MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
2, 31, 1993

THE EFFECT OF AXIAL LOADS ON TRANSVERSE VIBRATIONS OF AN EULER-BERNOULLI BEAM

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In this study the solution to the problem of vibrations of an Euler-Bernoulli beam, which is loaded by an axial force varying along the length of the beam, has been presented. The formula for vibration frequency and free vibration modes, for the case of a beam loaded by the force constant at intervals and for the load given in the form of the power series, has been determined. In the case of forced vibrations the flux of energy, which is emitted by the vibrating beam, has been established. The effect of axial load on vibration of the beam for various ways of its ends attachment have been illustrated in the enclosed pictures.

1. Introduction

The relationship between vibrations and the causes which generate stresses in the vibrating system may have particular applications in engineering. As it may allow one to identify the state of stress on the grounds of system vibrations and also may lead to the establishment of the vibrations of desired parts which have been initially prestressed. Thus, for example, in the case of vibration of musical instrument parts there exist a possibility to give certain acoustic properties to the sound board of the instrument. Also dynamometer, elements of which have been working at various loads, as a result of relation between states of stress of its elements and vibrations may give some information about the existing state of stress (thus, giving indication on the loading forces).

In this work we have been dealing with vibrations of the beams, which characterize well a broad class of linear vibrations. In this connection on the grounds of the obtained characteristics one may conclude on the motion of such elements as plates, shells, etc. Since some properties are resulting from the quality assessments which are common.

In the case of the Euler-Bernoulli beam effect on vibration, besides the boundary and initial conditions, have also flexural rigidity, mass per length unit, elasticity of foundation, rotary inertia, shear deformation, axial pre-stress and geometric parameters. If structural parameters are constant (time and spatial variable independent), solution to the problem is quite easy, and is prettly well known in the literature (cf Gürgöze, 1991). But in the case, where these parameters are spatial variable dependent, solution is obtained by using the approximate methods and only for the chosen forms of functions, which describe variation of the parameters, exact methods may be applied.

The problem of effect of parameters varying along the length of the beam on vibrations of the beam is the object of many studies. Afagh and Leipholz (1990) has investigated the influence of the uniformly distributed tangential follower force on vibrations of clamped-free rod. The influence of elasticity of foudation being the Winkler type (which varies along the length of the beam) on beam vibrations has been shown by Eisenberger and Clastornik (1987), Kukla (1991). Gladwell et al. (1987) has discused the problem of determining the phisical properties (the cross-sectional area of the beam $A(x)$ and the second moment of the area about the natural axis $I(x)$) of the beam from its vibrational behaviour, especially from the natural frequency data. Gottlieb (1988) presents explicit examples of operators with discontinuous coefficient functions which displays eigenvalue spectra identical to each other and to the corresponding operator with continuous coefficients. The investigations in the case of fourth-order differential equation deal with the stepped-density beams. The fundamental frequency of a cantilever beam subjected to a constant direction force at an intermediate point is approximately calculated by Gürgöze (1991). The effect of stepwise change of the cross-section of the beam on the first six natural frequencies has been studied for ten different boundary conditions by Jang and Bert (1989). The effect of constant axial force on vibrations of doublespan beams for three different cases of their beam ends attachements has been discussed by Laura et al. (1983). Skalmierski (1984), (1985) and (1986) has been studied the problem of axial loads and damping effects on vibrations of elastic systems. The relationship between the state of stress and the amplitude characteristic and the possibility of stress state introduction to the systems (in order to achieve desired vibrations) has been shown. This question is of considerable importance in elastic vibrating systems, e.g. in resonant boards of musical instruments and also in diagnostic problems.

In the present work we investigate the effect of axial loads on vibrations of the beam under the assumption that the axial force (tensile or compressive) is changing along the length of the beam. Let us consider two cases of the axial force: the force constant at intervals and the force given in the form of the power series. In both cases, obtaining of the solution to the problem by using the exact method is possible. The solutions to the problem of free vibrations of the beam, which have been obtained, concern various methods of the beam ends attachments. In the case of forced vibrations the flux of energy, which has been emited by a vibrating beam, has been determined.

2. Free vibration of a beam loaded by a variable axial force

For small amplitude free vibrations of a beam loaded by an axial force, the differential equation of motion in dimensionless coordinates is the following

$$\frac{d^4Y(\zeta)}{d\zeta^4} - \frac{d}{d\zeta}\Big[P(\zeta)\frac{dY(\zeta)}{d\zeta}\Big] - \lambda^4 Y(\zeta) = 0 \tag{2.1}$$

Let us assume that on both ends of the beam the boundary conditions are fulfilled which may be symbolically written as

$$U_0[Y(\zeta)]\Big|_{\zeta=0} = 0 \qquad\qquad U_1[Y(\zeta)]\Big|_{\zeta=1} = 0 \tag{2.2}$$

where U_0, U_1 are two-dimensional "vectors", components of which are linear, spatial differential operators. Our aim is to determine the nondimensional frequencies λ of free vibrations of the beam and corresponding to them non-trivial solutions $Y(\zeta)$ of the differential problem Eqs (2.1) and (2.2).

2.1. Problem of free vibration of the beam with step-wise axial load

We have assumed that axial load of the beam is the constant at intervals function, i.e. function which may be described by the formula

$$P(\zeta) = \sum_{i=1}^{m} P_i[H(\zeta - l_{i-1}) - H(\zeta - l_i)] \tag{2.3}$$

where $H(\cdot)$ denotes the Haeviside function and $0 = l_0 < l_1 < ... < l_m = 1$. Taking into account Eq (2.6) and assuming

$$\xi_i = \frac{\zeta - l_{i-1}}{l_i - l_{i-1}} \tag{2.4}$$

and $Y_i(\xi_i) = Y(\zeta)$ for $l_{i-1} < \zeta < l_i$, $0 < \xi_i < 1$, one can replace Eq (2.4) by the following system of m differential equations

$$\frac{d^4Y_i(\xi_i)}{d\xi_i^4} - 2\bar{P}_i\frac{d^2Y_i(\xi_i)}{d\xi_i^2} - \bar{\lambda}_i^4 Y_i(\xi_i) = 0 \tag{2.5}$$

where $\bar{\lambda}_i^4 = \lambda^4(l_i - l_{i-1})^4$ and $\bar{P}_i = P_i(l_i - l_{i-1})^2/2$, $i = 1,2,...,m$.

The functions $Y_i(\xi_i)$ fulfill the following continuity conditions

$$\frac{d^kY_i(1)}{d\xi_i^k} = \frac{d^kY_{i+1}(0)}{d\xi_{i+1}^k} \tag{2.6}$$

for $k = 0,1,2,3$ and $i = 1,2,...,m-1$. Moreover, the functions $Y_1(\xi_1)$ and $Y_m(\xi_m)$ are satisfying conditions, which follow from the boundary conditions (2.2) and the relationship (2.4). The conditions may be expressed in the following form

$$\bar{U}_0[Y_1(\xi_1)]\Big|_{\xi_1=0} = 0 \qquad \bar{U}_1[Y_m(\xi_m)]\Big|_{\xi_m=1} = 0 \tag{2.7}$$

The general solution to Eq (2.5) is the following

$$Y_i(\xi_i) = C_{i1}\cos\alpha_i\xi_i + C_{i2}\sin\alpha_i\xi_i + C_{i3}\cosh\beta_i\xi_i + C_{i4}\sinh\beta_i\xi_i \tag{2.8}$$

where

$$\alpha_i = \sqrt{\sqrt{\bar{P}_i^2 + \bar{\lambda}_i^4} - \bar{P}_i} \qquad \beta_i = \sqrt{\sqrt{\bar{P}_i^2 + \bar{\lambda}_i^4} + \bar{P}_i}$$

Substituting the solution (2.8) into conditions (2.6) and (2.7), we obtain $4m$ of the homogeneous equations with $4m$ arbitrary constants C_{ij}. Non-trivial solution to the problem exist only when the determinant of the coefficient matrix of the system of equations is equal to zero

$$\det(\mathbf{A}) = 0 \tag{2.9}$$

Eq (2.9) has been already the formula for frequency of free vibrations, ω. In this equation $\mathbf{A}$ denotes the matrix with the size $4m \times 4m$, which is defined as follows

$$\mathbf{A} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \vdots & & & \\ \mathbf{C}_1^1 & \mathbf{C}_2^0 & \mathbf{0} & \vdots & & \mathbf{0} & \\ \mathbf{0} & \mathbf{C}_2^1 & \mathbf{C}_3^0 & \vdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \vdots & \mathbf{C}_{m-2}^1 & \mathbf{C}_{m-1}^0 & \mathbf{0} \\ & \mathbf{0} & & \vdots & \mathbf{0} & \mathbf{C}_{m-1}^1 & \mathbf{C}_m^0 \\ & & & \vdots & \mathbf{0} & \mathbf{0} & \mathbf{B}_m \end{bmatrix} \tag{2.10}$$

Matrices $\mathbf{C}_1^1, \mathbf{C}_2^0, ..., \mathbf{C}_m^0$ have the size 4×4 and are derived on the grounds of the continuity conditions (2.6). These matrices are the following

$$\mathbf{C}_i^1 = \begin{bmatrix} \cos\alpha_i & \sin\alpha_i & \cosh\beta_i & \sinh\beta_i \\ -\alpha_i\sin\alpha_i & -\alpha_i\cos\alpha_i & \beta_i\sinh\beta_i & \beta_i\cosh\beta_i \\ -\alpha_i^2\cos\alpha_i & -\alpha_i^2\sin\alpha_i & \beta_i^2\cosh\beta_i & \beta_i^2\sinh\beta_i \\ \alpha_i^3\sin\alpha_i & -\alpha_i^3\cos\alpha_i & \beta_i^3\sinh\beta_i & \beta_i^3\cosh\beta_i \end{bmatrix} \tag{2.11}$$

for $i = 1, 2, \ldots, m-1$ and

$$\mathbf{C}_i^0 = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -\alpha_i & 0 & -\beta_i \\ \alpha_i^2 & 0 & -\beta_i^2 & 0 \\ 0 & \alpha_i^3 & 0 & -\beta_i^3 \end{bmatrix} \tag{2.12}$$

for $i = 2, \ldots, m$.

The matrices $\mathbf{B}_0$ and $\mathbf{B}_m$ have the size 2×4 and are obtained from the boundary conditions (2.7). The boundary conditions for functions $y_1(\xi_1)$ and $y_m(\xi_m)$ and also matrices $\mathbf{B}_0$ and $\mathbf{B}_m$ for the four various cases of beam ends attachement, are given below

— clamped beam

$$Y_1(0) = \frac{dY_1(0)}{d\xi_1} = 0 \qquad Y_m(1) = \frac{dY_m(1)}{d\xi_m} = 0$$

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}_m = \begin{bmatrix} \cos\alpha_m & \sin\alpha_m & \cosh\alpha_m & \sinh\alpha_m \\ -\sin\alpha_m & \cos\alpha_m & \sinh\alpha_m & \cosh\alpha_m \end{bmatrix} \tag{2.13}$$

— free beam

$$\frac{d^2Y_1(0)}{d\xi_1^2} = \frac{d^3Y_1(0)}{d\xi_1^3} = 0 \qquad \frac{d^2Y_m(1)}{d\xi_m^2} = \frac{d^3Y_m(1)}{d\xi_m^3} = 0$$

$$\mathbf{B}_0 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}_m = \begin{bmatrix} -\cos\alpha_m & -\sin\alpha_m & \cosh\alpha_m & \sinh\alpha_m \\ \sin\alpha_m & -\cos\alpha_m & \sinh\alpha_m & \cosh\alpha_m \end{bmatrix} \tag{2.14}$$

— pinned beam

$$Y_1(0) = \frac{d^2Y_1(0)}{d\xi_1^2} = 0 \qquad Y_m(1) = \frac{d^2Y_m(1)}{d\xi_m^2} = 0$$

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B}_m = \begin{bmatrix} \cos\alpha_m & \sin\alpha_m & \cosh\alpha_m & \sinh\alpha_m \\ -\cos\alpha_m & -\sin\alpha_m & \cosh\alpha_m & \sinh\alpha_m \end{bmatrix} \tag{2.15}$$

— sliding beam

$$\frac{dY_1(0)}{d\xi_1} = \frac{d^3Y_1(0)}{d\xi_1^3} = 0 \qquad \frac{dY_m(1)}{d\xi_m} = \frac{d^3Y_m(1)}{d\xi_m^3} = 0$$

$$\mathbf{B}_0 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}_m = \begin{bmatrix} -\sin\alpha_m & \cos\alpha_m & \sinh\alpha_m & \cosh\alpha_m \\ \sin\alpha_m & -\cos\alpha_m & \sinh\alpha_m & \cosh\alpha_m \end{bmatrix} \tag{2.16}$$

The eigenfunctions $\psi_n(\zeta)$ for the problem (2.1) and (2.2) corresponding to the frequencies λ_n are described as follows

$$\psi_n(\zeta) = \sum_{i=1}^{m} Y_{in}(\xi_i)[H(\zeta - l_{i-1}) - H(\zeta - l_i)] \tag{2.17}$$

where $Y_{in}(\xi_i)$ functions are given by the formula (2.8), whereas $C_{11}^{(n)} = C^{(n)}$ and $C_{ij}^{(n)}$ are defined in such a way that the conditions (2.6) and (2.7) are satisfied. Constant $C^{(n)}$ is selected so as to satisfy the condition: $\int_0^1 [\psi_n(\zeta)]^2 d\zeta = 1$.

2.2. Problem of free vibration of the beam with the axial load varying continuously

In the case of continuous load varying along the length of the beam we assumed that this load may be expressed in form of the power series

$$P(\zeta) = \sum_{i=1}^{\infty} \frac{P_i^*}{i!} \zeta^i \tag{2.18}$$

To solve the problem, each of the four independent partial solutions $Y_k^*(\zeta)$, $k = 1,2,3,4$ of Eq (2.1) can be written in the form of a power series

$$Y_k^*(\zeta) = \sum_{i=1}^{\infty} \frac{Y_{ki}^*}{i!} \zeta^i \qquad k = 1,2,3,4 \tag{2.19}$$

Let us assume that the functions $Y_k^*(\zeta)$ satisfy the following conditions

$$\frac{d^i Y_k^*(0)}{d\zeta^i} = \delta_{ki+1} \qquad i = 0,1,2,3 \qquad k = 1,2,3,4 \tag{2.20}$$

where δ_{mn} denotes the Kronecker delta. It means, that

$$Y_{ki}^* = \delta_{ki+1} \qquad i = 0,1,2,3 \qquad k = 1,2,3,4 \tag{2.21}$$

Considering Eqs (2.18) and (2.19) in Equation (2.1), we get, after some transformations, the following formula for the unknown coefficients Y_{ki}^* of expansion (2.19)

$$Y_{ki+4}^* + \sum_{j=1}^{i+1} \binom{i+1}{j} Y_{kj+1}^* P_{i+1-j} - \lambda^4 Y_{ki}^* = 0 \tag{2.22}$$

for $i = 0,1,2,...;\ k = 1,...,4$.

The general solution for Eq (2.1) may thus be written in the form

$$Y(\zeta) = \sum_{k=1}^{4} C_k Y_k^{*}(\zeta) \tag{2.23}$$

Then, using the boundary conditions (e.g. in the form of Eqs $(2.13)_1 \div (2.16)_1$) one can get the homogeneous system of four equations with respect to the unknown C_k. For existence of non-trivial solution to the problem it is necessary that the determinant of the coefficient matrix for that system of equations must be equal to zero

$$\det(\mathbf{K}) = 0 \tag{2.24}$$

In matrix $\mathbf{K}$ some sub-matrices may be distinguished

$$\mathbf{K} = \begin{bmatrix} \mathbf{L}_0 \\ \mathbf{L}_1 \end{bmatrix} \tag{2.25}$$

which are defined in dependence on the boundary conditions. Matrices $\mathbf{L}_0$ and $\mathbf{L}_1$ have the size 2×4. For writting the elements of matrix $\mathbf{L}_1$ the following denotation has been used

$$\bar{Y}_k^j = \sum_{i=j}^{\infty} \frac{Y_{ki}^{*}}{(i-j)!} \qquad k = 1,2,3,4 \qquad j = 0,1,2,3 \tag{2.26}$$

Below, matrices $\mathbf{L}_0$ and $\mathbf{L}_1$ have been given for the four cases of the beam ends attachements

— clamped beam

$$\mathbf{L}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{L}_1 = \begin{bmatrix} \bar{Y}_1^0 & \bar{Y}_2^0 & \bar{Y}_3^0 & \bar{Y}_4^0 \\ \bar{Y}_1^1 & \bar{Y}_2^1 & \bar{Y}_3^1 & \bar{Y}_4^1 \end{bmatrix} \tag{2.27a}$$

— free beam

$$\mathbf{L}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{L}_1 = \begin{bmatrix} \bar{Y}_1^2 & \bar{Y}_2^2 & \bar{Y}_3^2 & \bar{Y}_4^2 \\ \bar{Y}_1^3 & \bar{Y}_2^3 & \bar{Y}_3^3 & \bar{Y}_4^3 \end{bmatrix} \tag{2.27b}$$

— pinned beam

$$\mathbf{L}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{L}_1 = \begin{bmatrix} \bar{Y}_1^0 & \bar{Y}_2^0 & \bar{Y}_3^0 & \bar{Y}_4^0 \\ \bar{Y}_1^2 & \bar{Y}_2^2 & \bar{Y}_3^2 & \bar{Y}_4^2 \end{bmatrix} \tag{2.27c}$$

— sliding beam

$$\mathbf{L}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{L}_1 = \begin{bmatrix} \bar{Y}_1^1 & \bar{Y}_2^1 & \bar{Y}_3^1 & \bar{Y}_4^1 \\ \bar{Y}_1^3 & \bar{Y}_2^3 & \bar{Y}_3^3 & \bar{Y}_4^3 \end{bmatrix} \tag{2.27d}$$

Using $\mathbf{L}_0$ and $\mathbf{L}_1$ matrices, the formula for the frequencies of vibration (Eq (2.24)) may be written for ten various conditions of beam ends attachment.

The eigenfunctions $\psi_n(\zeta)$ for the differential problem Eqs (2.1) and (2.2), which correspond to the frequencies λ_n have the form

$$\psi_n(\zeta) = \sum_{k=1}^{4} C_k^{(n)} Y_k^{*(n)}(\zeta) \tag{2.28}$$

where $C_1^{(n)} = C^{(n)}$ and all other constants are obtained from the conditions (2.2). Constant $C^{(n)}$ should be choosen in such a way that condition: $\int_0^1 [\psi_n(\zeta)]^2 d\zeta = 1$ has to be satisfied.

3. Spectral density of the forced vibration

Let us consider vibrations of a beam forced by a transverse force varying in time and with the intensity of $z(\zeta, t)$. Assuming additionally the existence of the dissipation of energy effect characterized by forces which are proportional to the velocity of particular points of the system. The differential equation of the beam motion is thus as follows

$$\frac{\partial^4 W}{\partial \zeta^4} - \frac{\partial}{\partial \zeta}\Big[P(\zeta)\frac{\partial W}{\partial \zeta}\Big] + \alpha(\zeta)\frac{\partial W}{\partial t} + r\frac{\partial^2 W}{\partial t^2} = z(\zeta, t) \tag{3.1}$$

where $W = \frac{w}{L}$, $r = \frac{\rho A L^4}{EI}$. Assume that the boundary conditions are fulfilled in the form of Eq (2.2) and the zero initial conditions.

Solution to the problem is searched for in the form of sum of products from the eigenfunctions $\psi_k(\zeta)$ and functions $T_k(t)$, dependent exclusively upon time

$$w(\zeta, t) = \sum_{k=1}^{\infty} \psi_k(\zeta) T_k(t) \tag{3.2}$$

Function $T_k(t)$ will be determined from the equation, which we get by introducing the expansion Eq (3.2) into Eq (3.1), multiplying the obtained equation by the function $\psi_l(\zeta)$, $l = 1, 2, ...$ and by integration in the interval (0,1). Using the orthogonality of the eigenfunctions, the equation has the following form

$$\ddot{T}_k + 2\sum_{n=1}^{\infty} h_{nk}\dot{T}_n + b_k T_k = a_k(t) \tag{3.3}$$

where

$$h_{nk} = \frac{1}{2r}\int_0^1 \alpha(\zeta)\psi_n(\zeta)\psi_k(\zeta)d\zeta \tag{3.4}$$

$$a_k(t) = \frac{1}{r}\int_0^1 z(\zeta,t)\psi_k(\zeta)d\zeta \tag{3.5}$$

and $b_k = \lambda_k^4/r$. Furthermore, assume that the condition: $h_{nk} \cong 0$ for $n \neq k$, is satisfied (cf Skalmierski, 1985).

Applying the Fourier tranformation to Eq (3.3) and making use of the convolution theorem, the solution to Eq (3.1) may be expressed as follows

$$W(\zeta,t) = \sum_{k=1}^{\infty}\int_{-\infty}^{t} G_k(t-\tau)a_k(\tau)d\tau\psi_k(\zeta) \tag{3.6}$$

where $G_k(\cdot)$ is the impulse delta function.

Correlation function of the output signal is defined in the following way (cf Skalmierski, 1985)

$$K_w(t_1,t_2) = \sum_{k=1}^{\infty}\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} G_k(t_1-\tau_1)G_k(t_2-\tau_2)K_{a_k}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{3.7}$$

where $K_{a_k}(\tau_1,\tau_2)$ is the correlation function of component of the signal $a_k(\tau)$. Assuming, that the input signal is a stationary one in a broad sense, e.g.

$$K_{a_k}(\tau_1,\tau_2) = K_{a_k}(\tau_1-\tau_2) \tag{3.8}$$

we can determine the spectral density as the Fourier transform of the correlation function

$$S_w(\omega) = \sum_{k=1}^{\infty}|\bar{G}_k(\mathrm{i}\omega)|^2 S_{a_k}(\omega) \tag{3.9}$$

where

$$|\bar{G}_k(\mathrm{i}\omega)|^2 = \frac{1}{(\omega^2-b_k)^2+4\omega^2 h_{kk}} \tag{3.10}$$

Function $S_{a_k}(\omega)$ is the specıral density of the input signal and is described by the formula

$$S_{a_k}(\omega) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega t}K_{a_k}(t)dt \qquad \mathrm{i} = \sqrt{-1} \tag{3.11}$$

Vibrating body is a source of the sound waves. Energy flux I radiated by the vibrating beam is defined by the formula (cf Skalmierski, 1986)

$$I = \frac{1}{2\pi} c\rho_0 \int\limits_{-\infty}^{+\infty} S_w(\omega)\omega^2 d\omega \tag{3.12}$$

where c denotes the velocity of wave propagation and ρ_0 is the density of the medium.

Let us assume, that the function (input signal) existing on the right hand side of Eq (3.1) has the form

$$z(\zeta,t) = \sum_{k=1}^{K} a_k \psi_k(\zeta) \sin(\nu t + \varphi) \tag{3.13}$$

The correlation function of the input signal is thus defined by the relationship

$$\begin{aligned} K_z(\tau) &= \sum_{k=1}^{K}\sum_{n=1}^{K} \int\limits_0^1 \psi_k(\zeta)\psi_n(\zeta)d\zeta a_k a_n \frac{\nu}{2\pi} \int\limits_0^{\frac{2\pi}{\nu}} \sin(\nu t + \varphi)\sin(\nu(t+\tau)+\varphi)dt = \\ &= \frac{1}{2}\sum_{k=1}^{K} a_k^2 \cos \nu\tau \end{aligned} \tag{3.14}$$

Considering the relation (3.11), we can find the spectral densities of the components of the input signal

$$S_{a_k}(\omega) = \frac{\pi}{2} a_k^2 [\delta(\omega - \nu) + \delta(\omega + \nu)] \tag{3.15}$$

On the other hand on the grounds of Eqs (3.9), (3.10), (3.12) and (3.14) we get

$$I(\nu) = \frac{1}{2} c\rho_0 \nu^2 \sum_{k=1}^{\infty} \frac{a_k^2}{(\nu^2 - b_k)^2 + 4\nu^2 h_{kk}} \tag{3.16}$$

4. Results of the numerical computation

Numerical computations concern the investigation of axial force distribution influence on beam vibration. The influence of the axial force on the frequency change of free vibration of the beam has been shown, and their influence on the change of flux energy, which has been radiated by the vibrating beam forced by

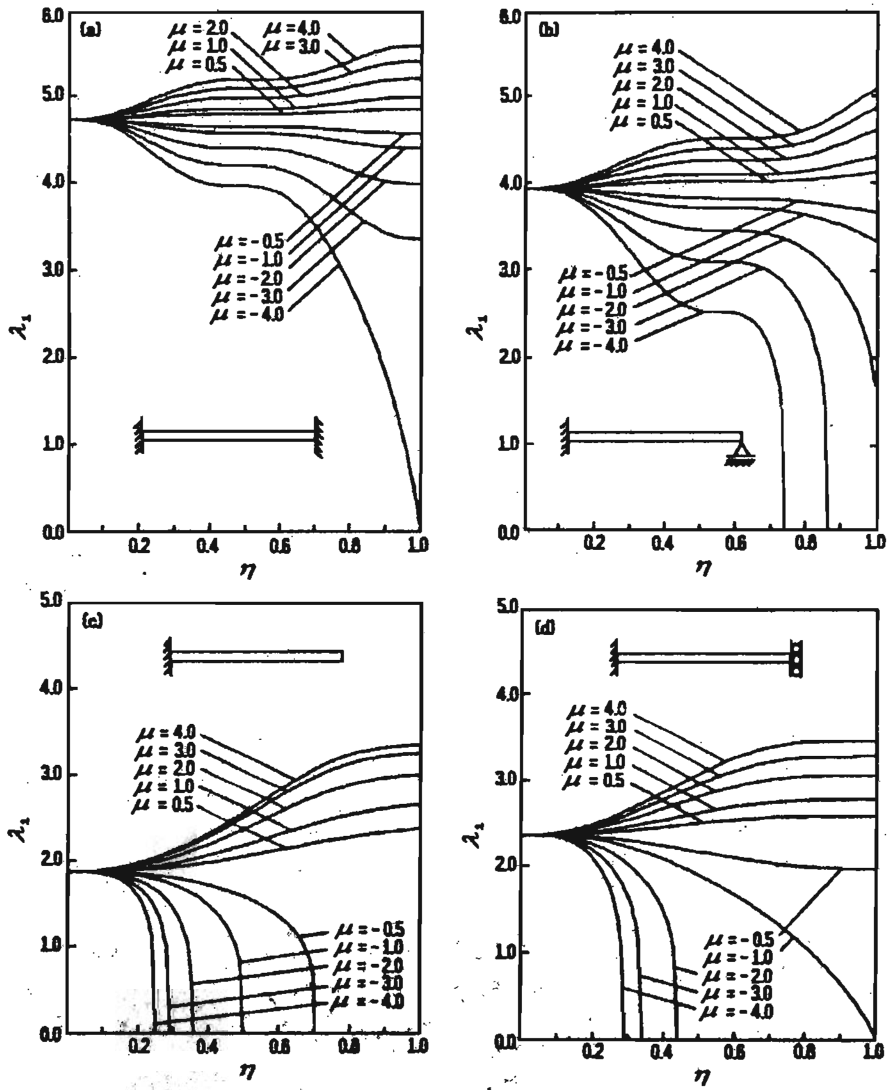

Fig. 1. First natural frequency of the beam loaded by the axial force: $P(\zeta) = \mu\pi^2[1 - H(\zeta - \eta)]$; (a) clamped-clamped beam, (b) clamped-pinned beam, (c) clamped-free beam, (d) clamped-sliding beam

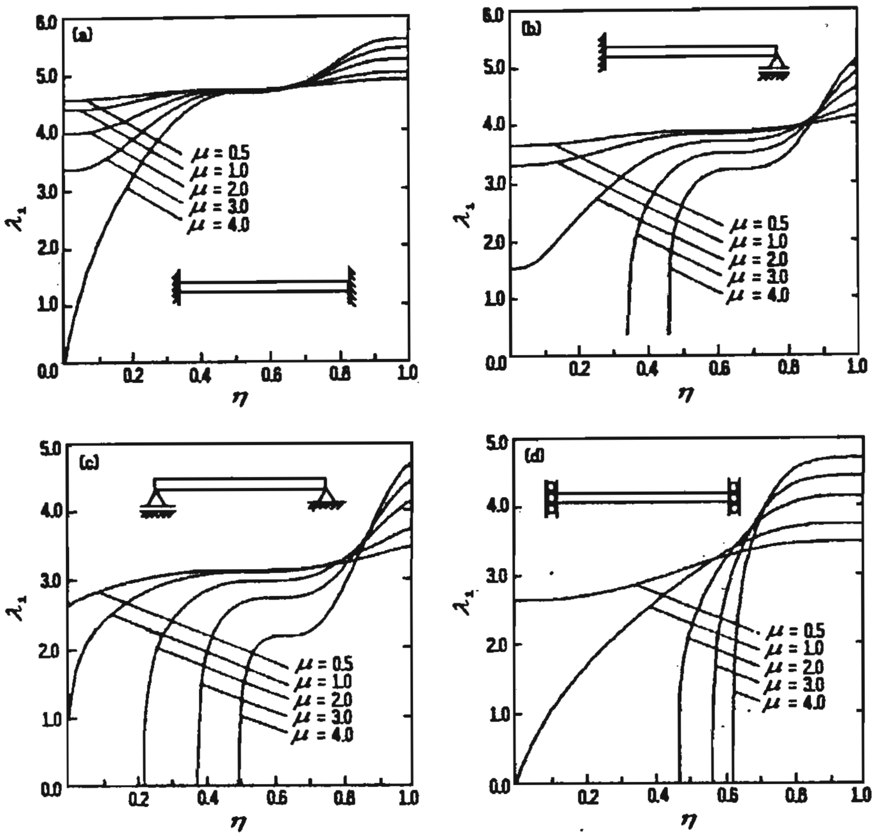

Fig. 2. First natural frequency of the beam loaded by the axial force: $P(\zeta) = \mu\pi^2[1 - 2H(\zeta - \eta)]$; (a) clamped-clamped beam, (b) clamped-pinned beam, (c) pinned-pinned beam, (d) sliding-sliding beam

the sine signal. Calculations have been performed assuming the following data: $L = 4.0$ m, $EI = 4.084 \cdot 10^5$ Nm^2, $\rho A = 30.394$ Ns^2/m^2.

Fig.1 ÷ 3 concern the influence of the force, changing in stepwise manner along the length of the beam, on free vibration frequencies. In Fig.1a÷d there are the curves, which depict this influence in the four cases of beam loaded by the axial force of constant value (which is characterized by dimensionless quantity $\mu\pi^2$) in the interval $(0, \eta)$. Considerable changes in first frequency of vibration ($\lambda_n = L \sqrt[4]{\rho A \omega_n^2/(EI)}$ are the dimensionless vibration frequency) have been observed not only while compression $(\mu < 0)$ but with tension as well $(\mu > 0)$.

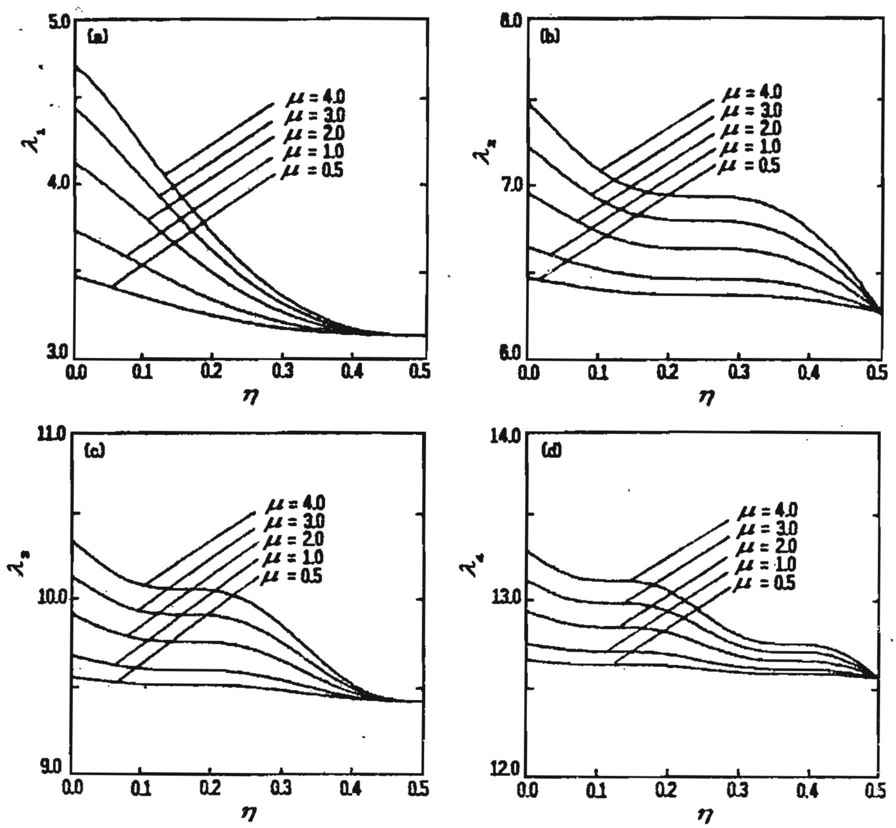

Fig. 3. First four natural frequencies of pinned-pinned beam loaded by the tensile force:
$P(\zeta) = \mu\pi^2[H(\zeta-\eta) - H(\zeta+\eta-1)]$

Figure 2 is for the beam, which is subjected to tension with the constant force $\mu\pi^2$ in the $(0,\eta)$ intrval and in $(\eta,1)$ is compressed by the constant force $-\mu\pi^2$. Calculations have been performed for three various ways of beam ends attachments. It is seen here the effect, on free vibration frequency, both the force magnitude and the way of beam ends attachments.

In Table 1 results of the first, dimensionless vibration frequency obtained by the Rayleigh method (cf Skalmierski, 1985) and these obtained by using the method proposed in this work, are listed. Comparison of the results concerns the pinned-pinned beam loaded by the axial force

$$P(\zeta) = \mu\pi^2 \text{ for } \zeta \in (0,\eta) \text{ and } P(\zeta) = -\mu\pi^2 \text{ for } \zeta \in (\eta,1)$$

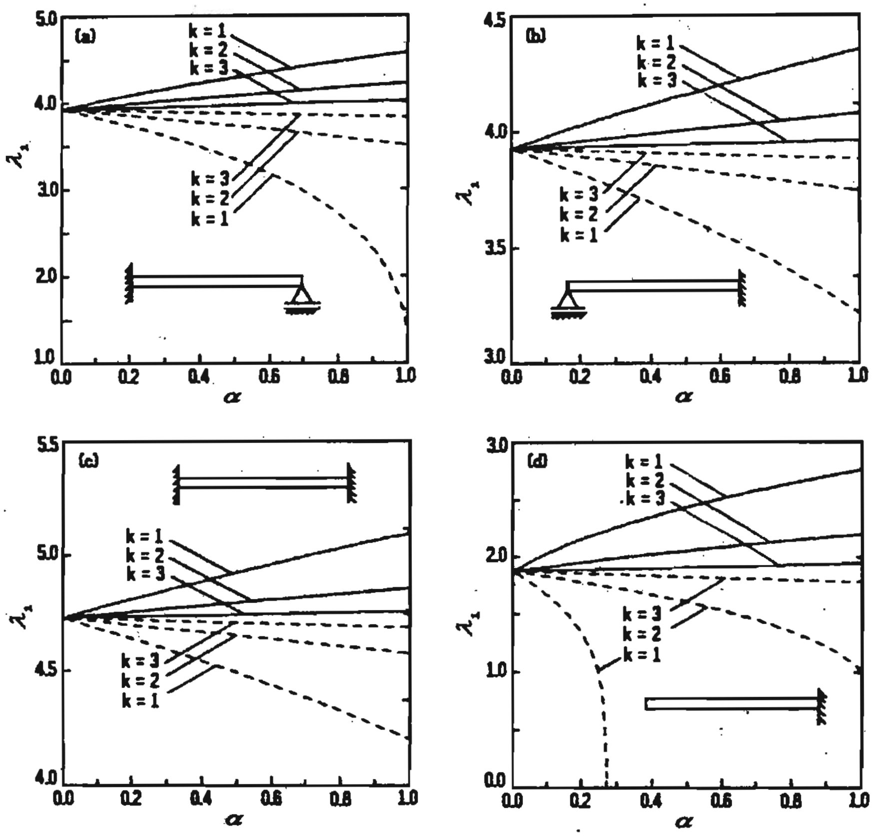

Fig. 4. First natural frequency of the beam loaded by the tensile force: $P(\zeta) = 3\alpha\pi^2\zeta^k$ (solid line) or compressing force: $P(\zeta) = -3\alpha\pi^2\zeta^k$ (dashed line) versus parameter α $(0 \le \alpha \le 1)$; (a) clamped-pinned beam, (b) pinned-clamped beam, (c) clamped-clamped beam, (d) free-clamped beam

Table 1

η	λ_1			
	$\mu = 0.5$		$\mu = 1.0$	
	Present method	Rayleigh method	Present method	Rayleigh method
0.1	2.8514	2.8669	2.3754	2.4780
0.3	3.0818	3.1027	2.9671	3.0622
0.5	3.1320	3.1416	3.1022	3.1416
0.7	3.1615	3.1791	3.1488	3.2153
0.9	3.3492	3.3587	3.5078	3.5404

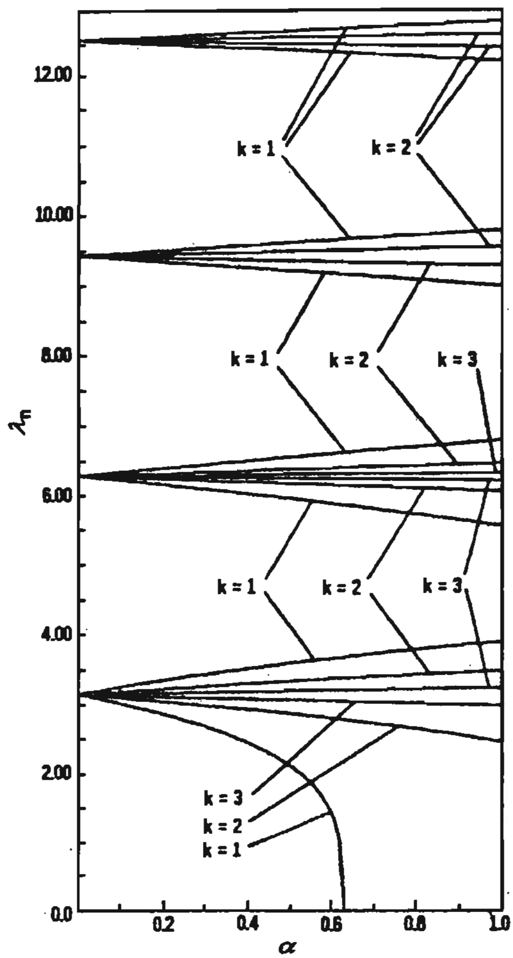

Fig. 5. First four natural frequencies of the pinned-pinned beam loaded by the axial force: $P(\zeta) = \pm 3\alpha\pi^2\zeta^k$

The curves in Fig.3 present changes of the first four vibration frequencies for pinned-pinned beam loaded by tensile force $(\mu\pi^2)$ in interval $(\eta, 1-\eta)$ with respect to η $(0 \le \eta \le 1)$ and the force magnitude (parameter μ).

Changes of the first natural frequency of the beam loaded by the axial force $P(\zeta) = \pm 3\alpha\pi^2\zeta^k$ versus parameter α $(0 \le \alpha \le 1)$ have been shown in Fig.4. Solid line denotes dimensionless frequency values for the tensile force and dashed line for the compressive force. Tensile force changes itself along the length of the beam in the linear way $(k = 1)$, with second $(k = 2)$ or third $(k = 3)$ power of the variable ζ in such a way that in all the cases – points $(0,0)$ and $(1, 3\pi^2\alpha)$

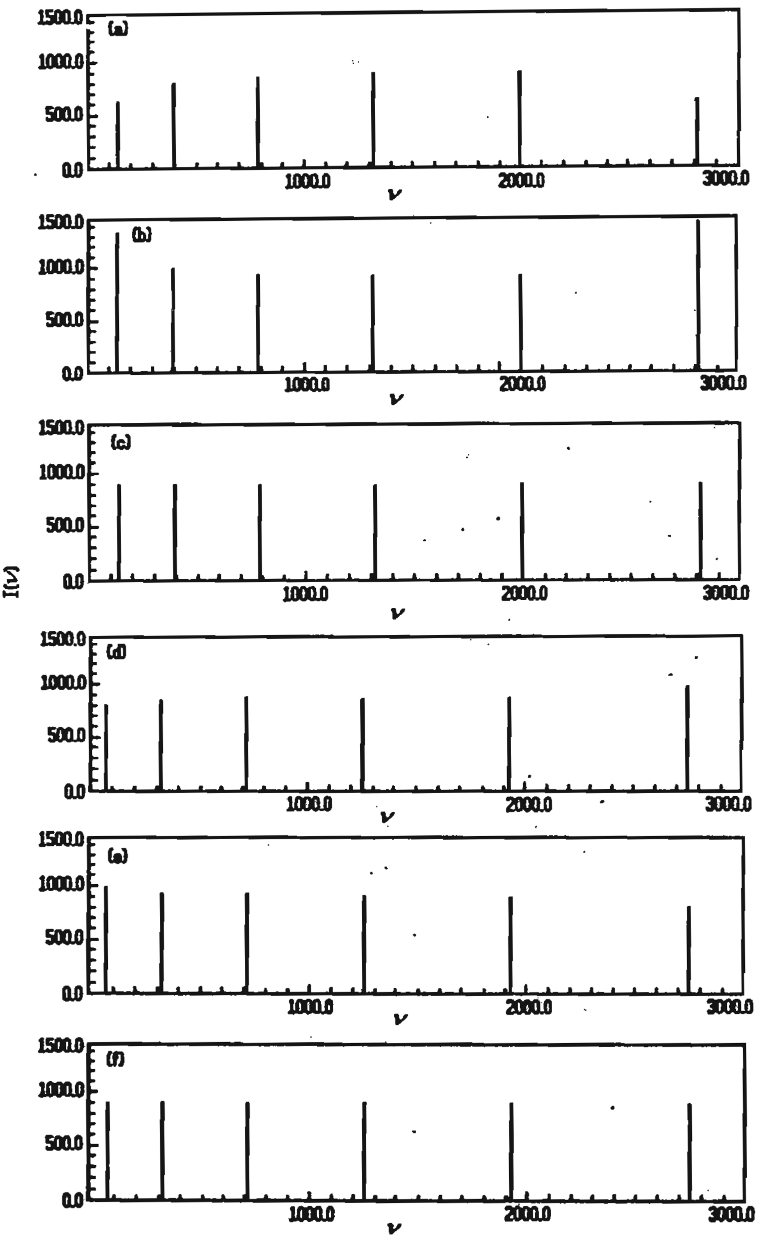

Fig. 6. Energy flux emitted by the vibrating clamped-pinned beam

belong to the force diagram.

Curves in Fig.5 illustrate first four natural frequency changes of the pinned-pinned beam. The beam is loaded by an axial force like in the case discussed with Fig.4.

In Fig.6 the energy flux emitted by the clampd-pinned beam has been presented graphically. The calculations have been performed on the grounds of formula (3.16) for the forced vibrations determinated by sinusoidal signal given by (3.14). For computation it have been assumed: $K = 6$, $a_k = 1$ for $k = 1, ..., 6$, $c\rho_0 = 221$ kg s^{-1} m^{-2}. Curves in Fig.6a,b,c have been obtained for the case of tensile force: $P(\zeta) = 2\pi^2\zeta$, and Fig.6d,e,f have been drawn assuming the compressive force: $P(\zeta) = -2\pi^2\zeta$. For computational purposes various damping characteristics: $\alpha(\zeta) = 1 - \zeta$ (Fig.6a and Fig.6d), $\alpha(\zeta) = \zeta$ (Fig.6b and Fig.6e), $\alpha(\zeta) = 0.5$ (Fig.6c and Fig.6f). Irregular distribution attenuating characteristics along the length of the beam has an effect on energy flux both in the case of tension (Fig.6a,b) and compression (Fig.6d,e).

5. Conclusions

In this work effect of axial force varying along the length of the beam on the vibration of the beam has been studied. Solution of the problem of free vibration of the beam has been obtained in the closed form for the case of the force being stepwise constant and force given in the form of the power series. It has been proved that the change in the axial force distribution causes the change of free vibration frequency. That effect is substantial irrespective of boundary conditions both in case of the force being stepwise constant and in case of the force varying in the continuous form (in form of poynomial). It has been stated that distribution of the axial force has an effect on energy flux emitted by the vibrating beam. It has been also proved that the change of the irregular distribution of vibration dumping characteristics along the length of the beam cause the change of energy flux emitted both with existence of tensile and compressive forces as well.

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Wpływ obciążeń osiowych na drgania poprzeczne belki Bernoulli'ego-Eulera

Streszczenie

W pracy przedstawiono rozwiązanie problemu drgań belki Bernoulli'ego-Eulera obciążonej zmienną wzdłuż długości siłą osiową. Wyznaczono równanie na częstości drgań i postacie drgań własnych dla przypadku obciążenia siłą osiową przedziałami stałą oraz obciążenia danego w postaci szeregu potęgowego. W przypadku drgań wymuszonych określono strumień energii wyemitowanej przez drgającą belkę. Wpływ obciążenia osiowego na drgania belki dla różnych sposobów zamocowania jej końców przedstawiono na wykresach.

Manuscript received January 8, 1992; accepted for print January 20, 1993