top} = \mathbf{0} \tag{3.13}$$

a set of independent vectors w_i $(i = 1, ..., p)$ can be introduced, and the vectors are orthogonal to u_j $(j = 1, ..., l)$ and tangent to the control forces represented in the controlled subspace; $w_i = \mathbf{w}_i^{*\top}\mathbf{e}_c^* = \mathbf{w}_i^{*\top}\mathbf{M}_c^{-1}\mathbf{e}_c$, where $\mathbf{w}_i^*$ is the ith column of $\mathbf{W}^{\top}$ and $\mathbf{M}_c = \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^{\top}$ is the metric tensor matrix of the base $\mathbf{e}_c = [c_1, ..., c_m]^{\top}$.

The vectors $\mathbf{e}_{uw} = [\mathbf{e}_u^{\top}\ \mathbf{e}_w^{\top}]^{\top} = [u_1, ..., u_l, w_1, ..., w_p]^{\top}$ form a new base of the constrained subspace, and

$$\mathbf{e}_{uw} = \begin{bmatrix} \mathbf{e}_u \\ \mathbf{e}_w \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ \mathbf{W}\mathbf{M}_c^{-1} \end{bmatrix} \mathbf{e}_c = \mathbf{H}_{uw}\mathbf{e}_c \tag{3.14}$$

where $\mathbf{H}_{uw}$ is the transformation matrix. Since the constraint equation (2.3) can be interpreted as the (covariant) representation of quasi-acceleration vector $\dot{\mathbf{v}}$ in the base $\mathbf{e}_c^*$ (the values of the corresponding components are gathered in the column matrix $-\mathbf{c}_0^*$), the projection of this equation into the base $\mathbf{e}_{uw}$ is equivalent to the left-sided multiplication of these equations by $\mathbf{H}_{uw}$. After considering the preceding formulation, this leads to the following partition of the constraint equations

$$\mathbf{g}(\mathbf{v}, \mathbf{x}, t) = \mathbf{U}(\mathbf{Q}^{\top}\mathbf{M}_d^{-1}\mathbf{D}\mathbf{h} + \mathbf{c}_0^*) = \mathbf{0} \tag{3.15}$$

$$\mathbf{W}\mathbf{M}_c^{-1}\mathbf{C}\dot{\mathbf{v}} = -\mathbf{W}\mathbf{M}_c^{-1}\mathbf{c}_0^* \tag{3.16}$$

The l algebraic equations (3.16) express the projection of the constraint conditions (2.3) into the base $\mathbf{e}_u$, and the new acceleration balance conditions are not affected by any control forces. Then, since the matrix $\mathbf{W}\mathbf{M}_c^{-1}\mathbf{C}$ can be interpreted as an inner product of the constrained and controlled subspaces, the matrix

$\mathbf{E} = [(\mathbf{W}\mathbf{M}_c^{-1}\mathbf{C})^{\mathsf{T}}\ (\mathbf{D}\mathbf{M})^{\mathsf{T}}]^{\mathsf{T}}$ is of maximal rank, and the governing equations (3.5) can be rewritten in the following semi-explicit DAE form

$$\begin{aligned} \mathbf{E}(\mathbf{v},\mathbf{x},t)\dot{\mathbf{v}} &= \mathbf{h}_e(\mathbf{v},\mathbf{x},t) \\ \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x},t)\mathbf{v} \\ \mathbf{g}(\mathbf{v},\mathbf{x},t) &= \mathbf{0} \end{aligned} \tag{3.17}$$

where $\mathbf{h}_e = [(-\mathbf{W}\mathbf{M}_c^{-1}\mathbf{c}_0^*)^{\mathsf{T}}\ (\mathbf{D}\mathbf{h}^*)^{\mathsf{T}}]^{\mathsf{T}}$. The problem arising now is whether Eqs (3.17) are solvable and, if so, how to solve the DAEs. The existence of a solution of these equations amounts to the controllability of the system in the prespecified motion.

A trivial case of nonsolvability of Eqs (3.17) is when at least one of the algebraic equations $(3.17)_3$ depends neither on $\mathbf{v}$ nor $\mathbf{x}$, and an example of such a case is mentioned in Section 4 (in the discussion to Example 1). In a general case, one may detect the solvability of Eqs (3.17) by inspecting the differentiated forms of Eqs $(3.17)_3$, i.e.

$$\mathbf{R}_s(\mathbf{v},\mathbf{x},t)\dot{\mathbf{v}} + \boldsymbol{\xi}_s^*(\mathbf{v},\mathbf{x},t) = \mathbf{0} \qquad (s = 1,2) \tag{3.18}$$

where s relates to the number of differentiations (up to 2),

$$\begin{aligned} \mathbf{R}_1 &= \mathbf{g}_v \\ \boldsymbol{\xi}_1^* &= \mathbf{g}_x\mathbf{A}\mathbf{v} + \mathbf{g}_t \\ \mathbf{R}_2 &= (\mathbf{g}_v\mathbf{M}^{-1}\mathbf{h}^*)_v + \mathbf{A}\mathbf{v} \\ \boldsymbol{\xi}_2^* &= (\mathbf{g}_v\mathbf{M}^{-1}\mathbf{h}^*)_x\mathbf{A}\mathbf{v} + (\mathbf{g}_v\mathbf{M}^{-1}\mathbf{h}^*)_t + (\mathbf{g}_x\mathbf{A})^{\cdot}\mathbf{A} + (\mathbf{g}_t)^{\cdot} \end{aligned}$$

and the subscripts $\mathbf{v}$, $\mathbf{x}$ and t denote partial differentiation with respect to the corresponding variable. The DAEs (3.17) are solvable if (for $s = 1$ or, if not, for $s = 2$) the $n \times n$ matrix

$$[\mathbf{E}^{\mathsf{T}}\ \mathbf{R}_s^{\mathsf{T}}]^{\mathsf{T}} \tag{3.19}$$

is nonsingular, and s indicates the number of times the algebraic equations have to be differentiated in order to transform the DAEs into ODEs. When two differentiations are required $(s = 2)$, after the first differentiation $\mathbf{M}^{-1}\mathbf{h}^*$ should be substituted for $\dot{\mathbf{v}}$. Evidently, the initial value problem of the produced ODEs must be in agreement with the additional conditions, i.e. $\mathbf{g}(\mathbf{v}_0,\mathbf{x}_0,t_0) = \mathbf{0}$, and eventually (for $s = 2$) $\dot{\mathbf{g}}(\mathbf{v}_0,\mathbf{x}_0,t_0) = \mathbf{0}$.

The above conclusions can be interpreted as follows. As previously stated, Eq $(3.17)_3$ (= Eq (3.15)) expresses the balance condition of the system accelerations due to the forces conserved in $\mathbf{h}^*$ and accelerations due to the constraint requirements (conserved in $\mathbf{c}_0^*$), both projected into the base $\mathbf{e}_u$, i.e. in those directions of the constrained subspace where the control forces are not represented. Thus, the control forces cannot influence directly the balance conditions. However, since

Eq $(3.17)_3$ depends on $\mathbf{v}$ and $\mathbf{x}$, an appropriate accommodation of state variables may assure the maintenance of the balance conditions. This must be ensured by the (*tangent*) control forces which do not project into the constrained subspace, as the (*orthogonal*) control forces projecting into this subspace are involved to govern the first p equations of Eqs (3.17)a (= Eqs (3.16)). This can be done only if the gradients of the algebraic equations $(3.17)_3$ are represented explicitly in the directions of tangent control forces. Mathematically, the matrix $[\mathbf{E}^\top\ \mathbf{R}_1^\top]^\top$ is nonsingular if the gradients with respect to $\mathbf{v}$ satisfy this condition, and the matrix $[\mathbf{E}^\top \mathbf{R}_2^\top]^\top$ is nonsingular when the combined gradients with respect to $\mathbf{v}$ and $\mathbf{x}$ satisfy this condition. When neither $[\mathbf{E}^\top\ \mathbf{R}_1^\top]^\top$ nor $[\mathbf{E}^\top\ \mathbf{R}_2^\top]^\top$ is nonsingular, the imposed program motion cannot be realized with the available control of the system as assumed in this paper.

According to the definitions given by Campbell (1982), Brenan and Enquist (1988), Galyullin (1989), Blajer (1992c), s defined in Eq (3.18) is the index value of the DAEs (3.17), and for the case at hand $s \leq 2$ (solvable case). Thus, instead of transforming the DAEs into the corresponding ODEs by differentiating the algebraic equations $(3.17)_3$, which may be a laboruos task, Eqs (3.17) can be solved directly using a range of DAE solvers. However, as discussed by Gear and Petzold (1984), Brenan and Enquist (1988), Brenan, Campbell and Petzold (1989), it is usually difficult to solve DAEs with index greater than one. Thus, in the case when $s = 2$, the DAEs (3.17) can eventually be transformed to a corresponding set of DAEs with index one by replacing Eqs $(3.17)_3$ by Eq (3.18); $\mathbf{g}(\mathbf{v}_0, \mathbf{x}_0, t_0) = \mathbf{0}$ must be satisfied (for some general remarks relating the initial value problem of DAEs referred to by Brenan, Campbell and Petzold (1989), Leimkuhler, Petzold and Gear (1991)). Then a range of automatic codes for solving index-one DAE systems, see e.g. Gear and Petzold (1984), Brenan and Enquist (1988), Brenan, Campbell and Petzold (1989), can be applied to solve the problem considered in this paper.

By solving the governing equations of program motion in the form of DAEs, the time histories $\mathbf{v}(t)$ and $\mathbf{x}(t)$ will be obtained. Then the program control can be synthesized from Eq (3.7), where $\dot{\mathbf{v}}(t)$ has to be determined numerically using $\mathbf{v}(t)$. Such an approach is suggested in Example 2 of this paper and was used by Blajer (1990) and (1991) for numerical simulation of aircraft predetermined trajectory flight.

4. Some applications

4.1. Example 1

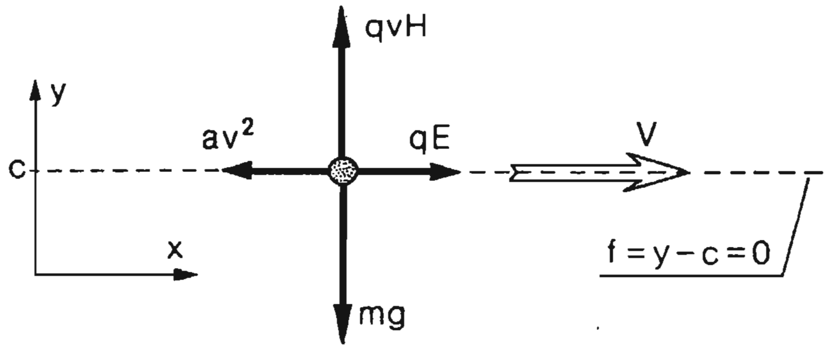

Fig. 1. Prticle motion along a horizontal trajectory

Consider a particle of mass m and charge q moving in the gravitational, electric, and magnetic fields, respectively. Let us limit ourselves to the planar motion case, and assume that the vectors of gravity acceleration $\boldsymbol{g}$, electric field intensity $\boldsymbol{E}$, and magnetic induction $\boldsymbol{H}$, respectively, act as follows: $\boldsymbol{g} = -g\boldsymbol{e}_y$, $\boldsymbol{E} = E\boldsymbol{e}_x$, and $\boldsymbol{H} = -H\boldsymbol{e}_z$ (see Fig.1). Then, suppose that the particle is to move along the path $f = y - c = 0$, where c is a constant value. The governing equations of the problem at hand, corresponding to Eqs (2.1) ÷ (2.3), are

$$
\begin{aligned}
m\dot{v}_x &= -avv_x + qE - qv_yH \\
m\dot{v}_y &= -avv_y - mg + qv_xH
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\dot{x} &= v_y \\
\dot{y} &= v_y
\end{aligned}
\tag{4.2}
$$

$$
\dot{v}_y = 0 \tag{4.3}
$$

where a is a constant value (av^2 denotes the drag force); $v = \sqrt{\dot{x}^2 + \dot{y}^2}$; and $y_0 = c$ and $v_{y0} = 0$ must be assured. Obviously, for the case at hand the equations can be simplified by applying $v_y = 0$ and $v_x = v$, which will be used in the following.

Let us consider two possible ways of control of the program motion: by changes in H (Case A), and by changes in E (Case B); and when one of these variables is considered as a control parameter, the other is an arbitrary function of time. Following the formulation of Section 3, it is easy to state that

$$\mathbf{M} = \mathrm{diag}(m, m) \tag{4.4}$$
$$\mathbf{C} = [0 \quad 1] \tag{4.5}$$
$$\mathbf{B}^A = [0 \quad qv] \qquad \mathbf{B}^B = [q \quad 0] \tag{4.6}$$
$$\mathbf{D}^A = [1 \quad 0] \qquad \mathbf{D}^B = [0 \quad 1] \tag{4.7}$$
$$\mathbf{P}^A = [qv/m] \qquad \mathbf{P}^B = [0] \tag{4.8}$$
$$\mathbf{Q}^A = [0] \qquad \mathbf{Q}^B = [1] \tag{4.9}$$

The results obtained in Eqs (4.8) and (4.9) are evident as the control forces, respectively for Cases A and B, are orthogonal and tangent to the imposed trajectory. Then Eq $(3.5)_1$ takes the forms

$$\begin{bmatrix} 0 & 1 \\ m & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} 0 \\ -av^2 + qE(t) \end{bmatrix} \qquad \text{(Case A)} \tag{4.10}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} 0 \\ -mg + gvH(t) \end{bmatrix} \qquad \text{(Case B)} \tag{4.11}$$

Thus, Eqs (4.10) and (4.2) are ODEs, whereas Eqs (4.11) and (4.2) are DAEs. Then, for the Case A, Eq (3.6) takes the form

$$H = \frac{mg}{qv} \tag{4.12}$$

i.e. following the solution $v(t)$ to the ODEs formed by Eqs (4.10) and (4.2), the program control by changes in H can be synthesized. Note that $v \neq 0$ is required both for maximal rank of $\mathbf{P}$ and for validity of Eq (4.12).

For the Case B, since Eqs (4.11) and (4.2) are DAEs, the program control has to be synthesized from Eq (3.7) which takes the form

$$E = \frac{m\dot{v}_x + av^2}{q} \tag{4.13}$$

In order to solve the problem of Case B, first the implicit DAEs formed by Eqs (4.11) and (4.2) have to be transformed to the semi-explicit form (3.17). For the case at hand this leads to

$$\begin{aligned} &\begin{bmatrix} 0 & m \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = -mg + qvH(t) \\ &\text{equations} \quad (4.2) \\ &-g + \frac{1}{m} qv_x H(t) = 0 \end{aligned} \tag{4.14}$$

and v_x $(= v)$ has been introduced intentionally to show that single differentiation of the algebraic equation $(4.14)_3$ transforms the above DAEs into ODEs. In other words, the index of the above set of DAEs is one, and the problem can be solved conveniently either by integrating Eqs (4.14) directly (by applying a range of DAE solvers suited to index-one systems) or Eqs (4.14) can be transformed to an ODE form, and then solved. In the first case, $v_x = v(t)$ and $x(t)$ will be obtained, and $\dot{v}_x(t)$ has to be determined numerically in order to synthesize the program control from Eq (4.13). In the second case, $\dot{v}_x(t)$ can be determined analytically which, for this simple case of study, leads to

$$E = \frac{m\dot{v}_x\dot{H}(t) + av^2H(t)}{qH(t)} \tag{4.15}$$

The above result can be interpreted as follows. Since the control force qE does not affect the acceleration balance $(4.14)_3$ explicitly, any changes in H which will lead to the violation of this balance condition have to be compensated by the appriopriate changes in $v_x = v$, i.e. assuring that Eq $(4.14)_3$ is satisfied at a given instant, $\frac{q}{m}(\dot{v}_xH + v_x\dot{H}) = 0$ (the differentiated form of $(4.14)_3$) must be fulfilled. Using the dynamic equation $(4.1)_1$, this can be assured by appropriate changes in E, which is reflected in Eq (4.15). Note also that $H \neq 0$ must be assured for existance of the tangent control. In this nondirect way the tangent control force qE assures the balance condition of orthogonal accelerations.

It may be worth noting that the tangent control could be impossible if the condition $(4.14)_3$ would not depend on v_x. For instance, removing the magnetic field and assuming that the electric field is such that $\boldsymbol{E} = E_x\boldsymbol{e}_x + E_y\boldsymbol{e}_y$, where $E_x(t)$ and $E_y(t)$ are independent, the control of the program motion by changes in E_x would be impossible; Eq $(4.14)_3$ would be of the form $-g + \frac{q}{m}E_y(t) = 0$. Given an arbitrary function $E_y(t)$, the balance of orthogonal accelerations can be assured only by adding a control force which gives an orthogonal projection.

4.2. Example 2

In order to illustrate some other aspects of the formulation introduced in Section 3, let us consider the problem of aircraft controlled flight along a predetermined trajectory. The problem has already been studied by Blajer (1990) and (1991), Blajer and Parczewski (1989), (1990) and (1991), hereinafter it is reformulated and solved systematically using the improved formulation reported in this paper.

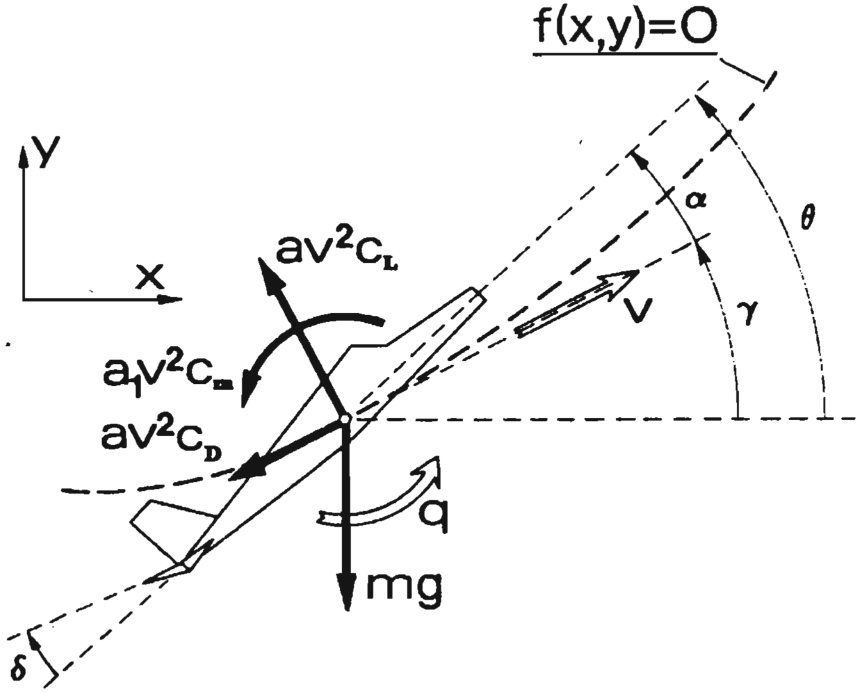

Fig. 2. Aircraft trajectory motion

Assume that the aircraft is moving in the vertical plane, and the program requirements are a predetermined trajectory and a prespecified speed history (see Fig.2), i.e.

$$
\begin{aligned}
&f(x,y) = 0 \\
&\varphi(v,x,y) = v - \phi(x,t) = 0
\end{aligned}
\tag{4.16}
$$

Then introduce two control parameters: the thrust force T, and the elevator displacement angle δ. Following some other assumptions epitomized in Fig.2, the governing equations of the problem, corresponding to Eqs (2.1) ÷ (2.3), can be written as follows

$$
\begin{aligned}
m\dot{v} &= -av^2c_D(\alpha) - mg\sin\gamma + T\cos\alpha \\
mv\dot{\gamma} &= av^2c_L(\alpha) - mg\cos\gamma + T\sin\alpha \\
J\dot{q} &= a_1v^2\Big(c_{m0}(\alpha) + c_{m1}(\alpha)\delta\Big)
\end{aligned}
\tag{4.17}
$$

$$\begin{aligned} \dot{x} &= v\cos\gamma \\ \dot{y} &= v\sin\gamma \\ \dot{\theta} &= q \end{aligned} \tag{4.18}$$

$$\begin{aligned} v\dot{\gamma} - \kappa v^2 &= 0 \\ \dot{v} - b &= 0 \end{aligned} \tag{4.19}$$

where m and J are the aircraft mass and moment of inertia, respectively; $a = \frac{1}{2}\rho S$ and $a_1 = c_a a$, ρ – air density (assumed constant), S – wing area, c_a – mean chord value; c_D, c_L, and $c_m = c_{m0} + c_{m1}$ are the coefficients of drag force, lift force, and pitching moment, respectively; α is the angle of attack; γ is the angle of inclination of velocity vector $\boldsymbol{v}$, and q is the aircraft angular velocity. In the constraint equations (second-order kinematic form): $\kappa = (f_{xx}f_y^2 + f_{yy}f_x^2 + 2f_{xy}f_xf_y)/(f_x^2 + f_y^2)^{3/2}$ is the curvature of the trajectory $f(x,y) = 0$, and $b = v(\phi_x\cos\gamma + \phi_y\sin\gamma) + \phi_t$. Obviously, the initial value problem of Eqs (4.17) $\div$ (4.19) must satisfy the lower-order constraint conditions: $f(x_0,y_0) = 0$; $f_x(x_0,y_0)\cos\gamma_0 + f_y(x_0,y_0)\sin\gamma_0 = 0$; and $v_0 - \phi(x_0,t_0) = 0$. Since $\alpha = \theta - \gamma$ (see Fig.2), the state variables of Eqs (4.17) $\div$ (4.19) are v, γ, q, x, y, and θ, and $v\dot{\gamma}$ should be interpreted as a quasi-acceleration (the corresponding quasi-velosity has no physical meaning). The path axes for the translatory dynamic equations $(4.17)_1$ and $(4.17)_2$ have been chosen for the convenience in formulating the orthogonal and tangent directions to the imposed constraints.

Following the formulation of Section 3, it is easy to find that

$$\mathbf{M} = \operatorname{diag}(m, m, J) \tag{4.20}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & a_1v^2c_{m1} \\ \cos\alpha & \sin\alpha & 0 \end{bmatrix} \tag{4.21}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{4.22}$$

$$\mathbf{D} = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \end{bmatrix} \tag{4.23}$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m}\sin\alpha & \frac{1}{m}\cos\alpha \end{bmatrix} \tag{4.24}$$

$$\mathbf{Q} = \begin{bmatrix} \cos\alpha & -\sin\alpha \end{bmatrix} \tag{4.25}$$

As $\operatorname{rank}(\mathbf{P}) = 1$ ($<$ maximal), only one program constraint (velocity requirement $(4.16)_2$) can be realized in an orthogonal way (by changes in T value), whereas the other constraint (trajectory requirement $(4.16)_1$) has to be realized by a tangent control reaction (changes in δ displacement).

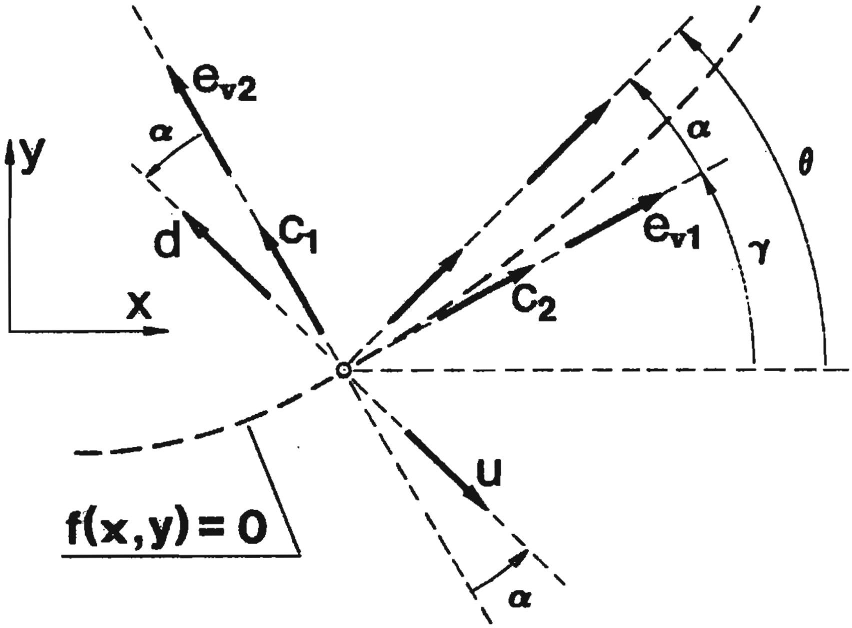

Fig. 3. Vector representations in the constrained subspace

In accordance with Eqs (3.13) and (3.14), the matrices $\mathbf{U}$ and $\mathbf{W}$ can be defined as follows

$$\mathbf{U} = \begin{bmatrix} -\cos\alpha & \sin\alpha \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} \sin\alpha & \cos\alpha \end{bmatrix} \tag{4.26}$$

and the directions of $\boldsymbol{u}$ and $\boldsymbol{w}$, as well as the other involved vectors, are demonstrated in Fig.3. Then the governing equations (3.17) take the form

$$\begin{bmatrix} m\cos\alpha & m\sin\alpha & 0 \\ -m\sin\alpha & m\cos\alpha & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ v\dot{\gamma} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} mv^2\kappa\sin\alpha + mb\cos\alpha \\ av^2(c_D\sin\alpha + c_L\cos\alpha) - mg\cos\theta \end{bmatrix}$$

equations (4.18) (4.27)

$$-\frac{av^2}{m}(c_D\sin\alpha + c_L\cos\alpha) + g\cos\theta + v^2\kappa\cos\alpha - b\sin\alpha = 0$$

According to the comments in Section 3, the algebraic equation $(4.27)_3$ expresses the balance of accelerations projected in the direction of $\boldsymbol{u}$, i.e. in such a direction of the constrained subspace where the control forces are not represented, see Fig.3. On the other hand, the first equation of Eqs $(4.27)_1$ expresses the total of the constraint conditions (4.19) projected in the direction of w, i.e. in the direction of the control force T.

It is easy to prove that the index of the DAEs (4.27) is two; after differentiating twice Eq $(4.27)_3$ and substituting for $\ddot{\theta} = \dot{q}$ and $\dot{\alpha} = \dot{q} - \ddot{\gamma}$, $\dot{q}$ will appear explicitly in the resultant formulation of this equation, and the corresponding equations will be ODEs. Such an approach is not recommended however, as leading to complexity in mathematical formulation. Instead, Eqs (4.27) can be solved directly using a range of DAE solvers, see e.g. Gear and Petzold (1984), Brenan, Campbell and Petzold (1989). It is worth noting that, for the case at hand, Eqs $(4.27)_1$, $(4.18)_1$, $(4.18)_2$ and $(4.27)_3$ form a subset of DAEs of index one with the state variables v, γ, x, y and θ. From the solution of this DAEs, $\dot{v}(t)$, $\dot{\gamma}(t)$ and $\dot{q}(t) = \ddot{\theta}(t)$ can be determined numerically, and the program control can be synthesized from the following relations (corresponding to Eq (3.7))

$$\begin{aligned} \delta &= \frac{J\dot{q}}{a_1 v^2 c_{m1}} - \frac{c_{m0}}{c_{m1}} = \delta(\dot{q}, v, \gamma, \theta) \\ T &= m\Big(\dot{v}\cos\alpha + v\dot{\gamma}\sin\alpha + av^2(c_D\cos\alpha - c_L\sin\alpha) + g\sin\theta\Big) = T(\dot{v}, \dot{\gamma}, v, \gamma, \theta) \end{aligned} \tag{4.28}$$

Some examples of numerical simulation of aircraft program motion in prespecified trajectory flight are demonstrated by Blajer (1990) and (1991), and the above approach to the solution of the problem has been used there.

5. Conclusions

A unified and systematic approach to the dynamic analysis and synthesis of control of mechanical systems with prespecified motions has been presented. It is proved theoretically and illustrated by examples that the program constraints may be realized by control reactions which have any directions relative the program constraint manifolds, and in the extreme, which are tangent to the constraints. Criteria of controllability of systems in prespecified motions are formulated, and a general mathematical formulation for the solution of the problem is proposed.

One of the principles (and advantages) of the proposed formulation is the partition of the general problem into two subproblems: the dynamic analysis of the system prespecified motion, and the synthesis of control ensuring the realization of the motion. The latter subproblem bases on the solution of the former one and is solvable only if that solution exist. This partition technique reduces the dimension of the governing equation which have to be solved, and is equivalent to one-step index reduction technique for DAEs (cf Gear and Petzold, 1984; Gear, 1988; Brenan, Campbell and Petzold, 1989).

The proposed formulation is valid for systems depending linearly on control parameters. This comprises a wide range of systems encountered in practice. In general, control forces may depend nonlinearly on control parameters and/or control parameters may influence indirectly the forces acting on the system.

In the mathematical formulation of the paper, the linear algebra/tensor formalism is applied. This gives a geometrical insight into the problems being solved and enables one to systematize the formulation. This may be, however, combersome to the readers which are unfamiliar with the formalism. Those readers are recommended to follow only the matrix notation of the proposed formulation.

The orthogonal complement matrix $\mathbf{D}$ to the control matrix $\mathbf{B}$, and then matrices $\mathbf{U}$ and $\mathbf{W}$ are introduced. One may face difficulties in determination of these matrices. As shown in examples, for small systems the matrices can be simply guessed, however, in a general case the difficulties may arise. These aspects are not referred in this paper.

The synthesized program control can be considered as a feed-forward control of the system, and the time histories $\lambda(t)$ are important both from the investigative and practical points of view. Obviously in practical applications, a feed-back control should be added to stabilize the system program motion.

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Appendix

Consider an n-dimensional metric space. A vector $\boldsymbol{a}$ can be expressed by its contravariant components $\mathbf{a} = [a_1, ..., a_n]^{\mathsf{T}}$ in the covariant base of this space $\mathbf{e} = [e_1, ..., e_n]^{\mathsf{T}}$, or by its covariant components $\mathbf{a}^* = [a_1^*, ..., a_n^*]^{\mathsf{T}}$ in the contravariant base $\mathbf{e}^* = [e_1^*, ..., e_n^*]^{\mathsf{T}}$ (cf Sokolnikoff, 1962; Pobedrya, 1974).

$$\boldsymbol{a} = \mathbf{a}^{\mathsf{T}}\mathbf{e} = \mathbf{a}^{*\mathsf{T}}\mathbf{e}^*$$

With the use of the metric tensor matrix $\mathbf{M}$ of the base $\mathbf{e}$

$$\mathbf{M} = \mathbf{e}\mathbf{e}^{\mathsf{T}}$$

the interdependences between the contravariant and covariant bases and vector components are as follows

$$\mathbf{e} = \mathbf{M}\mathbf{e}^* \qquad\qquad \mathbf{a}^* = \mathbf{M}\mathbf{a}$$

A dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can then be wtitten in four possible ways

$$\boldsymbol{a} \circ \boldsymbol{b} = \mathbf{a}^{\mathsf{T}}\mathbf{M}\mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b}^{*} = \mathbf{a}^{*\mathsf{T}}\mathbf{M}^{-1}\mathbf{b}^{*} = \mathbf{a}^{*\mathsf{T}}\mathbf{b}$$

and the orthogonality condition is defined as $\boldsymbol{a} \circ \boldsymbol{b} = \mathbf{0}$.

When the reference frame changes from the given one to another (denoted by the superscript $\widehat{\ }$), the transformation formulae are

$$\widehat{\mathbf{e}} = \mathbf{H}\mathbf{e} \qquad \mathbf{e}^{*} = \mathbf{H}^{\mathsf{T}}\widehat{\mathbf{e}}^{*}$$
$$\widehat{\mathbf{a}}^{*} = \mathbf{H}\mathbf{a}^{*} \qquad \mathbf{a} = \mathbf{H}^{\mathsf{T}}\widehat{\mathbf{a}}$$

and the metric tensor matrix of the base $\widehat{\mathbf{e}}$ is

$$\widehat{\mathbf{M}} = \mathbf{H}\mathbf{M}\mathbf{H}^{\mathsf{T}}$$

where $\mathbf{H}$ is an invertible $n \times n$ transformation matrix.

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O syntezie sterowania ukladami mechanicznymi w ruchu programowym

Streszczenie

Praca dotyczy dynamicznej analizy i syntezy sterowania układami mechanicznymi w ruchu programowym. Warunki nakładane przez program ruchu traktowane są jako więzy programowe i wyznaczane są reakcje sterowania zapewniające realizację tych więzów. Ponieważ siły sterujące ukladem mogą w ogólności nie być reprezentowane na kierunkach prostopadłych do uogólnionych powierzchni wiezów, rozważane zagadnienie wykracza poza klasyczny problem dynamicznej analizy ukł adów skrępowanych więzami i wyznaczania reakcji więzów. Pokazano (i zilustrowano przykładami), że więzy programowe mogą być realizowane siłami sterującymi, które mogą mieć dowolne kierunki względem uogólnionych powierzchni tych więzów, a w granicznym przypadku rownież poprzez styczne siły sterujące. Sformułowano kryteria istnienia rozwiązania rozważanego zagadnienia (= sterowalności układu w ruchu programowym) i zaproponowano sformułowanie matematyczne prowadzące do znalezienia tego rozwiązania. Rozważania teoretyczne zilustrowano dwoma przykładami poglądowymi.

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